Tracing, mixing and entropy II

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Topological transitivity and its variants

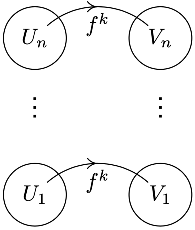
Definition

A dynamical system (X, T) is (topologically)

1. transitive if for all $U, V \neq \emptyset$ open in X there is $n > 0$ such that $T^n(U) \cap V \neq \emptyset$.
2. totally transitive if T^n is transitive for every n
3. weakly mixing if $T \times T$ is transitive.
4. mixing if for all $U, V \neq \emptyset$ open in X there is $N > 0$ such that $T^n(U) \cap V \neq \emptyset$ for every $n \geq N$.
5. exact if for all $U \neq \emptyset$ open in X there is $N > 0$ such that $T^N(U) = X$.

1. On spaces without isolated points, T is transitive iff there is a point with dense orbit
2. Trans(T) - the set of points with dense orbit (residual in transitive maps)

Weak Mixing



Definition

T is weak (topologicall) mixing if $T \times T$ is transitive. Equivalently:

1. For open $U_1, U_2, V_1, V_2 \neq \varnothing$, $(\exists k \in \mathbb{N})\ T^k(U_1) \cap V_1,\ T^k(U_2) \cap V_2 \neq \varnothing$.
2. For open $U_1, \ldots, U_n, V_1, \ldots, V_n \neq \varnothing$ $(n \geq 2)$ $(\exists k \in \mathbb{N})(\forall i \leq n)\ T^k(U_i) \cap V_i \neq \varnothing$.

- Mixing $\implies$ weakly mixing $\implies$ totally trans. $\implies$ transitive
- Implications cannot be reversed in general.
- Totally transitive + dense periodic points $\implies$ weak mixing
- In some settings:
  - Weak mixing $\implies$ Mixing
  - Totally transitive $\implies$ Weak mixing

Full (two-sided) shift on two symbols

We define the following metric space:

1. $\Sigma_2 = \{0,1\}^{\mathbb{Z}}$.
2. where a metric d on Σ_2 is defined as follows:

$$d(x,y) = \begin{cases} 2^{-k} & \text{, if } x \neq y \\ 0 & \text{, if } x = y \end{cases}$$

where k minimal integer such that $x_k \neq y_k$ or $x_{-k} \neq y_{-k}$.

	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6
$x=$	0	1	1	1	0	0	1	0	1	1	0	1	0
$y=$	1	1	1	1	0	0	1	0	1	1	1	0	0

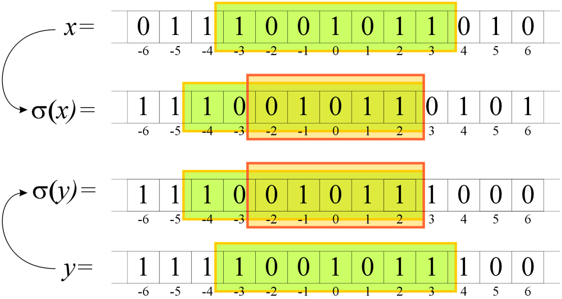
$$d(x,y) = 2^{-4}$$

Full (two-sided) shift on two symbols

1. We define the shift map $\sigma : \Sigma_2 \to \Sigma_2$ by:

$$\sigma(x)_i = x_{i+1}.$$

2. The map σ is continuous.



Forbidden words

1. Let $\mathcal{F}$ be a set of finite sequences, i.e.

$$\mathcal{F} \subset \bigcup_{n=1}^{\infty} \{0,1\}^n .$$

2. Define the following set

$$X_{\mathcal{F}} = \{x \in \Sigma_2 \ : \ (x_i \dots x_{i+n}) \notin \mathcal{F} \text{ for every } i \in \mathbb{Z}, n \geq 0\}$$

Theorem

A set $X \subset \Sigma_2$ is a shift (i.e. is closed and $\sigma(X) = X$) iff $X = X_{\mathcal{F}}$ for some set of forbidden words $\mathcal{F}$.

Spacing subshifts - Lau & Zame

1. Given $\mathcal{P} \subset \mathbb{N}$ we define a set $X_{\mathcal{P}} \subset \Sigma_2$ by

$$X_{\mathcal{P}} = \{x \in \Sigma_2 \ : \ x_i = x_j = 1, i < j \quad \Longrightarrow \quad j - i \in \mathcal{P}\}.$$

2. It follows from the definition that $X_{\mathcal{P}} = \sigma(X_{\mathcal{P}})$ and obviously $X_{\mathcal{P}}$ is closed (so it is a subshift of Σ_2). So $(X_{\mathcal{P}}, \sigma)$ is an invertible DS.
3. $\mathcal{P} \subset \mathbb{N}$ is thick when

$$\forall\, n \quad \exists\, i \qquad \{i, i+1, \ldots, i+n\} \subset \mathcal{P}$$

4. and syndetic when $\mathbb{N} \setminus \mathcal{P}$ is not thick.

Theorem

1. If the map $\sigma_{\mathcal{P}}$ is mixing iff $\#\left(\mathbb{N} \setminus \mathcal{P}\right) < \infty$ ($\mathcal{P}$ is co-finite),
2. If $\mathcal{P}$ is thick iff the map $\sigma_{\mathcal{P}}$ is weakly mixing.

- as a consequence weak mixing $\not\Longrightarrow$ mixing.

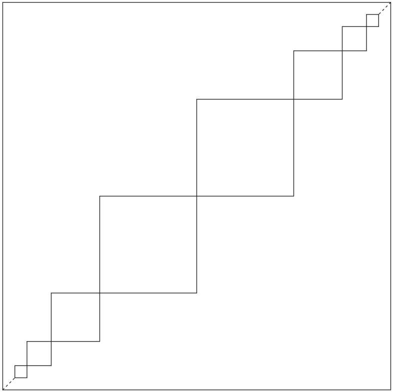
Transitive maps of $I = [0, 1]$

1. If (I, T) is transitive then it has dense periodic points
2. If (I, T^2) is transitive then (I, T) is mixing.
3. If (I, T) is transitive but (I, T^2) is not then
   - There is a fixed point $p \in (0, 1)$ such that
   - intervals $I_1 = [0, p]$ and $I_2 = [p, 1]$ are permuted by T that is
   - $T(I_1) = I_2$, $T(I_2) = I_1$ and
   - $T^2|_{I_i}$ is transitive for $i = 1, 2$.
4. In the circle (or more widely, topological graphs) it is similar, if we rule out irrational rotations (or map is non-invertible)

Piecewise linear maps

1. (I, T) is piecewise linear if
   - there is division $P = 0 = a_0 < a_1 < \ldots < a_n < a_{n+1} = 1$ such that
   - $T|_{[a_i, a_{i+1}]}$ is linear for every $i = 0, 1, \ldots, n$
2. Piecewise linear map is Markov if $T(P) \subset P$
3. Denote $I_i = [a_i, a_{i+1}]$. We can define a transition graph G_T for T where
   - I_i are vertices
   - there is an edge $I_i \longrightarrow I_j$ if $I_j \subset T(I_i)$
4. Graph is strongly connected if there is path between any two vertices.
5. If G_T is strongly connected but not a cycle and T is Markov then T is transitive.
6. If matrix A associated with G_T is primitive (i.e. A^n has all entries positive) then T is exact (mixing Markov maps are always exact).
7. If λ is leading eigenvalue of G_T then $h_{\mathrm{top}}(T) = \log \lambda$.

Mixing but not exact example



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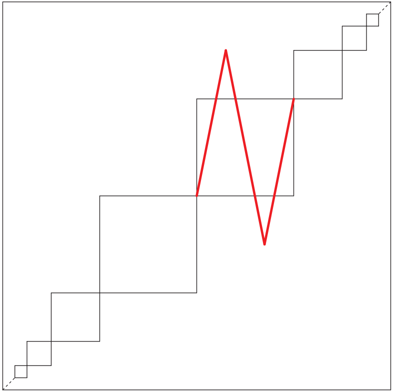

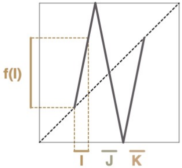

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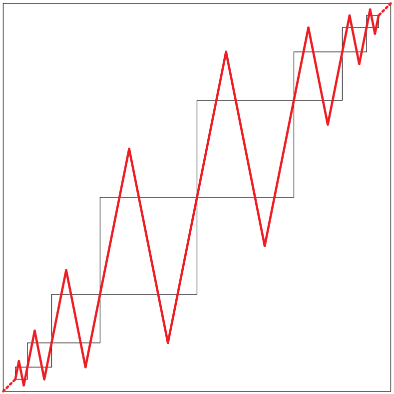

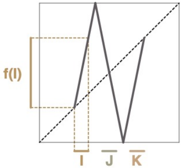

Mixing but not exact maps - an entropy gap

1. if T is mixing interval map then $h_{\text{top}}(T) > \frac{1}{2}\log(2)$ and this number is infimum of possible entropies
2. if T is mixing circle map then $h_{\text{top}}(T) > 0$ and this number is infimum of possible entropies
3. if T is mixing interval or circle map which is not exact then $h_{\text{top}}(T) > \frac{1}{2}\log(3)$.

Characterization of Weak Mixing by Xiong and Yang

1. Let (T, X) be a dynamical system.
2. A set $S \subset X$ is *chaotic with respect to a sequence* $\{p_i\}_{i=1}^{\infty}$ when
   1. for every continuous function $F : S \to X$
   2. there is a subsequence $\{q_j\}_{j=1}^{\infty} \subset \{p_i\}_{i=1}^{\infty}$ such that

$$\forall\, x \in S \qquad \lim_{j\to\infty} T^{q_j}(x) = F(x).$$

Theorem

The following conditions are equivalent:

1. (X, T) is weakly mixing (mixing)
2. There is a dense set S which is at most countable union of Cantor sets which is chaotic with respect to an increasing sequence $\{p_i\}_{i=1}^{\infty}$ (respectively: every increasing sequence).

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Uniformly rigid systems

1. Dynamical system (X, T) is uniformly rigid if for every $\varepsilon > 0$ there is $n > 0$ such that $d(x, T^n(x)) < \varepsilon$ for every $x \in X$.
2. Natural examples:
   - Periodic orbit
   - Irrational rotation of the circle (or $\mathbb{T}^n$).
3. Uniformly rigid systems cannot be mixing (if nontrivial).
4. Uniformly rigid systems do not contain (proper) asymptotic pairs.
5. But uniformly rigid system can be weakly mixing (Glasner & Maon, 1989)
   - On various spaces of the form $\mathbb{S}^1 \times Y$ (including all $\mathbb{T}^n$, $n \geq 2$) there exists weakly mixing, uniformly rigid, minimal dynamical system.
6. Fathi and Herman proved (1977) residual set of maps with weak mixing in:

$$\overline{O(\mathbb{T}^2)} = \mathrm{cl}\{h \circ R_\alpha \circ h^{-1} : h \in \mathrm{Diff}_\infty(\mathbb{T}^2), \alpha \in \mathbb{T}^2\}$$

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Theorem (Glasner & Maon)

Let $\mathcal{H}_0(Y)$ be a path component of identity in $\mathcal{H}(Y)$. If Y is nontrivial and action of $\mathcal{H}_0(Y)$ is minimal on Y then there is weakly mixing, uniformly rigid and minimal homeomorphism (on $\mathbb{S}^1 \times Y$) in

$$\overline{\{G^{-1} \circ (R_\alpha \times \mathrm{id}) \circ G : G \in \mathcal{H}(\mathbb{S}^1 \times Y)\}}$$

- (1982) M. Handel: pseudo-circle as minimal set and attractor

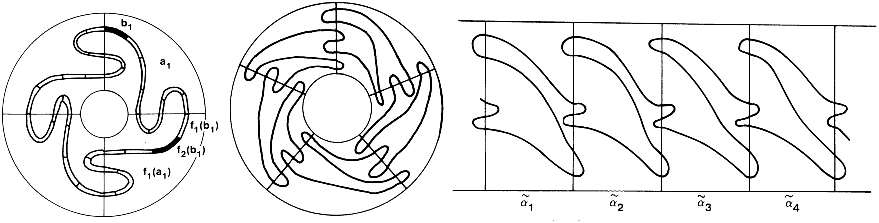

Figure: by M. Handel

Theorem (Handel, 1982)

There exists a C^∞-smooth diffeomorphism of the plane F with pseudo-circle as an attracting minimal set. In addition, F has a well defined irrational rotation number but is not semi-conjugate to a circle rotation (Thurston?).

Eigenfunctions and weak mixing

1. $\chi \colon X \to \mathbb{S}^1$ is an eigenfunction of T if it is continuous and $\chi \circ T = R_\alpha \circ \chi$ for some α.
2. Suppose T is a minimal homeomorphism. The following are equivalent:
   - (X, T) is weakly mixing
   - all eigenfunctions are constant.

   Additionally, if X is connected and χ is not constant, then χ is surjective.
3. Then, if T is a minimal homeomorphism and X is connected, (X, T) is weakly mixing iff it is not semi-conjugate to a rotation.

Corollary

Handel's example is weakly mixing.

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Uniformly rigid systems (Katznelson & Weiss example)

1. If (X, f) is uniformly rigid and proximal then it is completely scrambled.
2. Katznelson and Weiss provided a method of construction of uniformly rigid, proximal, transitive systems.
3. Constructed system is a subset of the Hilbert cube $[0,1]^{\mathbb{N}}$ with metric

$$d(\alpha, \beta) = \sum_{n=0}^{\infty} \frac{|\alpha(n) - \beta(n)|}{2^n}$$

and left shift $\sigma(\alpha)(i) = \alpha(i+1)$ on it.

Uniformly rigid systems (Katznelson & Weiss example)

1. Fix $L \geq 2$ and start with function $a_0\colon [-1,1] \to [0,1]$ such that

$$|a_0(s_1) - a_0(s_2)| \leq L|s_1 - s_2| \quad \text{and } a_0(1) = a_0(-1) = 1, \quad a_0(0) \neq 1.$$

2. Define $a_1\colon \mathbb{R} \to [0,1]$ by $a_1(s) = a_0(s)$ when $|s| \leq 1$, and $a_1(s+2) = a_1(s)$ for all $s \in \mathbb{R}$.
3. Put $a_p(s) = a_1(s/p)$, i.e. "stretch" graph of a_1.
4. Finally $a_\infty(s) = \sup_n a_{p_n}(s)$ for a sequence p_n where $p_{n+1} = p_n k_n$, $\{k_n\}_{n=1}^\infty$ is strictly increasing and 8 divides each k_n.
5. Define $\alpha(n) = a_\infty(n)$ and $\mathbb{X} = \overline{\{\sigma^n(\alpha) : n \geq 0\}} \subset [0,1]^\mathbb{N}$.

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Uniformly rigid systems (Katznelson & Weiss example)

Main features of K-W construction:

1. $|a_p(s_1) - a_p(s_2)| \leq |a_1(s_1/p) - a_1(s_2/p)| \leq \frac{L}{p}|s_1 - s_2|$ hence
$$|a_\infty(s + 2p_i) - a_\infty(s)| \leq \sup_{j>i} |a_j(s + 2p_m) - a_j(s)| \leq \sup_{j>i} \frac{2Lp_i}{p_j} \leq \frac{2L}{k_i}.$$
2. for any $\varepsilon > 0$, all odd m and $|s| < \varepsilon p_i/L$ we have
$$1 - \varepsilon < a_\infty(mp_i + s) \leq 1.$$

Theorem

Dynamical system $(\mathbb{X}, \sigma)$ (which is orbit closure of α) is uniformly rigid, and $\{\theta\}$ is its unique minimal subsystem, where $\theta(n) = 1$ for all n.

Theorem (Akin, Auslander, Berg)

If a_0 has strict minimum in 0 then $(\mathbb{X}, \sigma)$ is almost equicontinuous.

Uniformly rigid systems (Katznelson & Weiss example)

Theorem (Akin, Auslander, Berg)

Every almost equicontinuous dynamical system is uniformly rigid.

1. Defining $a_0(t) = 0$ for $|t| \leq 1/2$ and $a_0(t) = 2|t| - 1$ for $|t| > 1/2$ we obtain $(\mathbb{X}, \sigma)$ which is not almost equicontinuous.

Theorem

Let (X, f) be transitive and pointwise recurrent. If (X, f) contains a minimial set that is connected, then X is connected.

1. Then $\mathbb{X}$ is always connected. However dimension of $\mathbb{X}$ is unknown.