Celest Mech Dyn Astr (2009) 104:103–128
DOI 10.1007/s10569-009-9184-7

ORIGINAL ARTICLE

Stability of the planar full 2-body problem

D. J. Scheeres

Received: 12 September 2008 / Accepted: 10 January 2009 / Published online: 24 February 2009

Abstract The stability of the Full Two-Body Problem is studied in the case where both bodies are non-spherical, but are restricted to planar motion. The mutual potential is expanded up to second order in the mass moments, yielding a highly symmetric yet non-trivial dynamical system. For this system we identify all relative equilibria and determine their stability properties, with an emphasis on finding the energetically stable relative equilibria and conditions for Hill stability of the system. The energetically stable relative equilibria always correspond to the classical "gravity gradient" configuration with the long ends of the two bodies pointed at each other, however there always exists a second equilibrium in this configuration at a closer separation that is unstable. For our model system we precisely map out the relations between these different configurations at a given value of angular momentum. This analysis identifies the fundamental physical constraints and limitations that exist on such systems, and has immediate applications to the stability of asteroid systems that are fissioned due to a rapid spin rate. Specifically, we find that all contact binary asteroids which are spun to fission will initially lie in an unstable dynamical state and can always re-impact. If the total system energy is positive, the fissioned system can disrupt directly from this relative equilibrium, while if it is negative the system is bound together.

Keywords Coupled rotational and translational motion · Asteroids · Stability

1 Introduction

The dynamics of two arbitrary rigid bodies in orbit about each other is a fundamental problem in Celestial Mechanics which has been studied extensively in the past. Classical studies, characterized by Kinoshita (1972), have often focused on applying perturbation theory to the problem, under the assumption that one or both of the bodies are nearly spherical and that the coupling between rotational and translational motion is weak and may even be

D. J. Scheeres (✉)
Department of Aerospace Engineering Sciences, The University of Colorado, Boulder, CO, USA
e-mail: scheeres@colorado.edu

separated from each other. A series of more recent studies have revisited this classical problem allowing these restrictions on the bodies to be relaxed. Notable papers in this area are those by Wang et al. (1991); Maciejewski (1995); Koon et al. (2004); Cendra and Marsden (2005). We have also been studying this problem, under the name of the Full Two-Body Problem, and have identified a number of fundamental stability criteria that tell when the two rigid bodies are Hill stable and are stable against impact (Scheeres 2002). In (Scheeres 2006) we developed a general analysis for computing relative equilibria in the problem when one of the bodies is a sphere and the other has an arbitrary gravity field, and developed explicit methods for evaluating the energetic and spectral stability of these relative equilibria. A specific study of orbital equilibria for a sphere-ellipsoid system was made in Bellerose and Scheeres (2008). This previous work has found application in studying the stability of asteroidal systems when they are spun to rotation rates at which different parts of the body achieve orbit relative to each other (Scheeres 2007, 2009).

In the current paper we continue to extend our explicit analysis of this problem for arbitrary gravity fields. This paper reports our investigations of the stability of a Full 2-Body System when the mutual potential between the two bodies is expanded to second order. In this case the dynamical system can have a symmetry which allows the system to be constrained to lie in a plane. This restriction is justified in that it allows us to easily reach definite conclusions on the relative equilibria and their stability for this system. It will also serve as a first order result that can lead to improved understanding of the more general problem. By emphasizing the relation between fundamental system quantities such as energy and angular momentum to the stability of the system, the methods we use to analyze the current, simplified problem, should be transferrable to the full problem.

In addition to describing a complete analysis of the stability of this system, we also discover some new results that may have further implications for the evolution of small asteroidal systems that have been spun to disruption. Specifically, we find that any contact binary that has been spun to fission will initially lie in an unstable relative equilibrium. A study of the properties of this equilibrium state show that, in the absence of energy dissipation, the system may undergo disruption (i.e., mutual escape) or be trapped in a chaotic dynamical state as a function of the body moments of inertia and the mass fraction between the bodies. We also find that it is always possible for a fissioned contact binary to re-impact. If energy dissipation occurs, we see that there always exists a second, stable relative equilibria at a larger distance where the system is at its minimum energy configuration for a give value of angular momentum. The implications of these mathematical results for physical systems will be explored in the future.

2 Physical system and its equations of motion

Consider two bodies, P_1 and P_2, interacting gravitationally with each other, with an assumed symmetry about a common equatorial plane allowing us to restrict our system to planar motion. Thus we can consider the system as having two relative coordinates between the centers of mass and for each body to only rotate about one of its moments of inertia, which we always assume to be the maximum moment of inertia. We start in an inertial frame and write the kinetic and potential energy. Then we transform to a rotating frame. Following this we find the angular momentum integral and eliminate it from the system, resulting in a reduced dynamical system. Figure 1 shows the physical set-up of the system, specified using the degrees of freedom introduced later in this paper.

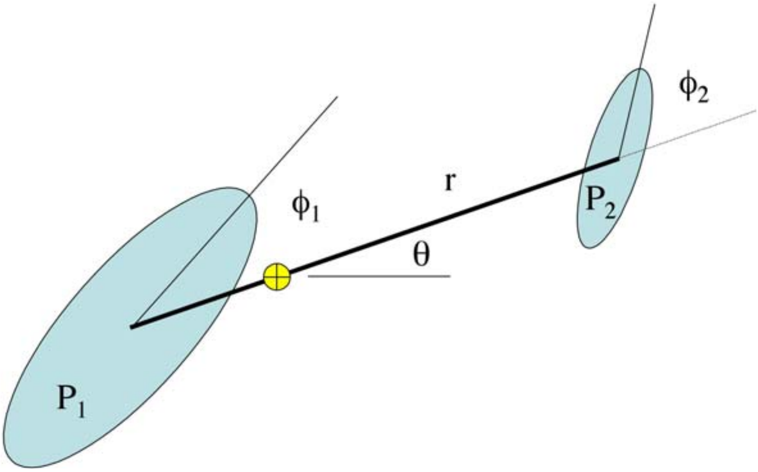

Fig. 1 Physical specification of the system, using the degrees of freedom of body separation, r, rotation of the line connecting the mass centers, θ, and rotation of the bodies relative to the line connecting the centers, ϕ_i

For the nominal, planar, system the entire assembly has 6 degrees of freedom, the center of mass of the system (which we ignore), relative motion of the centers of mass in the plane, $\boldsymbol{r}$, and rotation of each body about its maximum moment of inertia, θ_i. Then, given the kinetic energy T and the potential energy V expressed in these variables and their time derivatives we can form the Lagrangian, $L = T - V$, and hence can define the equations of motion for the system. We will also see that this system has an energy integral and an angular momentum integral.

2.1 Potential energy

First, we state the potential energy of the system. In the current paper we take this to be a 2nd order expansion in the moments of inertia, although in the future we plan to consider higher-order expansions:

$$V(\boldsymbol{r}, \boldsymbol{A}_1, \boldsymbol{A}_2) = -\frac{\mathcal{G}M_1M_2}{|\boldsymbol{r}|} - \frac{\mathcal{G}}{2|\boldsymbol{r}|^3}\left[M_2\mathrm{Tr}(\boldsymbol{I}_1) + M_1\mathrm{Tr}(\boldsymbol{I}_2)\right] + \frac{3\mathcal{G}}{2|\boldsymbol{r}|^5}\boldsymbol{r}\cdot\left[M_2\boldsymbol{A}_1^T\cdot\boldsymbol{I}_1\cdot\boldsymbol{A}_1 + M_1\boldsymbol{A}_2^T\cdot\boldsymbol{I}_2\cdot\boldsymbol{A}_2\right]\cdot\boldsymbol{r} \tag{1}$$

where $\boldsymbol{r}$ is the relative position between the centers of mass in an inertial frame, M_1 and M_2 are the masses of bodies P_1 and P_2, respectively, $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are the inertia dyads of these bodies, specified in a body-fixed frame as is usual, and $\boldsymbol{A}_1$ and $\boldsymbol{A}_2$ are the attitude dyads, transforming a vector from an inertial frame to a body-fixed frame. For mutually planar motion we note that each of these dyads is defined by a single rotation angle, θ_1 and θ_2, as:

$$\boldsymbol{A}_i = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\sin\theta_i & \cos\theta_i & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2}$$

The inertia dyads in the above equation are fully dimensional, with units of kg m^2. We will also find it convenient to define a mass-normalized inertia dyad by dividing by the body's mass, $\bar{\boldsymbol{I}}_i = \boldsymbol{I}_i/M_i$. With these definitions we can define a slightly different version of the mutual potential:

$$V(\boldsymbol{r}, \boldsymbol{A}_1, \boldsymbol{A}_2) = -\frac{\mathcal{G}M_1M_2}{|\boldsymbol{r}|}\left[1 + \frac{1}{2|\boldsymbol{r}|^2}\left[\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1\right) + \mathrm{Tr}\left(\bar{\boldsymbol{I}}_2\right)\right] - \frac{3}{2|\boldsymbol{r}|^4}\boldsymbol{r}\cdot\left[\boldsymbol{A}_1^T\cdot\bar{\boldsymbol{I}}_1\cdot\boldsymbol{A}_1 + \boldsymbol{A}_2^T\cdot\bar{\boldsymbol{I}}_2\cdot\boldsymbol{A}_2\right]\cdot\boldsymbol{r}\right] \quad (3)$$

A convenient decomposition is to take the relative position vector as the frame of reference, and separately track the orientation of the two bodies relative to this vector. Let us call this vector the $\hat{\boldsymbol{r}}$ axis. Then we have a more symmetric form of the potential energy:

$$V(r, \boldsymbol{A}) = -\frac{\mathcal{G}M_1M_2}{r}\left\{1 + \frac{1}{2r^2}[\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1\right) + \mathrm{Tr}\left(\bar{\boldsymbol{I}}_2\right) - 3\hat{\boldsymbol{r}}\cdot\left(\boldsymbol{A}_1^T\cdot\bar{\boldsymbol{I}}_1\cdot\boldsymbol{A}_1 + \boldsymbol{A}_2^T\cdot\bar{\boldsymbol{I}}_2\cdot\boldsymbol{A}_2\right)\cdot\hat{\boldsymbol{r}}]\right\} \quad (4)$$

For our specific case of planar motion these can be simplified to:

$$V(r, \phi_1, \phi_2) = -\frac{\mathcal{G}M_1M_2}{r}\left\{1 + \frac{1}{2r^2}\left[\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1\right) + \mathrm{Tr}\left(\bar{\boldsymbol{I}}_2\right) - \frac{3}{2}\left(I_{1_x} + I_{1_y} - \cos 2\phi_1\left(I_{1_y} - I_{1_x}\right) + I_{2_x} + I_{2_y} - \cos 2\phi_2\left(I_{2_y} - I_{2_x}\right)\right)\right]\right\} \quad (5)$$

where the angles ϕ_i define the orientation of body i relative to the $\hat{\boldsymbol{r}}$ line (see Fig. 1). For 3-dimensional rotational motion we need to add two additional angles to specify the orientation of each body.

2.2 Kinetic energy

For the kinetic energy we again start relative to the inertial frame. For the relative motion of the two mass centers with respect to each other, the kinetic energy is stated as:

$$T = \frac{1}{2}I_{1_z}\dot{\theta}_1^2 + \frac{1}{2}I_{2_z}\dot{\theta}_2^2 + \frac{1}{2}\frac{M_1M_2}{M_1+M_2}\boldsymbol{v}\cdot\boldsymbol{v} \quad (6)$$

where $\boldsymbol{v}$ is the relative velocity of the mass centers. We neglect the motion of the system's center of mass. Also, the rotational kinetic energies assume that rotation only occurs about principal moments of inertia of both bodies.

We again choose the relative position vector as the frame of reference. Now denote the orbit angular rate as $\dot{\theta}$, defined by $\dot{\theta} = |\boldsymbol{r}\times\boldsymbol{v}|/|\boldsymbol{r}|^2$, and the individual body rotation rates relative to the orbit rate as $\dot{\phi}_i$. Then we see that $\dot{\theta}_i = \dot{\phi}_i + \dot{\theta}$, that the inertial speed of the system is $\boldsymbol{v} = \dot{r}\hat{\boldsymbol{r}} + r\dot{\theta}\hat{\boldsymbol{\theta}}$ and that the kinetic energy in these coordinates is:

$$T = \frac{1}{2}I_{1_z}(\dot{\phi}_1 + \dot{\theta})^2 + \frac{1}{2}I_{2_z}(\dot{\phi}_2 + \dot{\theta})^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}m(r\dot{\theta})^2 \quad (7)$$

$$= \frac{1}{2}I_{1_z}\dot{\phi}_1^2 + \frac{1}{2}I_{2_z}\dot{\phi}_2^2 + \frac{1}{2}m\dot{r}^2 + \frac{1}{2}\left(I_{1_z} + I_{2_z} + mr^2\right)\dot{\theta}^2 + \left(I_{1_z}\dot{\phi}_1 + I_{2_z}\dot{\phi}_2\right)\dot{\theta} \quad (8)$$

For notational convenience we define:

$$m = \frac{M_1 M_2}{M_1 + M_2} \tag{9}$$

$$\nu = \frac{M_1}{M_1 + M_2} \tag{10}$$

$$1 - \nu = \frac{M_2}{M_1 + M_2} \tag{11}$$

$$I_z(r) = I_{1_z} + I_{2_z} + mr^2 \tag{12}$$

where I_z is a function of r^2 and is the total moment of inertia of the system about the z axis.

2.3 Lagrangian equations of motion

The Lagrangian of the system can then be formed as $L = T - V$. We choose to use as coordinates the distance, orbit angle, and body rotation angles relative to the orbit frame, denoted as $r, \theta, \phi_1, \phi_2$, respectively. We find:

$$L = \frac{1}{2} I_{1_z} \dot{\phi}_1^2 + \frac{1}{2} I_{2_z} \dot{\phi}_2^2 + \frac{1}{2} m\dot{r}^2 + \frac{1}{2}\left(I_{1_z} + I_{2_z} + mr^2\right)\dot{\theta}^2 + \left(I_{1_z}\dot{\phi}_1 + I_{2_z}\dot{\phi}_2\right)\dot{\theta} - V(r, \phi_1, \phi_2) \tag{13}$$

The general form of the equations are $\mathrm{d}/\mathrm{d}t\,(\partial L/\partial \dot{q}_i) = \partial L/\partial q_i$. We note that the equation for $\ddot{r}$ is decoupled from the other equations, and so state that first:

$$\ddot{r} = \dot{\theta}^2 r - \frac{1}{m}\frac{\partial V}{\partial r} \tag{14}$$

The remaining three equations are coupled to each other:

$$\begin{bmatrix} I_{1_z} & 0 & I_{1_z} \\ 0 & I_{2_z} & I_{2_z} \\ I_{1_z} & I_{2_z} & I_z \end{bmatrix} \begin{bmatrix} \ddot{\phi}_1 \\ \ddot{\phi}_2 \\ \ddot{\theta} \end{bmatrix} = \begin{bmatrix} -\frac{\partial V}{\partial \phi_1} \\ -\frac{\partial V}{\partial \phi_2} \\ -2mr\dot{r}\dot{\theta} \end{bmatrix} \tag{15}$$

The system can be decoupled by multiplying by the inverse of the mass matrix, yielding three independent equations of motion for the system:

$$\ddot{\phi}_1 = -\left(1 + \frac{mr^2}{I_{1_z}}\right)\frac{1}{mr^2} V_{\phi_1} - \frac{1}{mr^2} V_{\phi_2} + 2\frac{\dot{r}\dot{\theta}}{r} \tag{16}$$

$$\ddot{\phi}_2 = -\left(1 + \frac{mr^2}{I_{2_z}}\right)\frac{1}{mr^2} V_{\phi_2} - \frac{1}{mr^2} V_{\phi_1} + 2\frac{\dot{r}\dot{\theta}}{r} \tag{17}$$

$$\ddot{\theta} = \frac{1}{mr^2} V_{\phi_1} + \frac{1}{mr^2} V_{\phi_2} - 2\frac{\dot{r}\dot{\theta}}{r} \tag{18}$$

2.4 Integrals of motion and reduction of the system

The system, in either inertial or relative coordinates, is time invariant and thus has a Jacobi integral. The standard form for the Jacobi integral is:

$$h = \dot{\boldsymbol{q}} \cdot \frac{\partial L}{\partial \dot{\boldsymbol{q}}} - L \tag{19}$$

When applied to the inertial system we see that this recovers the classical energy of the system, $h = T + V$. By necessity, this carries over to the relative system as well, providing us with the first integral of motion.

For the second integral of motion note that the orbit angle θ is ignorable, and thus that $\partial L/\partial\theta = 0$. From Lagrange's equations we immediately see that $\mathrm{d}/\mathrm{d}t(\partial L/\partial\dot{\theta}) = 0$ or that the angular momentum associated with the angle θ (which is also the total angular momentum of the system) is conserved:

$$K = \frac{\partial L}{\partial\dot{\theta}} \tag{20}$$

$$= \left(I_{1_z} + I_{2_z} + mr^2\right)\dot{\theta} + I_{1_z}\dot{\phi}_1 + I_{2_z}\dot{\phi}_2 \tag{21}$$

The simple relationship between K and $\dot{\theta}$ allows us to eliminate the angular velocity $\dot{\theta}$ from the Lagrangian, essentially performing a Routh reduction of the system. Solving for $\dot{\theta}$ as a function of K and the other state variables we find:

$$\dot{\theta} = \frac{1}{I_z(r)}\left[K - \left(I_{1_z}\dot{\phi}_1 + I_{2_z}\dot{\phi}_2\right)\right] \tag{22}$$

The rotation rate $\dot{\theta}$ only appears in the kinetic energy, so we substitute this relationship in T to find:

$$T = \frac{1}{2}\frac{\left[K^2 - \left(I_{1_z}\dot{\phi}_1 + I_{2_z}\dot{\phi}_2\right)^2\right]}{I_z} + \frac{1}{2}I_{1_z}\dot{\phi}_1^2 + \frac{1}{2}I_{2_z}\dot{\phi}_2^2 + \frac{1}{2}m\dot{r}^2 \tag{23}$$

So reduced, the total energy of the system is then the only remaining integral of motion, and is expressed as:

$$E = \frac{1}{2}\frac{\left[K^2 - \left(I_{1_z}\dot{\phi}_1 + I_{2_z}\dot{\phi}_2\right)^2\right]}{I_z} + \frac{1}{2}I_{1_z}\dot{\phi}_1^2 + \frac{1}{2}I_{2_z}\dot{\phi}_2^2 + \frac{1}{2}m\dot{r}^2 + V(r, \phi_1, \phi_2) \tag{24}$$

Note, this simple reduction is only possible in the planar problem.

It is instructive to rewrite the kinetic energy to separate the angular momentum term, K, from the other terms. Simplifying we find the result:

$$T = \frac{1}{2}\frac{K^2}{I_z} + \frac{1}{2I_z}\left[I_{1_z}I_{2_z}\left(\dot{\phi}_1 - \dot{\phi}_2\right)^2 + mr^2\left(I_{1_z}\dot{\phi}_1^2 + I_{2_z}\dot{\phi}_2^2\right)\right] + \frac{1}{2}m\dot{r}^2 \tag{25}$$

This formulation will allow us to investigate the zero-velocity surfaces of this problem later.

It is also possible to define a reduced set of equations of motion involving only r and ϕ_i, however they become rather algebraically complex. In practice it is often simpler to deal with the equations of motion as derived in Eqs. 14, 16–18, and then apply the angular momentum integral of motion to the system.

3 Equilibrium conditions

To solve for the equilibrium conditions the usual practice is to set all time derivative quantities in the equations of motion to zero. We will take a slightly different approach, however, that fits in better with our stability characterization.

3.1 Equilibrium from energy considerations

Instead of solving directly for the equilibrium conditions from the equations of motion, it is simpler to apply Proposition 7 from Scheeres (2006) to this system and find the conditions for the energy variations of the system to be stationary at a constant value of angular momentum. Specifically, we find these by finding the zeros to the equations:

$$E_r = 0 \tag{26}$$

$$E_{\dot{r}} = 0 \tag{27}$$

$$E_{\phi_i} = 0 \tag{28}$$

$$E_{\dot{\phi}_i} = 0 \tag{29}$$

where we must use the energy which has the angular momentum explicitly present, as given in Eq. 24. Because of the symmetry of the problem, we know that the equilibria will occur at the locally central points (defined in Scheeres 2006), when the principal axes of the two bodies are aligned with each other. Given such relative orientations the only real condition we need to evaluate is the level of angular momentum (i.e., spin rate $\dot{\theta}$) needed to maintain a relative equilibrium. To be specific, we assume the equilibrium position is at a given value of r, and that the orientation of the P_i take on one of the values $\phi_i = 0, \pm\pi/2, \pi$. All we need to find then is the spin rate to achieve relative equilibrium.

In the following we give the complete partial derivatives of the energy, as we will need these later in our stability discussions.

$$E_r = -mr\frac{K^2 - \left(I_{1_z}\dot{\phi}_1 + I_{2_z}\dot{\phi}_2\right)^2}{I_z^2} + V_r \tag{30}$$

$$E_{\phi_i} = V_{\phi_i} \tag{31}$$

$$E_{\dot{r}} = m\dot{r} \tag{32}$$

$$E_{\dot{\phi}_i} = I_{i_z}\dot{\phi}_i - \frac{I_{i_z}\left(I_{1_z}\dot{\phi}_1 + I_{2_z}\dot{\phi}_2\right)}{I_z} \tag{33}$$

The partials with respect to the potential energy are:

$$V_r = \frac{\mathcal{G}M_1M_2}{r^2}\left[1 + \frac{3}{2r^2}\left\{\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1\right) + \mathrm{Tr}\left(\bar{\boldsymbol{I}}_2\right) - \frac{3}{2}\left(I_{1_x} + I_{1_y} - \cos 2\phi_1\left(I_{1_y} - I_{1_x}\right) + I_{2_x} + I_{2_y} - \cos 2\phi_2\left(I_{2_y} - I_{2_x}\right)\right)\right\}\right] \tag{34}$$

$$V_{\phi_i} = \frac{3\mathcal{G}M_1M_2}{2r^3}\sin 2\phi_i(I_{i_y} - I_{i_x}) \tag{35}$$

It is simple to show that the coordinate velocities must all be zero for an equilibrium, thus $\dot{r}^* = \dot{\phi}_i^* = 0$, where the $*$ signifies a relative equilibrium. Next consider the term $E_{\phi_i} = 0$ which leads to the condition:

$$\frac{3\mathcal{G}M_1M_2}{2r^3}\sin 2\phi_i(I_{i_y} - I_{i_x}) = 0 \tag{36}$$

If $I_{i_y} > I_{i_x}$, which is our general assumption, we note that $\phi_i^* = 0, \pm\pi/2, \pi$ will satisfy this equation. If $I_{i_y} = I_{i_x}$ then the body is rotationally symmetric and the angle ϕ_i is an ignorable coordinate and the body preserves its rotational angular momentum and kinetic energy independent of the system.

Putting these constraints into the final condition, $E_r = 0$, yields:

$$\frac{mrK^2}{I_z^2} = \frac{\mathcal{G}M_1M_2}{r^2}\left[1 + \frac{3}{2r^2}\left\{\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1\right) + \mathrm{Tr}\left(\bar{\boldsymbol{I}}_2\right) - \frac{3}{2}\left(I_{1_x} + I_{1_y} - \pm_1(I_{1_y} - I_{1_x}) + I_{2_x} + I_{2_y} - \pm_2(I_{2_y} - I_{2_x})\right)\right\}\right] \quad (37)$$

where the $\pm_i$ corresponds to $\phi_i^* = 0, \pi/2$, respectively.

It is also of interest to relate this to the original rotation rate $\dot{\theta}$ at equilibrium:

$$\dot{\theta}^* = \frac{K}{I_z} \quad (38)$$

Substituting this into the above equations we find an explicit relationship for the equilibrium rotation rate:

$$\dot{\theta}^{*2} = \frac{\mathcal{G}(M_1 + M_2)}{r^3}\left[1 + \frac{3}{2r^2}\left\{\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1\right) + \mathrm{Tr}\left(\bar{\boldsymbol{I}}_2\right) - \frac{3}{2}\left(I_{1_x} + I_{1_y} - \pm_1(I_{1_y} - I_{1_x}) + I_{2_x} + I_{2_y} - \pm_2(I_{2_y} - I_{2_x})\right)\right\}\right] \quad (39)$$

where there are 4 possible values depending on the relative orientations of the bodies.

3.2 Number of equilibria

The above relations allow one to specify the separation between the bodies and body orientations, and then solve for the necessary angular momentum of the system for this to be an equilibria. We can also consider the converse problem, and ask how many relative equilibria exist for a given value of angular momentum. This is a significant question, as system angular momentum is nominally conserved unless there are significant exogenous perturbations acting on the system. This is especially relevant when we consider the system energy at various equilibria configurations, as under dissipative internal effects we will find that the system can transition from one equilibrium to a different, lower energy equilibrium with the same value of angular momentum.

Consider Eq. 37, the defining equation for the angular momentum of a given relative equilibrium. Rearranging this equation yields:

$$r^6 - \frac{K^2}{m^2\mu}r^5 + \left(2\frac{I_{1_z} + I_{2_z}}{m} + \frac{3}{2}\left(C_1^{\pm} + C_2^{\pm}\right)\right)r^4 + \frac{I_{1_z} + I_{2_z}}{m}\left(\frac{I_{1_z} + I_{2_z}}{m} + 3\left(C_1^{\pm} + C_2^{\pm}\right)\right)r^2 + \frac{3}{2}\left(C_1^{\pm} + C_2^{\pm}\right)\left(\frac{I_{1_z} + I_{2_z}}{m}\right)^2 = 0 \quad (40)$$

where we define

$$C_i^{\pm} = \mathrm{Tr}(\bar{I}_i) - 3\left\{\begin{array}{l}\bar{I}_{i_x} \\ \bar{I}_{i_y}\end{array}\right. \quad (41)$$

The roots of this polynomial are relative equilibrium solutions for the system. Application of the Routh-Hurwitz criterion to this polynomial shows that there are always two positive roots for a given set of $C_i^{\pm}$, corresponding to two possible equilibria values at a given angular momentum. Taking all possible non-trivial orientations between the two bodies we get 4 possible relative orientations, leading to a total of 8 possible relative equilibria at a given angular momentum. Among all of these equilibria we expect the one with minimum energy to be stable and all others to be unstable, with a larger energy. This will be borne out in our subsequent analysis, where we will show that only one configuration can be stable for a fixed value of angular momentum.

3.3 Orbit elements at equilibrium

It is of interest to compute the Keplerian orbit elements of the system at relative equilibrium. The Keplerian orbit elements of interest are the semi-major axis, eccentricity, longitude of periapsis and mean anomaly (or true anomaly). In classical tidal theories the assumption is usually made that the fully relaxed equilibrium rotation rate of a system equals the Keplerian mean motion, $\sqrt{\mu/r^3}$ (Ferraz-Mello et al. 2008). This is an excellent assumption for planetary satellites in general, however, in the following we can show that this is false to the first order for closely spaced systems.

We first will deal with the true anomaly and longitude of periapsis. Since the angles $\dot{\phi}_i^* = 0$, the only rotation occurs due to $\dot{\theta}^*$, and is computed using Eq. 39. We note that this is a steady rate, however we also note that it does not equal the usual mean motion. The difference between the usual mean motion squared and our equilibrium rotation rate squared, $\Delta^* = \dot{\theta}^{*2} - \frac{\mu}{r^3}$, equals:

$$\frac{3\mu}{4r^5}\left\{\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1\right) + \mathrm{Tr}\left(\bar{\boldsymbol{I}}_2\right) - \frac{3}{2}\left(I_{1_x} + I_{1_y} - \pm_1\left(I_{1_y} - I_{1_x}\right) + I_{2_x} + I_{2_y} - \pm_2\left(I_{2_y} - I_{2_x}\right)\right)\right\} \tag{42}$$

Consider the four terms separately:

$$\Delta_{++} = \frac{3\mu}{4r^5}\left\{\bar{I}_{1_z} + \bar{I}_{1_y} - 2\bar{I}_{1_x} + \bar{I}_{2_z} + \bar{I}_{2_y} - 2\bar{I}_{2_x}\right\} \tag{43}$$

$$\Delta_{+-} = \frac{3\mu}{4r^5}\left\{\bar{I}_{1_z} + \bar{I}_{1_y} - 2\bar{I}_{1_x} + \bar{I}_{2_z} + \bar{I}_{2_x} - 2\bar{I}_{2_y}\right\} \tag{44}$$

$$\Delta_{-+} = \frac{3\mu}{4r^5}\left\{\bar{I}_{1_z} + \bar{I}_{1_x} - 2\bar{I}_{1_y} + \bar{I}_{2_z} + \bar{I}_{2_y} - 2\bar{I}_{2_x}\right\} \tag{45}$$

$$\Delta_{--} = \frac{3\mu}{4r^5}\left\{\bar{I}_{1_z} + \bar{I}_{1_x} - 2\bar{I}_{1_y} + \bar{I}_{2_z} + \bar{I}_{2_x} - 2\bar{I}_{2_y}\right\} \tag{46}$$

Under our assumption for the mass distribution we note immediately that $\Delta_{++} \geq 0$ but that the others may either be positive or negative. If the quantity $\Delta > 0$, then we note that the system is at periapsis, since its rotation rate is greater than the mean motion for a circular orbit and $\dot{r} = 0$. The fact that this situation persists means that the orbit is trapped at periapsis and the true anomaly equals 0 while the rotation rate $\dot{\theta}$ equals the time rate of change of the argument of periapsis. Thus the osculating orbit is a constant ellipse with a constant precession rate in the longitude of periapsis. If $\Delta < 0$, then the system is trapped at apoapsis but the rate of change of the longitude of periapsis still equals $\dot{\theta}$. This is a modification that is generally not accounted for in tidally locked systems, yet which may be relevant when they are in close proximity.

Now, we note that the specific Keplerian energy $E^K = \frac{1}{2}v^2 - \mu/r$ and specific orbital angular momentum $H^K = |\boldsymbol{r} \times \boldsymbol{v}|$ are computed using the translational motion alone, and assuming a point-mass potential. Specifically, for our system at equilibrium where the speed equals $\dot{\theta} r$,

$$E^K = \frac{1}{2}\dot{\theta}^2 r^2 - \frac{\mu}{r} \tag{47}$$

$$= \frac{-\mu}{2r}\left[1 - \frac{3}{2r^2}\left(C_1^{\pm} + C_2^{\pm}\right)\right] \tag{48}$$

$$H^K = r^2\dot{\theta} \tag{49}$$

$$= \sqrt{\mu r}\left[1 + \frac{3}{2r^2}\left(C_1^{\pm} + C_2^{\pm}\right)\right]^{1/2} \tag{50}$$

Thus we define the semi-major axis as:

$$a = \frac{-\mu}{2E^K} \tag{51}$$

$$= r\left[1 - \frac{3}{2r^2}\left(C_1^{\pm} + C_2^{\pm}\right)\right]^{-1} \tag{52}$$

and the eccentricity as

$$e^2 = 1 + \frac{2E^K H^{K2}}{\mu^2} \tag{53}$$

$$= 1 - \left(1 - \frac{3}{2r^2}\left(C_1^{\pm} + C_2^{\pm}\right)\right)\left(1 + \frac{3}{2r^2}\left(C_1^{\pm} + C_2^{\pm}\right)\right) \tag{54}$$

$$= \left(\frac{3}{2r^2}\left(C_1^{\pm} + C_2^{\pm}\right)\right)^2 \tag{55}$$

Note that the eccentricity is linearly related to the moments of inertia as $e = \frac{3}{2r^2}\left(C_1^{\pm} + C_2^{\pm}\right)$, where a negative value implies that the system is at apoapsis. Relating this back to the spin rate for equilibrium, we find a simple relation between the equilibrium spin rate, the eccentricity of the equilibrium orbit, and the mass distribution:

$$\dot{\theta}^2 = \frac{\mu}{r^3}(1+e) \tag{56}$$

This can be a significant correction if the bodies are close to each other. We also note that the semi-major axis at equilibrium is a function of the separation radius and the eccentricity:

$$a = \frac{r}{1-e} \tag{57}$$

Physically, this makes sense, as the system is trapped at periapsis for $e > 0$ with $a > r$, and if $e < 0$ the system is trapped at apoapsis with $a < r$.

4 Stability

Having computed the conditions for the system to be in relative equilibrium, it is now of interest to determine the conditions which control the stability of the system. We will focus on four different types of stability, called energetic stability, spectral stability, Hill stability and impact stability. Energetic stability is the most fundamental, and will determine whether the system can seek out a lower energy state by dissipation of energy. If a system is energetically stable, than it has reached its lowest energy configuration for a given angular momentum (assuming that the bodies are rigid). Spectral stability, on the other hand, just determines whether the system will stay in the vicinity of its relative equilibria if perturbed from it, assuming that no energy dissipation occurs. Spectral stability is a necessary condition for energetic stability, but is not sufficient. Thus, we can sometimes find instances where an equilibrium solution is spectrally stable but is not energetically stable. For the motion and dynamics of natural bodies, we are generally more interested in energetic stability. Next we compute Hill stability. A system is Hill stable if it cannot undergo a mutual escape. If a system is energetically stable, then it is Hill stable, although the converse is not true in general. Hill stability is not as focused on motion in the vicinity of a relative equilibria, but instead is more concerned with placing absolute limits on where the system can evolve dynamically. Finally, a system has impact stability if it cannot, under its internal dynamics alone, result in the two bodies impacting each other. In general, systems that have energetic stability also have impact stability. In this case we find that the converse is also true, and that systems which are not energetically stable do not have impact stability, meaning that it is possible for them to re-impact.

4.1 Energetic stability

For energetic stability we follow the approach given in (Scheeres 2006), which can be derived from the more fundamental study given in (Simo et al. 1991). Since we have the angular momentum explicitly contained in our expression for the energy, we can directly take the second variation of the energy and check if it is positive definite. If it is, then we have shown that the relative equilibrium in question is stable. If we represent our coordinates as $\boldsymbol{q} = [r, \phi_1, \phi_2]$ and the state $\boldsymbol{X} = [\boldsymbol{q}, \dot{\boldsymbol{q}}]$, then we must compute the matrix

$$E_{\boldsymbol{XX}} = \begin{bmatrix} E_{\boldsymbol{qq}} & E_{\boldsymbol{q}\dot{\boldsymbol{q}}} \\ E_{\boldsymbol{q}\dot{\boldsymbol{q}}} & E_{\dot{\boldsymbol{q}}\dot{\boldsymbol{q}}} \end{bmatrix} \tag{58}$$

evaluated at the equilibrium condition. Due to the symmetry of the mutual potential, and the assumed planarity of the system, this matrix is simplified at equilibrium, which allows us to easily identify the conditions for stability.

We first note that there are several cross terms that are identically equal to zero. Specifically, these are: $E_{\dot{r}r} = E_{\dot{r}\phi_i} = E_{\dot{r}\dot{\phi}_i} = E_{\phi_i\phi_j} \equiv 0$, where $i \neq j$. Other coupling terms that are not identically zero, yet are when evaluated at equilibrium, include $E^*_{r\dot{\phi}_i} = E^*_{r\phi_i} = 0$. Thus, the only coupling term that is not zero is :

$$E^*_{\dot{\phi}_i\dot{\phi}_j} = I_{i_z}\left(\delta_{ij} - \frac{I_{j_z}}{I_z}\right) \tag{59}$$

where $\delta_{ij} = 0$ for $i \neq j$ and 1 for $i = j$.

For the second variation to be positive definite then requires the following conditions:

$$E^*_{rr} > 0 \tag{60}$$

$$E^*_{\dot{r}\dot{r}} > 0 \tag{61}$$

$$E^*_{\phi_i\phi_i} > 0 \tag{62}$$

$$\begin{bmatrix} E^*_{\dot{\phi}_1\dot{\phi}_1} & E^*_{\dot{\phi}_1\dot{\phi}_2} \\ E^*_{\dot{\phi}_2\dot{\phi}_1} & E^*_{\dot{\phi}_2\dot{\phi}_2} \end{bmatrix} > 0 \tag{63}$$

We first note that $E^*_{\dot{r}\dot{r}} = m$ and Eq. 61 is positive trivially. Next we compute the eigenvalues of the matrix in Eq. 63:

$$\begin{bmatrix} I_{1_z}\left(1-\frac{I_{1_z}}{I_z}\right) & -\frac{I_{1_z}I_{2_z}}{I_z} \\ -\frac{I_{1_z}I_{2_z}}{I_z} & I_{2_z}\left(1-\frac{I_{2_z}}{I_z}\right) \end{bmatrix} \tag{64}$$

The characteristic equation is:

$$I_z^2\lambda^2 - \left[2I_{1_z}I_{2_z} + (I_{1_z}+I_{2_z})mq^2\right] I_z\lambda - \left(I_{1_z}I_{2_z}\right)^2 = 0 \tag{65}$$

The solution is:

$$I_z\lambda = \frac{1}{2}\left[2I_{1_z}I_{2_z} + \left(I_{1_z}+I_{2_z}\right)mq^2\right] \pm \frac{1}{2}\sqrt{[2I_{1_z}I_{2_z} + \left(I_{1_z}+I_{2_z}\right)mq^2]^2 - 4\left(I_{1_z}I_{2_z}\right)^2} \tag{66}$$

$$= \frac{1}{2}\left[2I_{1_z}I_{2_z} + \left(I_{1_z}+I_{2_z}\right)mq^2\right] \pm \frac{1}{2}\sqrt{4I_{1_z}I_{2_z}\left(I_{1_z}+I_{2_z}\right)mq^2 + \left(I_{1_z}+I_{2_z}\right)^2 m^2q^4} \tag{67}$$

By inspection of Eq. 66 we note that the eigenvalues are always positive, and by inspection of Eq. 67 we note that they are always real. Hence Eq. 63 is always positive definite.

Next we consider the term $E^*_{\phi_i\phi_i} = V^*_{\phi_i\phi_i}$. The term $V_{\phi_i\phi_i}$ equals:

$$V_{\phi_i\phi_i} = \frac{3\mathcal{G}M_1M_2}{r^3}\cos 2\phi_i (I_{i_y} - I_{i_x}) \tag{68}$$

and we recall our assumption that $I_y \geq I_x$. Now we note that the solutions with the long axis pointing at the other body, $\phi_i = 0, \pi$, yield positive values and that the other solutions, $\phi_i = \pm\pi/2$, are negative definite and thus are always energetically unstable. We can note that the relative equilibria for $\phi_i = \pm\pi/2$ can sometimes be spectrally stable (Scheeres 2006), however they will always be unstable in the presence of energy dissipation. Thus we set $\phi_i = 0$ henceforth, unless otherwise noted.

Finally, consider the remaining term, E_{rr}. With our above assumptions, this term becomes:

$$E_{rr} = -\frac{mK^2}{I_z^2} + \frac{4m^2K^2r^2}{I_z^3} - \frac{2\mathcal{G}M_1M_2}{r^3}\left[1 + \frac{3}{r^2}C_{12}\right] \tag{69}$$

$$= \frac{mK^2}{I_z^3}\left[3mr^2 - I_{1_z} - I_{2_z}\right] - \frac{2\mathcal{G}M_1M_2}{r^3}\left[1 + \frac{3}{r^2}C_{12}\right] \tag{70}$$

where we recall the constants

$$C_i = \mathrm{Tr}(\bar{I}_i) - 3\bar{I}_{i_x} \tag{71}$$

$$C_{12} = C_1 + C_2 \tag{72}$$

and assume the $\phi_i = 0$ orientation unless specified otherwise. We note two properties of the quantity C_i. First, $C_i \geq 0$ for all possible mass distributions, given our assumptions on $I_y \geq I_x$. Second, $C_i = 0$ if and only if all of the moments of inertia of the body are equal, i.e., if the body is a sphere to 2nd order.

Substituting for the equilibrium conditions into the angular momentum, we wish to evaluate when the following inequality is satisfied:

$$\frac{3mr^2 - I_{1_z} - I_{2_z}}{I_z}\frac{\mathcal{G}M_1M_2}{r^3}\left[1 + \frac{3}{2r^2}C_{12}\right] - \frac{2\mathcal{G}M_1M_2}{r^3}\left[1 + \frac{3}{r^2}C_{12}\right] > 0 \tag{73}$$

Clearing terms and simplifying we find:

$$(3mr^2 - I_{1_z} - I_{2_z})\left(r^2 + \frac{3}{2}C_{12}\right) - 2(mr^2 + I_{1_z} + I_{2_z})(r^2 + 3C_{12}) > 0 \tag{74}$$

Now we extract the mass m from the equation, noting that $I_{1_z}/m = \bar{I}_{1_z}/\nu$ and $I_{2_z}/m = \bar{I}_{2_z}/(1-\nu)$. This leads to the final condition for stability:

$$r^4 - 3\left[\frac{\bar{I}_{1_z}}{\nu} + \frac{\bar{I}_{2_z}}{1-\nu} + \frac{1}{2}C_{12}\right]r^2 - \frac{15}{2}C_{12}\left(\frac{\bar{I}_{1_z}}{\nu} + \frac{\bar{I}_{2_z}}{1-\nu}\right) > 0 \tag{75}$$

Factoring the quadratic we find the following roots for the radius, $r^2 = q_\pm^2$ where:

$$q_\pm^2 = \frac{3}{2}\left[\frac{\bar{I}_{1_z}}{\nu} + \frac{\bar{I}_{2_z}}{1-\nu} + \frac{1}{2}C_{12}\right] \pm\frac{1}{2}\sqrt{9\left[\frac{\bar{I}_{1_z}}{\nu} + \frac{\bar{I}_{2_z}}{1-\nu} + \frac{1}{2}C_{12}\right]^2 + 30C_{12}\left(\frac{\bar{I}_{1_z}}{\nu} + \frac{\bar{I}_{2_z}}{1-\nu}\right)} \tag{76}$$

Only q_+^2 is positive, and we take the positive root of that, $+\sqrt{q_+^2}$, to find the limiting value of r for the system to be energetically stable. Any system with $r < q_+$ will be energetically unstable. Whenever this situation occurs it means that there exists a second equilibrium separation at the same value of angular momentum which has a lower energy, this second position always lying at a distance $r > q_+$. This situation was discussed earlier, and we will consider some specific examples of it later.

4.2 Spectral stability

It is also of interest to briefly consider the spectral stability of the system, as this provides insight into the motion of the system when perturbed from equilibrium. We will consider a system perturbed from the necessary configuration for energetic stability, i.e., with the long axes of the bodies facing each other. To find the equations for our system, we go back to the original equations of motion and linearize about an equilibrium solution. The equilibrium solution is represented as r^*, $\phi_i^* = 0$, $\dot{r}^* = \dot{\phi}_i^* = 0$ and with $\dot{\theta}^*$ being solved for from Eq. 39. The linear variations are expressed as δr, $\delta\phi_i$, and $\delta\dot{\theta}$.

The full equations of motion become, under this linearization:

$$\delta\ddot{r} = \left[\dot{\theta}^2 - \frac{1}{m}V_{rr}\right]\delta r + 2r\dot{\theta}\delta\dot{\theta} \tag{77}$$

$$\delta\ddot{\phi}_i = -\left(1 + \frac{mr^2}{I_{i_z}}\right)\frac{1}{mr^2}V_{\phi_i\phi_i}\delta\phi_i + \frac{2\dot{\theta}}{r}\delta\dot{r} \tag{78}$$

$$\delta\ddot{\theta} = \frac{1}{mr^2}V_{\phi_1\phi_1}\delta\phi_1 + \frac{1}{mr^2}V_{\phi_2\phi_2}\delta\phi_2 - \frac{2\dot{\theta}}{r}\delta\dot{r} \tag{79}$$

where $i = 1, 2$ in the second equation. We note that the system is closed in r and ϕ_i except for the $\delta\dot{\theta}$ in the first equation. In the following we use the conservation of angular momentum to remove this variation, thus giving us a 3 degree of freedom problem.

First, take the variation of the angular momentum integral given in Eq. 21 evaluated at equilibrium:

$$0 = I_z\delta\dot{\theta} + 2mr\dot{\theta}\delta r + I_{1_z}\delta\dot{\phi}_1 + I_{2_z}\delta\dot{\phi}_2 \tag{80}$$

Solving for $\delta\dot{\theta}$ and substituting into the first equation above yields:

$$\delta\ddot{r} = \left[\left(1 - \frac{4mr^2}{I_z}\right)\dot{\theta}^2 - \frac{1}{m}V_{rr}\right]\delta r - \frac{2r\dot{\theta}}{I_z}\left[I_{1_z}\delta\dot{\phi}_1 + I_{2_z}\delta\dot{\phi}_2\right] \tag{81}$$

Now define the state-space form of the system, using $\boldsymbol{q} = [r, \phi_1, \phi_2]$ and $\boldsymbol{X} = [\boldsymbol{q}, \dot{\boldsymbol{q}}]$, we find:

$$\delta\dot{\boldsymbol{X}} = A\delta\boldsymbol{X} \tag{82}$$

$$A = \begin{bmatrix} 0_{3x3} & I_{3x3} \\ a_{qq} & b_{q\dot{q}} \end{bmatrix} \tag{83}$$

where a_{qq} is a diagonal matrix with entries:

$$a_{rr} = \frac{\mu}{r^3}\left[\left(1 - \frac{4mr^2}{I_z}\right)\left(1 + \frac{3}{2r^2}C_{12}\right) + 2\left(1 + \frac{3}{r^2}C_{12}\right)\right] \tag{84}$$

$$a_{\phi_i\phi_i} = \frac{-3\mu}{r^5}\left(\bar{I}_{i_y} - \bar{I}_{i_x}\right)\left(1 + \frac{mr^2}{I_{i_z}}\right) \tag{85}$$

for $i = 1, 2$. The matrix $b_{q\dot{q}}$ has relatively few terms, defined to be:

$$b_{r\dot{\phi}_i} = \frac{-2rI_{i_z}}{I_z}\sqrt{\frac{\mu}{r^3}\left(1 + \frac{3}{2r^2}C_{12}\right)} \tag{86}$$

$$b_{\phi_i\dot{r}} = \frac{2\dot{\theta}}{r} \tag{87}$$

$$= \frac{2}{r}\sqrt{\frac{\mu}{r^3}\left(1 + \frac{3}{2r^2}C_{12}\right)} \tag{88}$$

where we recall that $\mu = \mathcal{G}(M_1 + M_2)$.

To evaluate the stability of the system we compute the characteristic equation, and to evaluate the dynamical motion of the system we compute the eigenvectors of the matrix A. We note that the spectral stability and energetic stability coincide for this configuration, as is expected. Further, when either of the bodies has an orientation $\phi_i = \pi/2$ it is possible for the system to be spectrally stable, however it will never be energetically stable. The eigenvectors are rather complex to represent, but by studying the characteristic directions in which $\delta\boldsymbol{q}$ and $\delta\dot{\boldsymbol{q}}$ move one may gain insight into the system dynamics.

4.3 Hill stability

Hill stability is distinguished from energetic and spectral stability. Those types of stability are focused on motion relative to a specific point in phase space, while Hill stability is focused on whether the general motion of the system is bounded or not. For our system, we can state that it will be Hill stable if $r < c < \infty$ for all time, where c is a constant. Conversely, it will be Hill unstable if $r \to \infty$ as $t \to \infty$. We find necessary conditions for our system to be Hill unstable, and can find sufficient conditions for it to be Hill stable.

4.3.1 Necessary conditions for Hill instability

Following from the analysis given in (Scheeres 2002), we note that the necessary condition for our system to be Hill unstable is that the Free Energy of the system be positive. The

Free Energy is defined as the total energy of the system minus the self-potential energy of the bodies and equals Eq. 24. For this particular paper we are especially interested in determining whether a given relative equilibrium point has a positive Free Energy. To do this we substitute our candidate equilibrium solution into the total energy and evaluate whether it is positive or negative.

$$E = \frac{1}{2}\frac{K^2}{I_{1_z} + I_{2_z} + mr^2} - \frac{\mathcal{G}M_1M_2}{r}\left[1 + \frac{1}{2r^2}\left\{\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1 + \bar{\boldsymbol{I}}_2\right) - 3\left(\bar{I}_{1_x} + \bar{I}_{2_x}\right)\right\}\right] \tag{89}$$

which, when the angular momentum is substituted and the expression simplified, yields

$$\begin{aligned} E = \frac{1}{2}\frac{\mathcal{G}M_1M_2}{r^3}\Bigg[&\frac{1}{\nu}\bar{I}_{1_z} + \frac{1}{1-\nu}\bar{I}_{2_z} - r^2 \\ &+ \frac{1}{2r^2}\left\{3\left(\frac{1}{\nu}\bar{I}_{1_z} + \frac{1}{1-\nu}\bar{I}_{2_z}\right) + 2r^2\right\}\left\{\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1 + \bar{\boldsymbol{I}}_2\right) - 3(\bar{I}_{1_x} + \bar{I}_{2_x})\right\}\Bigg] \end{aligned} \tag{90}$$

The specific condition of interest is the value of separation between the components when the energy goes from positive to negative. Again, we see that for large enough r the energy is always negative, meaning the system is bound, but is not necessarily stable. Setting the energy to zero, we can arrange this equation into a bi-quadratic for r again:

$$\begin{aligned} r^4 - &\left(\frac{1}{\nu}\bar{I}_{1_z} + \frac{1}{1-\nu}\bar{I}_{2_z} + \left\{\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1 + \bar{\boldsymbol{I}}_2\right) - 3(\bar{I}_{1_x} + \bar{I}_{2_x})\right\}\right) r^2 \\ &- \frac{3}{2}\left(\frac{1}{\nu}\bar{I}_{1_z} + \frac{1}{1-\nu}\bar{I}_{2_z}\right)\left\{\mathrm{Tr}\left(\bar{\boldsymbol{I}}_1 + \bar{\boldsymbol{I}}_2\right) - 3\left(\bar{I}_{1_x} + \bar{I}_{2_x}\right)\right\} = 0 \end{aligned} \tag{91}$$

Only taking the positive solution we find a similar limit for the zero-energy of the system as we found for the stability of the system:

$$\begin{aligned} q_E^2 = &\frac{1}{2}\left(\frac{1}{\nu}\bar{I}_{1_z} + \frac{1}{1-\nu}\bar{I}_{2_z} + \left\{\mathrm{Tr}(\bar{\boldsymbol{I}}_1 + \bar{\boldsymbol{I}}_2) - 3\left(\bar{I}_{1_x} + \bar{I}_{2_x}\right)\right\}\right) \\ &\left[1 + \sqrt{1 + 6\frac{\left(\frac{1}{\nu}\bar{I}_{1_z} + \frac{1}{1-\nu}\bar{I}_{2_z}\right)\left\{\mathrm{Tr}(\bar{\boldsymbol{I}}_1 + \bar{\boldsymbol{I}}_2) - 3\left(\bar{I}_{1_x} + \bar{I}_{2_x}\right)\right\}}{\left(\frac{1}{\nu}\bar{I}_{1_z} + \frac{1}{1-\nu}\bar{I}_{2_z} + \left\{\mathrm{Tr}(\bar{\boldsymbol{I}}_1 + \bar{\boldsymbol{I}}_2) - 3\left(\bar{I}_{1_x} + \bar{I}_{2_x}\right)\right\}\right)^2}}\right] \end{aligned} \tag{92}$$

If the equilibrium separation between the bodies, r, satisfies $r < q_E$, then the system is Hill unstable and can escape.

It can be shown that the separation between bodies for $E = 0$ is less than the separation for the bodies to be stable, or $q_E < q_+$. Thus, a relative equilibrium with a positive free energy is always unstable. This is interesting and implies that these positive energy equilibria can transition directly from an equilibrium state to a disrupted binary. More specifically, it implies that the unstable manifolds of these relative equilibria can escape to infinity, and conversely that a stable manifold may come in from infinity to the equilibrium configuration.

4.3.2 Sufficient conditions for Hill stability

A sufficient condition for Hill stability is always that the Free Energy be negative, as can be inferred by applying the converse to the necessary condition for instability. However, we can often state more than this and actually define the envelope of possible distances and orientations that the system is limited to. These more stringent constraints arise in the form of zero-velocity curves and surfaces, and act as a physical barrier that a system cannot cross, as a function of its energy.

To derive the zero-velocity surfaces recall that the energy of the system can be written in the form:

$$E = \frac{1}{2}\frac{K^2}{I_z} + \frac{1}{2I_z}\left[I_{1_z}I_{2_z}\left(\dot{\phi}_1 - \dot{\phi}_2\right)^2 + mr^2\left(I_{1_z}\dot{\phi}_1^2 + I_{2_z}\dot{\phi}_2^2\right)\right] + \frac{1}{2}m\dot{r}^2 + V(r, \phi_1, \phi_2) \tag{93}$$

We can rearrange this as:

$$\frac{1}{2I_z}\left[I_{1_z}I_{2_z}\left(\dot{\phi}_1 - \dot{\phi}_2\right)^2 + mr^2\left(I_{1_z}\dot{\phi}_1^2 + I_{2_z}\dot{\phi}_2^2\right)\right] + \frac{1}{2}m\dot{r}^2 = E - \frac{1}{2}\frac{K^2}{I_z} - V(r, \phi_1, \phi_2) \tag{94}$$

We note by inspection that the left-hand side is always positive, and thus we find the fundamental inequality:

$$E - \frac{1}{2}\frac{K^2}{I_z} - V(r, \phi_1, \phi_2) \geq 0 \tag{95}$$

Recall that $V < 0$, $V \propto -1/r$, and that $I_z \propto r^2$. Thus we immediately note that if $E > 0$, or if the free energy is positive, that there is no barrier to $r \to \infty$. Conversely, if $E < 0$, we note that it is not possible for the system to disrupt, and it is bound.

For the case where the relative equilibrium is unstable, but $E < 0$, the current constraint can help in delimiting the range of configuration space where the system can evolve. For this system, the surfaces defined by $E - \frac{1}{2}\frac{K^2}{I_z(r)} - V(r, \phi_1, \phi_2) = 0$ delimit the possible range of motion for the system. The surfaces that arise from this constraint define a surface in the 3-dimensional r, ϕ_1, ϕ_2 configuration space.

Finally, we note that if the system is stable and the energy and angular momentum are chosen at the equilibrium values, then E is at its absolute minimum and there are no reconfigurations such that $E > \frac{1}{2}\frac{K^2}{I_z} + V(r, \phi_1, \phi_2)$, meaning that only the equality can apply. This is so as $\boldsymbol{E_{qq}}$ is positive definite in this case. These three different cases are explicitly discussed at the end of the paper for a sphere-sphere system with graphical examples.

4.4 Impact stability

A counterpart to stability against escape is stability against impact. For this study we are specifically interested in whether there are any conditions which control whether two bodies started close to or at equilibrium can impact each other. In (Scheeres 2002), a sufficient condition for impact stability of two mass distributions was derived without the planar restriction, and is stated below. For a given system at a distance r with an energy E and angular momentum K, if a number $d < r$ can be found such that:

$$\frac{1}{2}\,\mathrm{Tr}\,(\boldsymbol{I}_1 + \boldsymbol{I}_2) + md^2 \leq \frac{K^2}{2(E - V_m(d))} \tag{96}$$

where $V_m(d) = \min_{\phi_i} V(d, \phi_1, \phi_2)$, and is the minimum value of the mutual potential at a given distance d, then $r > d$ in all subsequent motion. If d is greater than the separation between the bodies, the system cannot impact.

This condition shares similarities with the zero-velocity conditions, although it is subtly different. The two conditions can be made more similar if we literally treat our mass distributions as planar. Then the term $\frac{1}{2}\,\mathrm{Tr}\,(\boldsymbol{I}_1 + \boldsymbol{I}_2) = I_{z_1} + I_{z_2}$ and the equation is of the same form as the zero-velocity condition given in Eq. 95. Then, restating the sufficient condition for impact to mimic the zero-velocity condition we find:

$$E - \frac{1}{2}\frac{K^2}{I_z(d)} - V_m(d) \leq 0 \tag{97}$$

which goes in the opposite sense as the zero-velocity curve. A deeper analysis of this condition shows that the above inequality can be satisfied for a number $d < r$, where r is the relative equilibrium separation, when the equilibrium point is energetically stable. When the equilibrium point is energetically unstable, it is not possible to satisfy that condition, due to the energy decreasing as one moves away from the equilibrium point, as in that case $E_{rr} < 0$.

This can also be understood in terms of dynamical systems theory. For an unstable relative equilibrium, there will be unstable manifolds that extend interior and exterior to the equilibrium point. The manifold that extends interior to the equilibrium point constitutes the solution branch, at a given equilibrium value of E and K, than can lead to an impact between the bodies. Instead, when the system is energetically stable there are only central manifolds around the equilibrium, which cannot depart from the neighborhood of the equilibrium point due to the positive definite structure of the energy.

5 Example computations and special cases

In the following we specialize the above analysis to definite cases, that of two ellipsoids and the even simpler case of two spheres. These examples allow us to make some general observations about the physical relevance of this study, and point out some interesting results for the stability and fission of contact binary asteroids.

5.1 Ellipsoidal shapes

To provide some specific examples of these results, we will choose ellipsoid shapes for our bodies, compute the moments of inertia, and find, as a function of mass fraction, the value of separation for which the equilibrium configuration becomes stable. Before proceeding, however, we must account for the relation between the size of the bodies and the mass fraction, assuming constant density. To facilitate this, we will assume an ellipsoid shape for each of the bodies. To set this up, we note a few things. The general formula for a moment of inertia of an ellipsoid with semi-major axis $\alpha \geq \beta \geq \gamma$ is:

$$\bar{I}_{i_x} = \frac{1}{5}\left(\beta_i^2 + \gamma_i^2\right) \tag{98}$$

$$\bar{I}_{i_y} = \frac{1}{5}\left(\alpha_i^2 + \gamma_i^2\right) \tag{99}$$

$$\bar{I}_{i_z} = \frac{1}{5}\left(\alpha_i^2 + \beta_i^2\right) \tag{100}$$

Next, define the total effective radius of the system as $R^3 = \alpha_1\beta_1\gamma_1 + \alpha_2\beta_2\gamma_2$, where we note that $\nu = \alpha_2\beta_2\gamma_2/R^3$ and $1 - \nu = \alpha_1\beta_1\gamma_1/R^3$. From these relations we can define the size of the ellipsoid given its shape ratios:

$$\alpha_1 = \left(\frac{1-\nu}{\bar{\beta}_1\bar{\gamma}_1}\right)^{1/3} R \tag{101}$$

$$\alpha_2 = \left(\frac{\nu}{\bar{\beta}_2\bar{\gamma}_2}\right)^{1/3} R \tag{102}$$

where $\bar{\beta}_i = \beta_i/\alpha_i$ and $\bar{\gamma}_i = \gamma_i/\alpha_i$ and both of these follow the inequalities $1 \geq \bar{\beta}_i \geq \bar{\gamma}_i$. The proposed unit of length for each mass fraction value and set of shapes is then the minimum distance between the two bodies in their relative equilibrium configuration, or $l = \alpha_1 + \alpha_2$:

$$l = \left[\left(\frac{1-\nu}{\bar{\beta}_1\bar{\gamma}_1}\right)^{1/3} + \left(\frac{\nu}{\bar{\beta}_2\bar{\gamma}_2}\right)^{1/3}\right] R \tag{103}$$

Now introduce the normalized distance between the two bodies as $\bar{q} = q/l$. The normalized moments of inertia for the bodies are then I_i/l^2 and equal:

$$\bar{I}_{1_x}/l^2 = \frac{1}{5}\frac{\bar{\beta}_1^2 + \bar{\gamma}_1^2}{\left[1+\sigma^{1/3}\right]^2} \tag{104}$$

$$\bar{I}_{1_y}/l^2 = \frac{1}{5}\frac{1 + \bar{\gamma}_1^2}{\left[1+\sigma^{1/3}\right]^2} \tag{105}$$

$$\bar{I}_{1_z}/l^2 = \frac{1}{5}\frac{1 + \bar{\beta}_1^2}{\left[1+\sigma^{1/3}\right]^2} \tag{106}$$

$$\bar{I}_{2_x}/l^2 = \frac{1}{5}\frac{\bar{\beta}_2^2 + \bar{\gamma}_2^2}{\left[1+\frac{1}{\sigma^{1/3}}\right]^2} \tag{107}$$

$$\bar{I}_{2_y}/l^2 = \frac{1}{5}\frac{1 + \bar{\gamma}_2^2}{\left[1+\frac{1}{\sigma^{1/3}}\right]^2} \tag{108}$$

$$\bar{I}_{2_z}/l^2 = \frac{1}{5}\frac{1 + \bar{\beta}_2^2}{\left[1+\frac{1}{\sigma^{1/3}}\right]^2} \tag{109}$$

where

$$\sigma = \frac{\nu}{1-\nu}\frac{\bar{\beta}_1\bar{\gamma}_1}{\bar{\beta}_2\bar{\gamma}_2} \tag{110}$$

We note that $\sigma \to 0$ as P_2 becomes small and that $\sigma \to \infty$ as P_1 becomes small.

Then the normalized value of $\bar{q}_+^2$ where the system transitions from a stable to an unstable equilibrium, Eq. 76, is:

$$\bar{q}_+^2 = \frac{3}{10}\left[\frac{\left(1+\bar{\beta}_1^2+\nu-\frac{\nu}{2}\left(\bar{\beta}_1^2+\bar{\gamma}_1^2\right)\right)}{\nu(1+\sigma^{1/3})^2} + \frac{\left(1+\bar{\beta}_2^2+1-\nu-\frac{1-\nu}{2}\left(\bar{\beta}_2^2+\bar{\gamma}_2^2\right)\right)}{(1-\nu)(1+1/\sigma^{1/3})^2}\right]$$
$$\left[1+\sqrt{1+\frac{10}{3}\frac{\left[\frac{1+\bar{\beta}_1^2}{\nu(1+\sigma^{1/3})^2}+\frac{1+\bar{\beta}_2^2}{(1-\nu)(1+1/\sigma^{1/3})^2}\right]\left[\frac{2-(\bar{\beta}_1^2+\bar{\gamma}_1^2)}{(1+\sigma^{1/3})^2}+\frac{2-(\bar{\beta}_2^2+\bar{\gamma}_2^2)}{(1+1/\sigma^{1/3})^2}\right]}{\left[\frac{\left(1+\bar{\beta}_1^2+\nu-\frac{\nu}{2}(\bar{\beta}_1^2+\bar{\gamma}_1^2)\right)}{\nu(1+\sigma^{1/3})^2}+\frac{\left(1+\bar{\beta}_2^2+1-\nu-\frac{1-\nu}{2}(\bar{\beta}_2^2+\bar{\gamma}_2^2)\right)}{(1-\nu)(1+1/\sigma^{1/3})^2}\right]^2}}\right] \tag{111}$$

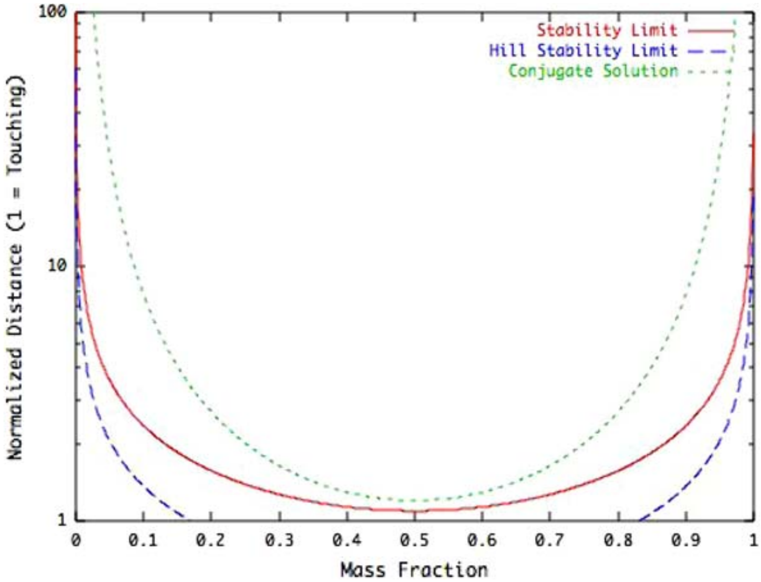

Fig. 2 Plot showing the stability limit, Hill stability limit, and conjugate solution for two spheres in relative equilibrium. The conjugate solution has the same angular momentum as the solution for a touching system, at a distance of 1

where $\bar{q} = 1$ corresponds to the two ellipsoids touching each other. We note that $1 < q_+$, which means that all binary ellipsoid systems that fission are formally unstable when the full coupling between the bodies is taken into account.

We can apply the same normalization to the condition for the Free Energy to be positive, Eq. 24, to find:

$$\bar{q}_E^2 = \frac{1}{10}\left[\frac{\left(1+\bar{\beta}_1^2+\nu-\frac{\nu}{2}\left(\bar{\beta}_1^2+\bar{\gamma}_1^2\right)\right)}{\nu(1+\sigma^{1/3})^2}+\frac{\left(1+\bar{\beta}_2^2+1-\nu-\frac{1-\nu}{2}\left(\bar{\beta}_2^2+\bar{\gamma}_2^2\right)\right)}{(1-\nu)(1+1/\sigma^{1/3})^2}\right]$$
$$\left[1+\sqrt{1+6\frac{\left[\frac{1+\bar{\beta}_1^2}{\nu(1+\sigma^{1/3})^2}+\frac{1+\bar{\beta}_2^2}{(1-\nu)(1+1/\sigma^{1/3})^2}\right]\left[\frac{2-\left(\bar{\beta}_1^2+\bar{\gamma}_1^2\right)}{(1+\sigma^{1/3})^2}+\frac{2-\left(\bar{\beta}_2^2+\bar{\gamma}_2^2\right)}{(1+1/\sigma^{1/3})^2}\right]}{\left[\frac{\left(1+\bar{\beta}_1^2+2\nu-\nu\left(\bar{\beta}_1^2+\bar{\gamma}_1^2\right)\right)}{\nu(1+\sigma^{1/3})^2}+\frac{\left(1+\bar{\beta}_2^2+2(1-\nu)-(1-\nu)\left(\bar{\beta}_2^2+\bar{\gamma}_2^2\right)\right)}{(1-\nu)(1+1/\sigma^{1/3})^2}\right]^2}}\right] \tag{112}$$

In the following we plot a number of different curves for a range of different ellipsoid shapes, for values of ν in the range of [0, 1]. In these plots the distance has been normalized to the distance of the two long-axes added together. Thus, at each value of ν a separation of 1 corresponds to when the two bodies are in contact. If the system is formed by fission of a contact binary, then this unity distance will define the initial orbital equilibrium that the system enters. We note that all such equilibria are apparently unstable and that those with a mass fraction close to 0 or 1 will also have a positive free energy.

The matter in which these curves scale with the different shapes is interesting and shows little deviation from the ideal case of two spheres. First consider Fig. 2, plotting the curves for two spherical bodies ($\alpha_i = \beta_i = \gamma_i$). We see that this simple situation defines the qualitative

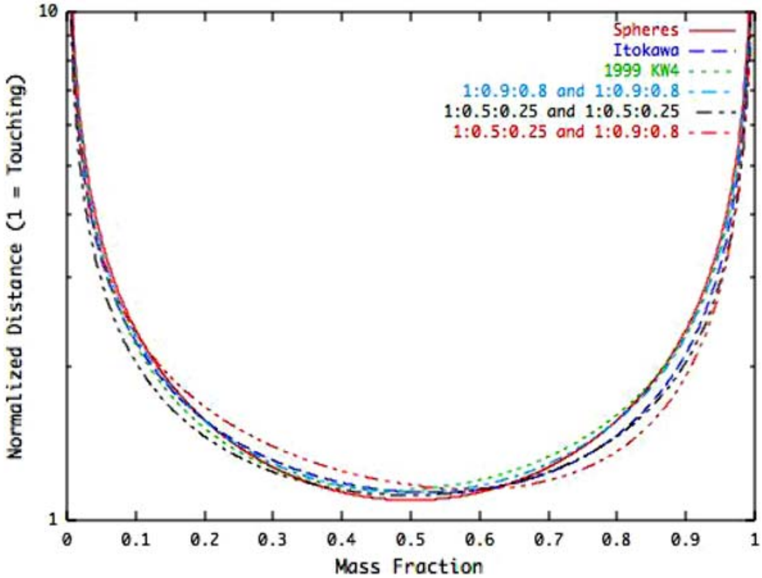

Fig. 3 Plot showing the stability limit for a range of different shape bodies. The Itokawa solution uses the shape of the "Head" and "Body" of that asteroid (Demura et al. 2006), and the 1999 KW4 solution uses the shapes of the primary and secondary for that binary asteroid (Ostro et al. 2006)

situation shown in Figs. 3, 4, 5, representing a diverse set of possible shapes, where we note that their properties are all qualitatively unchanged. However, we must be careful when introducing bodies with any rotational symmetry into the plots, as discussed below.

5.2 Axis-symmetric bodies

There are a few special cases that can result in a fundamental change in the model, and which indicate some limitations of these results. When one of the bodies is changed into being rotationally symmetric about the z-axis the body's rotational kinetic energy and angular momentum decouples from the rest of the system. While this decoupling occurs in a smooth way for the mutual potential (where the additional terms vanish when one body becomes a sphere) it does not smoothly occur in the total energy and angular momentum available for transfer within the system. What happens instead is that the rotational energy and angular momentum of the body in question remains conserved, but must be accounted for independent of the rest of the system energy. In fact, what must occur is a change in the analysis from the start, where these terms are not included in the system energy and angular momentum. For the stated results above, this means that the rotationally symmetric body's I_{z_i} moment of inertia must be removed from the system moment of inertia $I_{z_1} + I_{z_2} + mr^2$.

The biggest issue related to this is that the system's stability properties do not smoothly transition from the case of two non-spheroid bodies to the case of only one non-spheroid body. This can be seen in the paper (Scheeres 2007), where similar stability diagrams are given for relative equilibrium stability in a sphere-ellipsoid system. This abrupt transition occurs once one of the bodies is rotationally symmetric, with a marked difference between the case where one of the bodies is approaching a spheroid in the limit and once it reaches that

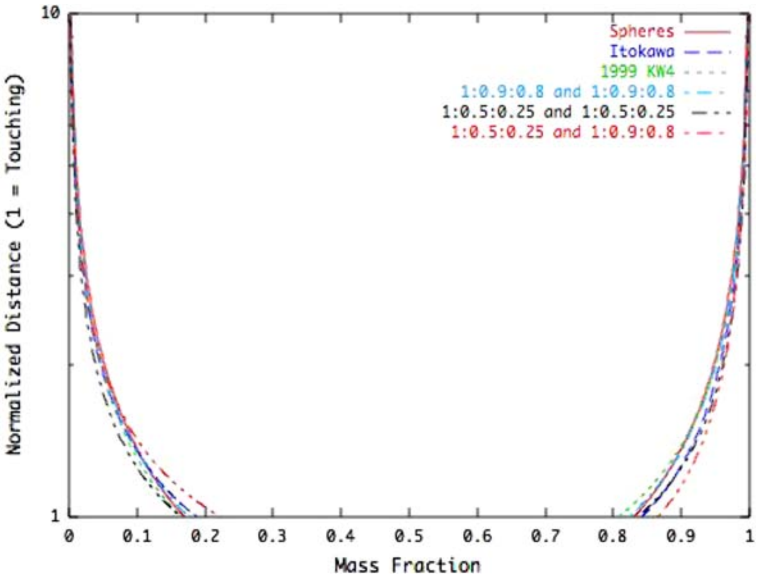

Fig. 4 Plot showing the Hill stability limit for a range of different shape bodies. See the caption of Fig. 3 for more details

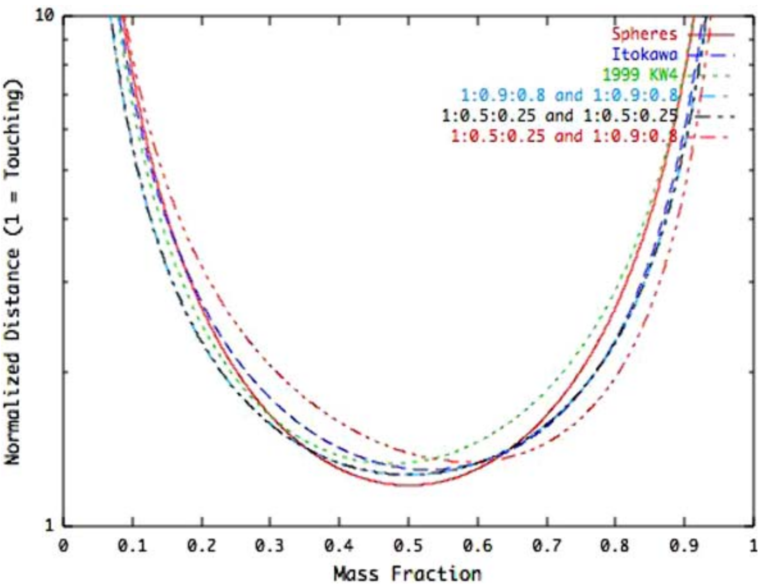

Fig. 5 Plot showing the conjugate stable solutions for a range of different shape bodies. The conjugate solution has the same angular momentum as the solution for a touching system, at a distance of 1, and is the stable counterpart to the unstable touching equilibrium

limit. Physically, there should be no noticeable distinction between these systems, however the presence of even a slight ellipticity implies that rotational coupling can occur and can modify the stability of the system.

In the current paper we do not formally decouple the body's rotational energy and angular momentum from the total system's when one body is made to be rotationally symmetric. This is done with a physical assumption in mind, namely that even for a perfect sphere one expects there to be small deviations in shape due to tidal effects. In the following section we make this assumption for the two-sphere system and, under this assumption, find some interesting results.

5.3 Sphere-sphere

It is instructive to examine the equilibrium and stability of a sphere-sphere system. This is of interest given that the scaled results for non-spherical shapes compare qualitatively to the sphere-sphere system. Technically, this system just decouples into a constant spin for each body and a Keplerian circular orbit for the system. However, if we invoke tidal theory, the presence of small deviations from sphericity can cause transfer of angular momentum and energy between the two bodies, allowing us to apply the results from this study. We find, surprisingly, that for a given angular momentum there are two system equilibria for a sphere-sphere system and that one is unstable and the other stable.

The simplest way to proceed is to reconsider the above, more general results for an ellipsoid-ellipsoid system and specialize it to the sphere-sphere case. To do this, we let $\bar{\beta}_i = \bar{\gamma}_i = 1$. Making this substitution we find that $\sigma = \nu/(1-\nu)$ and see that the above limits for transition between stable and unstable equilibria, and between negative and positive free energy cases, simplify to:

$$\bar{q}_+^2 = \frac{6}{5}\left[\frac{1}{\nu(1+\sigma^{1/3})^2} + \frac{1}{(1-\nu)(1+1/\sigma^{1/3})^2}\right] \tag{113}$$

$$\bar{q}_E^2 = \frac{2}{5}\left[\frac{1}{\nu(1+\sigma^{1/3})^2} + \frac{1}{(1-\nu)(1+1/\sigma^{1/3})^2}\right] \tag{114}$$

or that $q_E^2 = q_+^2/3$. Note that these are the normalized results. It is instructive to remove the normalization by multiplication by m and l^2. This yields the simple to interpret results:

$$mq_+^2 = 3\left(I_{z_1} + I_{z_2}\right) \tag{115}$$

$$mq_E^2 = \left(I_{z_1} + I_{z_2}\right) \tag{116}$$

Thus we see an extremely simple relationship between the moment of inertia of the two bodies treated as mass points and the rotational moment of inertia of the two rigid bodies. When the moment of inertia of the two bodies is greater than the rotational moments of inertia, the total energy of the system is negative, however it may still be unstable. It is only when the system moment of inertia is greater than three times the rotational moments of inertia does the system become stable.

It is also instructive to reconsider the number of relative equilibria that exist for a sphere-sphere system. Applying our simplifications to Eq. 40, noting that $C_{12} = 0$ for this case, we find:

$$r^2\left[r^4 - \frac{K^2}{m^2\mu}r^3 + \left(2\frac{I_{1_z} + I_{2_z}}{m}\right)r^2 + \frac{I_{1_z} + I_{2_z}}{m}\left(\frac{I_{1_z} + I_{2_z}}{m}\right)\right] = 0 \tag{117}$$

Application of the Routh-Hurwitz criterion to this polynomial again yields that there are two distinct, real solutions.

Finally, it is instructive to consider the limits imposed by our zero-velocity surfaces for this particular case. For a 2-sphere system the zero-velocity constraints reduce to:

$$E + \frac{m\mu}{r} \geq \frac{1}{2} \frac{K^2}{I_{z_1} + I_{z_2} + mr^2} \tag{118}$$

where, of course, the moments of inertia for body 1 and 2 are simplified, however we will leave them in their current form given our above realizations. We also assume that the energy and angular momentum are evaluated at a relative equilibrium at a distance r_o between the bodies, thus introducing the equilibrium distance as a parameter. This inequality can be reduced to a cubic equation:

$$\begin{aligned}&\left(I_{z_1} + I_{z_2} - mr_o^2\right)\left(\frac{r}{r_o}\right)^3 + 2mr_o^2\left(\frac{r}{r_o}\right)^2 - \left[3\left(I_{z_1} + I_{z_2}\right) + mr_o^2\right]\left(\frac{r}{r_o}\right)\\ &+2\left(I_{z_1} + I_{z_2}\right) \geq 0\end{aligned} \tag{119}$$

The equation has a double root at $r = r_o$, allowing the equation to be factored into final form:

$$\left[\left(\frac{r}{r_o}\right) - 1\right]^2 \left[\left(I_{z_1} + I_{z_2} - mr_o^2\right)\left(\frac{r}{r_o}\right) + 2\left(I_{z_1} + I_{z_2}\right)\right] \geq 0 \tag{120}$$

We note that the first factor will always be satisfied. This leaves the controlling equation to be:

$$\left(I_{z_1} + I_{z_2} - mr_o^2\right)\left(\frac{r}{r_o}\right) + 2\left(I_{z_1} + I_{z_2}\right) \geq 0 \tag{121}$$

There are three different cases to consider.

I $mr_o^2 \leq I_{z_1} + I_{z_2}$
In this case we see that the inequality is always satisfied, leading to the trivial constraint $r > 0$. For this case we note that there is no upper bound on r, meaning that escape is always an option, and that the free energy of the equilibrium point is positive. Figure 6 shows an explicit example of this situation.

II $I_{z_1} + I_{z_2} < mr_o^2 \leq 3(I_{z_1} + I_{z_2})$
In this case the controlling inequality for r is:

$$0 \leq r \leq r_o \frac{2\left(I_{z_1} + I_{z_2}\right)}{mr_o^2 - \left(I_{z_1} + I_{z_2}\right)} \tag{122}$$

Furthermore, given the constraint on r_o we find that the upper limit satisfies:

$$1 \leq \frac{2\left(I_{z_1}+I_{z_2}\right)}{mr_o^2-\left(I_{z_1}+I_{z_2}\right)} < \infty \tag{123}$$

This implies that there exists a finite upper bound on r, and that a system starting/disturbed from $r = r_o$ can potentially reach this upper limit. In this situation the free energy of the equilibrium point is negative, but the system is still unstable. Figure 7 shows an explicit example of this situation.

III $mr_o^2 > 3(I_{z_1} + I_{z_2})$
In this situation we can show that the radius is subject to the same restrictions:

$$0 \leq r \leq r_o \frac{2\left(I_{z_1} + I_{z_2}\right)}{mr_o^2 - \left(I_{z_1} + I_{z_2}\right)} \tag{124}$$

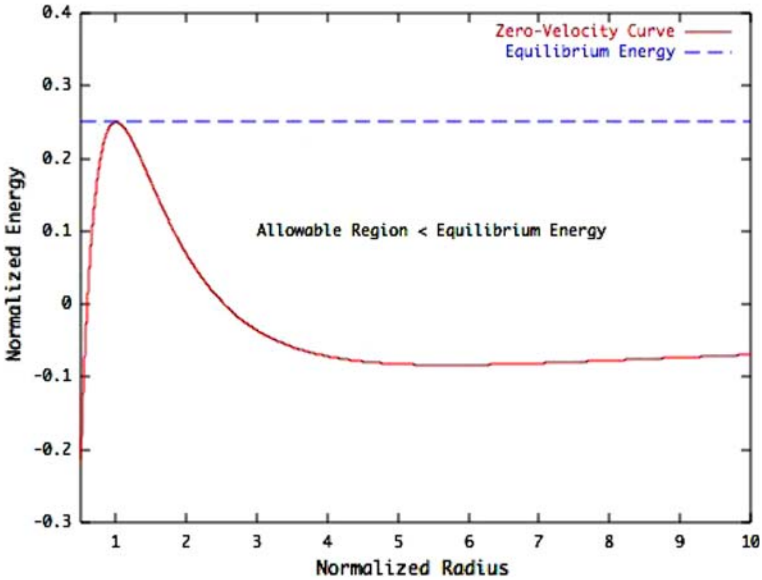

Fig. 6 The equilibrium point is at a radius of 1 and the energy of the equilibrium point equals 0.25. Motion is allowable on all parts of the curve beneath the equilibrium energy, and hence we see that all values of radius can be reached

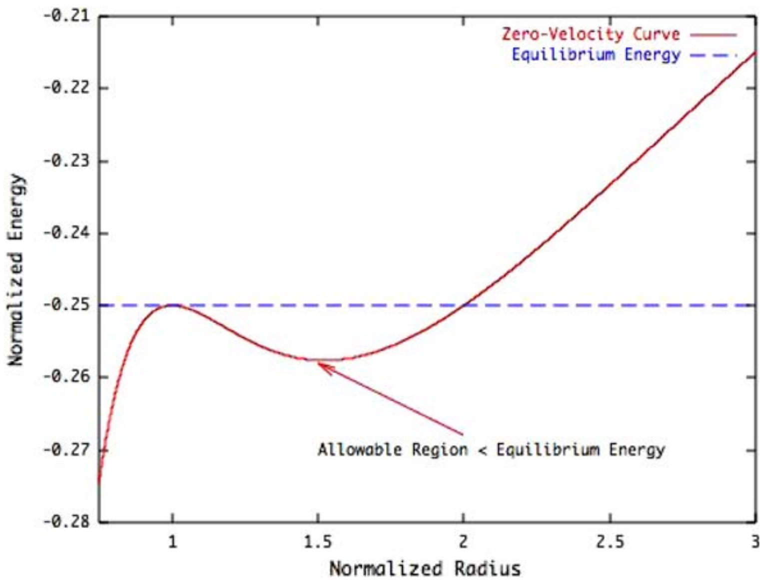

Fig. 7 The equilibrium point is at a radius of 1 and the energy of the equilibrium point equals −0.25. Motion is allowable on all parts of the curve beneath the equilibrium energy, and hence we see that only a limited range of radii can be reached

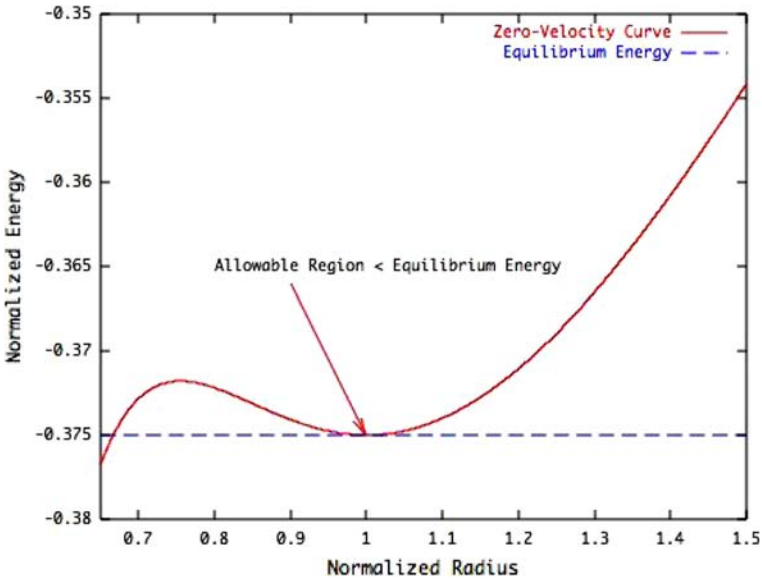

Fig. 8 The equilibrium point is at a radius of 1 and the energy of the equilibrium point equals −0.375. Motion is allowable on all parts of the curve beneath the equilibrium energy, and hence we see that the equilibrium point is isolated

However now, given the constraint on r_o we find that the upper limit satisfies:

$$\frac{2\left(I_{z_1}+I_{z_2}\right)}{mr_o^2-\left(I_{z_1}+I_{z_2}\right)} < 1 \tag{125}$$

Thus, a particle starting in the vicinity of $r = r_o$ is now separated from this allowable interval. Going back to the original, full inequality we see that in the vicinity of $r = r_o$ we will violate the inequality unless we keep $r = r_o$. This fits well with our knowledge that the relative equilibrium is the minimum energy equilibrium. Figure 8 shows an explicit example of this situation.

6 Conclusions

The stability of a planar full 2-body problem has been studied, accounting for a non-trivial mutual gravitational potential between the two bodies. For the case when the mutual potential is expanded to second order, we can make a complete discussion of the stability of the system, incorporating energetic stability, spectral stability, Hill stability and stability against impact. The current discussion reveals some interesting results when applied to constant density ellipsoidal bodies, expanded to second order. Specifically, we find that all binary systems that arise from the fission of the two components will be unstable, with a stable relative equilibrium in existence at a larger distance with a lower system energy (but same system angular momentum). This has implications for the evolution of an asteroid spun to fission, implying that it immediately enters a dynamic phase which can include reimpact of the bodies or separation of the bodies. If the total system energy is positive, the system may even disrupt. A surprising find is that these stability considerations also apply to a system of two spheres orbiting each other, when their rotational moments of inertia are incorporated into the energy and angular momentum constraints. This shows that not every circular orbit of a Keplerian system is stable, and that as a function of the rotational moments of inertia a different circular orbit may be found at the same value of angular momentum which has a lower value of system energy. A second important result is that the stability constraints and equilibrium separations between two strongly non-spherical bodies, when scaled appropriately, is close to the same results as stated for a two-sphere system.

References

Bellerose, J.E., Scheeres, D.J.: Energy and stability in the full two body problem. Celest. Mech. Dyn. Astron. **100**(1), 63–91 (2008)

Cendra, H., Marsden, J.E.: Geometric mechanics and the dynamics of asteroid pairs. Dyn. Syst. **20**(1), 3–21 (2005)

Demura, H., Kobayashi, S., et al.: Pole and Global Shape of 25143 Itokawa. Science **312**, 1347–1349 (2006)

Ferraz-Mello, S., Rodriguez, A., Hussmann, H.: Tidal friction in close-in satellites and exoplanets: The Darwin theory re-visited. Celest. Mech. Dyn. Astron. **101**(1–2), 171–201 (2008)

Kinoshita, H.: First-order perturbations of the two finite body problem. Publ. Astron. Soc. Jpn. **24**, 423–457 (1972)

Koon, W.-S., Marsden, J.E., Ross, S., Lo, M.W., Scheeres, D.J.: Geometric mechanics and the dynamics of asteroid pairs. In: Belbruno, E., Folta, D., Gurfil, P. (eds.) Astrodynamics, Space Missions, and Chaos, vol. 1017, pp. 11–38. Annals of the New York Academy of Science (2004)

Maciejewski, A.J.: Reduction, relative equilibria and potential in the two rigid bodies problem. Celest. Mech. Dyn. Astron. **63**, 1–28 (1995)

Ostro, S.J., Margot, J.-L., Benner, L.A.M. et al.: Radar imaging of binary near-earth asteroid (66391) 1999 KW4. Science **314**, 1276–1280 (2006)

Scheeres, D.J.: Stability in the full two-body problem. Celest. Mech. Dyn. Astron. **83**, 155–169 (2002)

Scheeres, D.J.: Stability of relative equilibria in the full two-body problem. In: Belbruno, E., Folta, D., Gurfil, P. (eds.) Astrodynamics, Space Missions, and Chaos, vol. 1017, pp. 81–94. Annals of the New York Academy of Science (2004)

Scheeres, D.J.: Relative equilibria for general gravity fields in the sphere-restricted full 2-body problem. Celest. Mech. Dyn. Astron. **94**, 317–349 (2006)

Scheeres, D.J.: Rotational fission of contact binary asteroids. Icarus **189**, 370–385 (2007)

Scheeres, D.J.: Minimum energy asteroid reconfigurations and catastrophic disruptions. Planetary and Space Science in press (2009). doi:10.1016/j.pss.2008.03.009

Simo, J.C., Lewis, D., Marsden, J.E.: Stability of relative equilibria. I. The reduced energy-momentum method. Arch. Ration. Mech. Anal. **115**(1), 15–59 (1991)

Wang, L.-I., Krishnaprasad, P.S., Maddocks, J.H.: Hamiltonian dynamics of a rigid body in a central gravitational field. Celest. Mech. Dyn. Astron. **50**, 349–386 (1991)