Coupled Systems of Differential Equations

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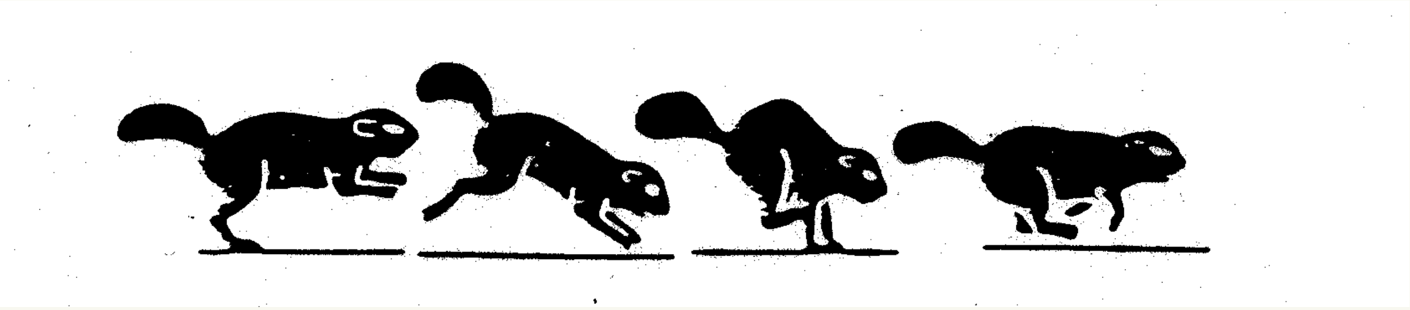
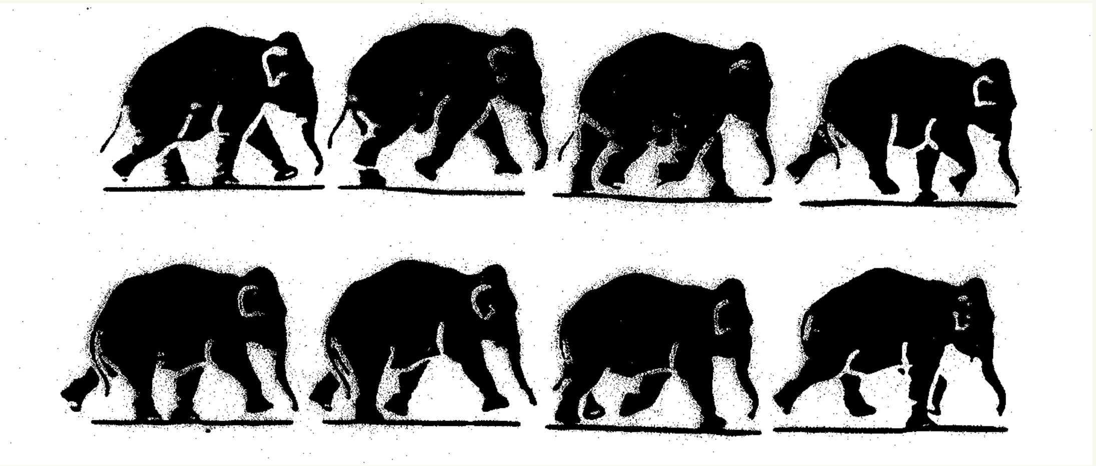
Quadruped Gaits

- Bound of the Siberian Souslik

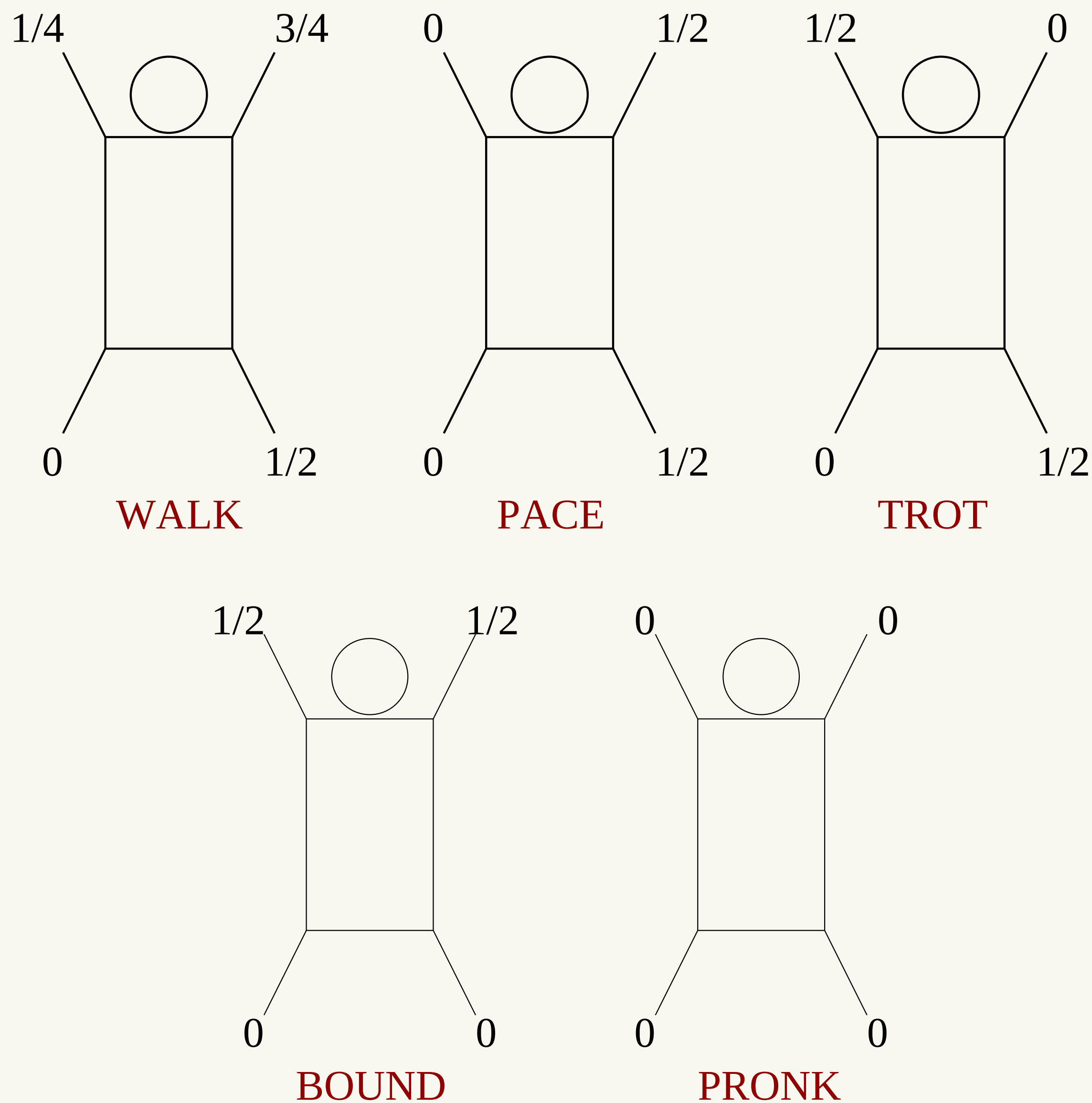


- Amble of the Elephant



- Trot of the Horse

Standard Gait Phases



Gait Symmetries / Central Pattern Generators

Gait	Spatio-temporal symmetries
Trot	(Left/Right, $\frac{1}{2}$) and (Front/Back, $\frac{1}{2}$)
Pace	(Left/Right, $\frac{1}{2}$) and (Front/Back, 0)
Walk	(Figure Eight, $\frac{1}{4}$)

- Network of neurons (CPG) that produces gait rhythms
- Hodgkin - Huxley (1952)

  Neuron modeled by system of differential equations
- Design simplest network to produce walk, trot, and pace

Collins and Stewart (1993)

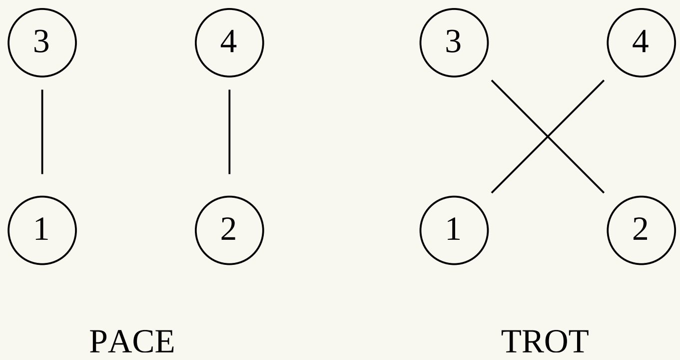
Four Cells Do Not Suffice

- $\Gamma =$ symmetry group of locomotor CPG network

- Network produces walk. There is a four-cycle symmetry

$$(1\ 3\ 2\ 4)$$

- Four-cycle permutes pace to trot



- CPG cannot be modeled by four-cell network where each cell gives rhythmic pulsing to one leg

G., Stewart, Buono, and Collins (1999)

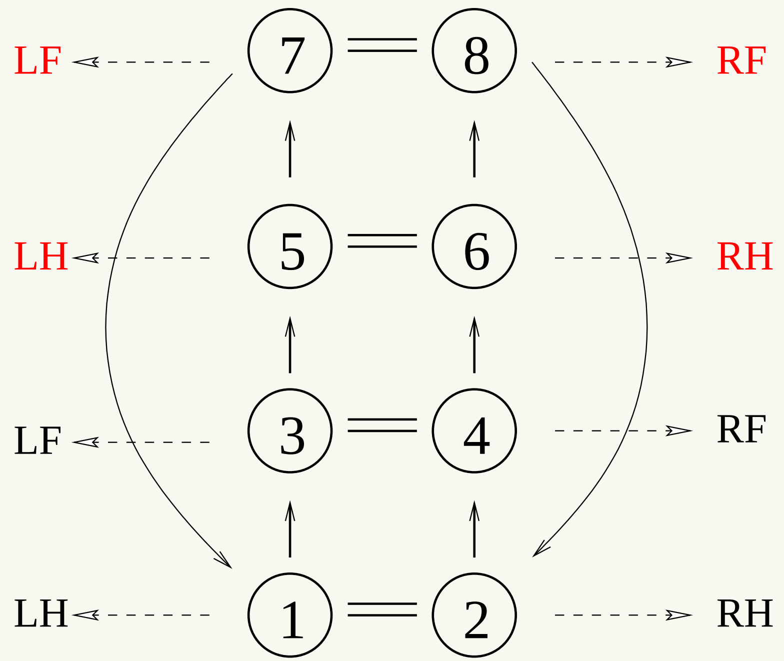
Central Pattern Generators (CPG)

- Use gait symmetries to construct coupled network

  1) walk $\Longrightarrow$ four-cycle ω in symmetry group

  2) pace or trot $\Longrightarrow$ transposition κ in symmetry group

- Simplest network has $\mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$ symmetry

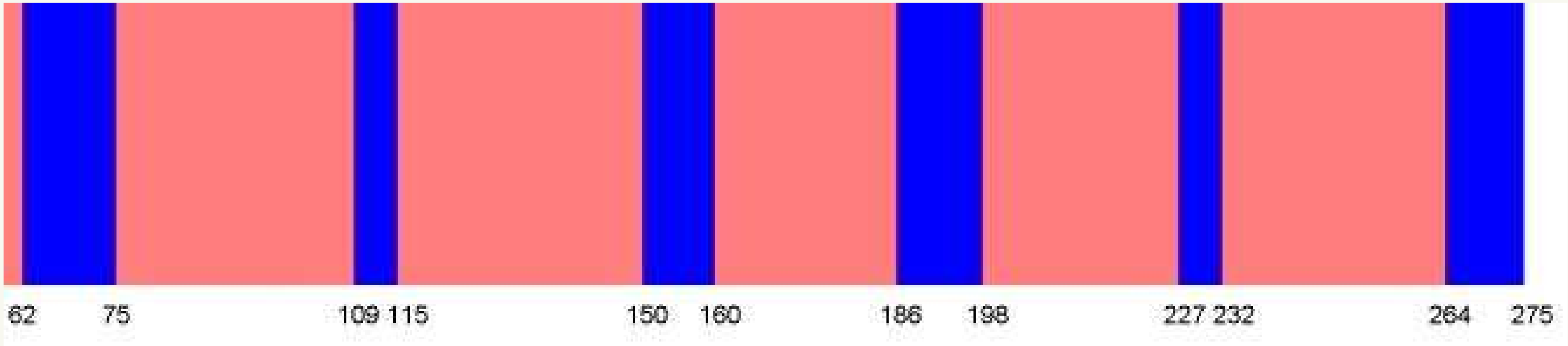


G., Stewart, Buono, and Collins (1999); Buono and G. (2001)

Primary Gaits or Hopf Bifurcation from Stand: $H = \mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$

K	Phase Diagram	Gait
$\mathbf{Z}_4(\omega) \times \mathbf{Z}_2(\kappa)$	$\begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix}$	pronk
$\mathbf{Z}_4(\omega)$	$\begin{matrix} 0 & \frac{1}{2} \\ 0 & \frac{1}{2} \end{matrix}$	pace
$\mathbf{Z}_4(\kappa\omega)$	$\begin{matrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{matrix}$	trot
$\mathbf{Z}_2(\kappa) \times \mathbf{Z}_2(\omega^2)$	$\begin{matrix} \frac{1}{2} & \frac{1}{2} \\ 0 & 0 \end{matrix}$	bound
$\mathbf{Z}_2(\kappa\omega^2)$	$\begin{matrix} \frac{1}{4} & \frac{3}{4} \\ 0 & \frac{1}{2} \end{matrix}$	walk
$\mathbf{Z}_2(\kappa)$	$\begin{matrix} 0 & 0 \\ \frac{1}{4} & \frac{1}{4} \end{matrix}$	jump

The Jump



- Average Right Rear to Right Front = 31.2 frames
- Average Right Front to Right Rear = 11.4 frames
- $\frac{31.2}{11.4} = 2.74$

G., Stewart, Buono, and Collins (2000)