Coupled Systems of Differential Equations

DANCE Winter School
Pamplona
January 27, 2012

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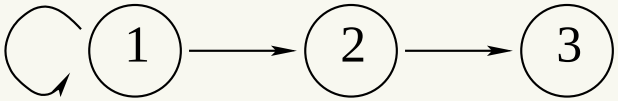
Outine

- Why $\frac{1}{6}$?
- Feedforward network as a motif
- Heteroclinic cycles in networks
- A model for rivalry

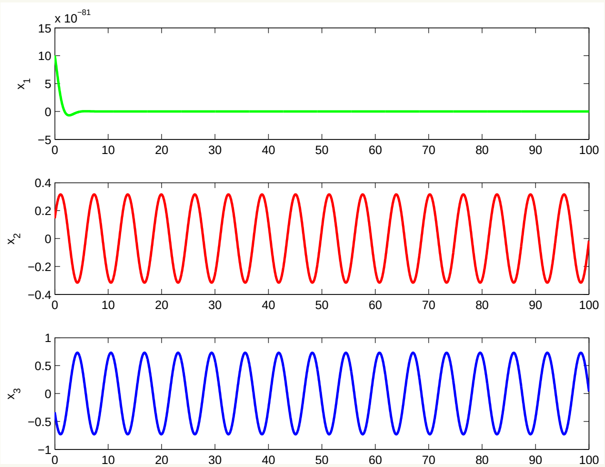
Three-Cell Feed-Forward Network

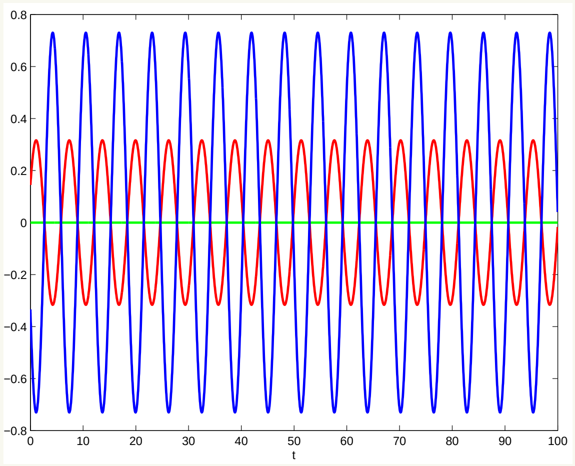
- 

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1) \\ \dot{x}_2 &= f(x_2, x_1) \\ \dot{x}_3 &= f(x_3, x_2) \end{aligned} \qquad J = \begin{bmatrix} \alpha+\beta & 0 & 0 \\ \beta & \alpha & 0 \\ 0 & \beta & \alpha \end{bmatrix}$$

- Network supports solution by Hopf bifurcation where
  $x_1(t)$ equilibrium $\qquad$ $x_2(t), x_3(t)$ time periodic

- $x_2(t) \approx \lambda^{1/2}$ $\qquad$ $x_3(t) \approx \lambda^{1/6}$





G., Nicol, and Stewart (2004); Elmhirst and G. (2005); G. and Postlethwaite (2012)

F.F. Ex. $f(u,v) = (\lambda + i - |u|^2)u - v$ where $u, v \in \mathbf{C}$

$$\dot{x}_1 = f(x_1, x_1) = (\lambda + i - |x_1|^2)x_1 - x_1$$

$x_1 = 0$ is a stable equilibrium for $\lambda < 1$

$$\dot{x}_2 = f(x_2, x_1) = (\lambda + i - |x_2|^2)x_2 - x_1$$

$$\dot{x}_2 = f(x_2, 0) = (\lambda + i - |x_2|^2)x_2$$

$x_2(t) = \sqrt{\lambda}e^{it}$ is stable periodic solution for $0 < \lambda < 1$

$$\dot{x}_3 = f(x_3, x_2) = (\lambda + i - |x_3|^2)x_3 - x_2$$

$$\dot{x}_3 = f(x_3, \sqrt{\lambda}e^{it}) = (\lambda + i - |x_3|^2)x_3 - \sqrt{\lambda}e^{it}$$

F.F. Ex. $f(u,v) = (\lambda + i - |u|^2)u - v$

$x_1 = 0$ is a stable equilibrium for $\lambda < 1$
$x_2(t) = \sqrt{\lambda}e^{it}$ is stable periodic solution for $0 < \lambda < 1$

$$\dot{x}_3 = (\lambda + i - |x_3|^2)x_3 - \sqrt{\lambda}e^{it}$$

Set $x_3(t) = y(t)e^{it}$

$$\dot{y}e^{it} + yie^{it} = (\lambda + i - |y|^2)ye^{it} - \sqrt{\lambda}e^{it}$$

$$\dot{y}e^{it} + \boxed{yie^{it}} = (\lambda + \boxed{i} - |y|^2)\boxed{ye^{it}} - \sqrt{\lambda}e^{it}$$

$$\dot{y}e^{it} = (\lambda - |y|^2)ye^{it} - \sqrt{\lambda}e^{it}$$

$$\dot{y}\boxed{e^{it}} = (\lambda - |y|^2)y\boxed{e^{it}} - \sqrt{\lambda}\boxed{e^{it}}$$

$$\dot{y} = (\lambda - |y|^2)y - \sqrt{\lambda}$$

F.F. Ex. $f(u, v) = (\lambda + i - |u|^2)u - v$

$x_1 = 0$ is a stable equilibrium for $\lambda < 1$
$x_2(t) = \sqrt{\lambda} e^{it}$ is stable periodic solution for $0 < \lambda < 1$
$x_3(t) = y(t)e^{it}$

$$\dot{y} = (\lambda - |y|^2)y - \sqrt{\lambda}$$

Set $y(t) = \lambda^{1/6} u(t)$

$$\lambda^{1/6}\dot{u} = (\lambda^{7/6} - \lambda^{3/6}|u|^2)u - \lambda^{3/6}$$

$$\dot{u} = (\lambda - \lambda^{1/3}|u|^2)u - \lambda^{1/3}$$

$$\dot{u} = -\lambda^{1/3}(|u|^2 u + 1) + \lambda u$$

F.F. Ex. $f(u,v) = (\lambda + i - |u|^2)u - v$

$x_1 = 0$ is a stable equilibrium for $\lambda < 1$
$x_2(t) = \sqrt{\lambda}e^{it}$ is stable periodic solution for $0 < \lambda < 1$
$x_3(t) = y(t)e^{it}$
$y(t) = \lambda^{1/6}u(t)$

$$\dot{u} = -\lambda^{1/3}(|u|^2 u + 1) + \lambda u$$

Solve $\dot{u} = 0$ for equilibria

$$-(|u|^2 u + 1) + \lambda^{2/3}u = 0$$

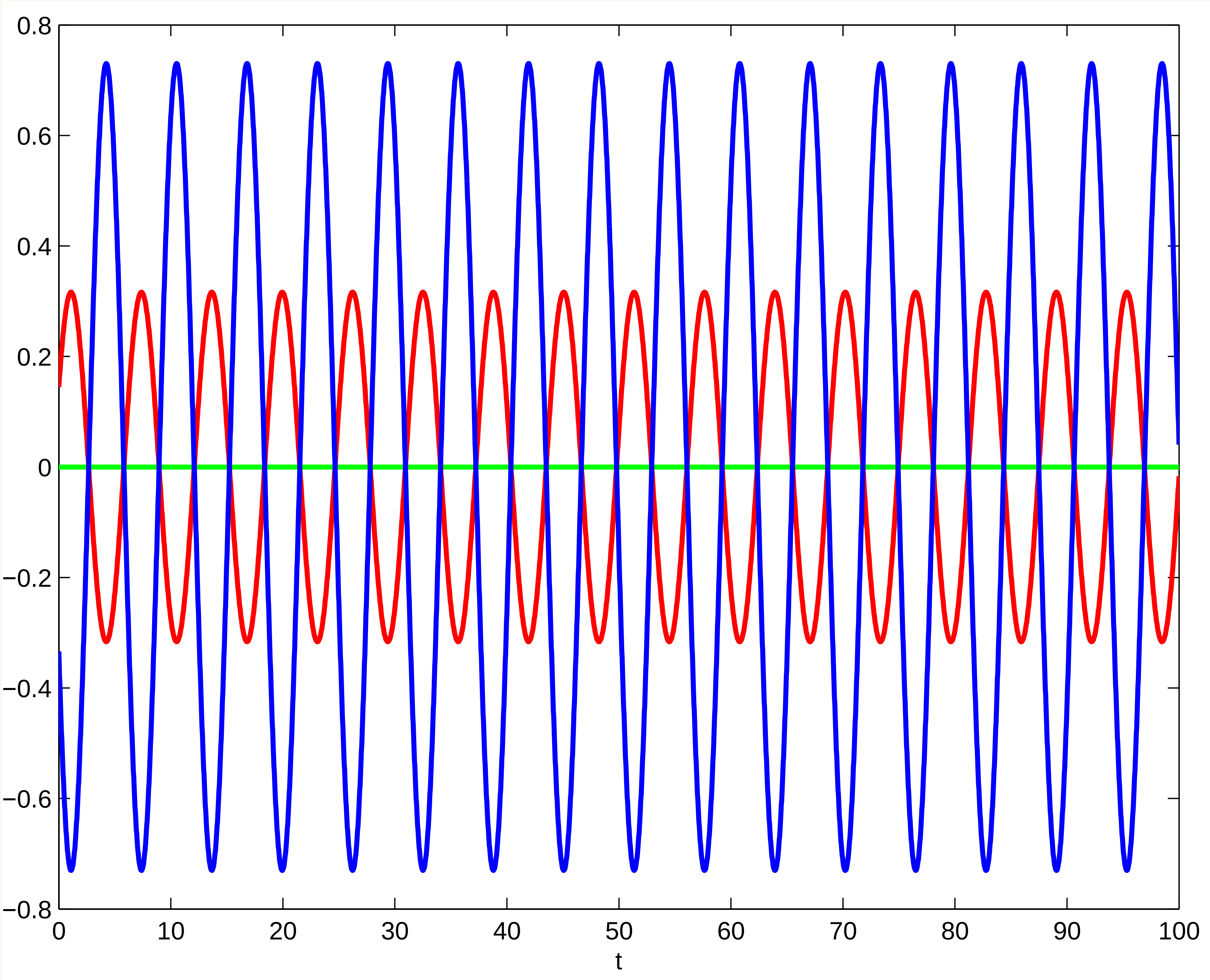
Use IFT to obtain branch of (stable) equilibria

$$u_0(\lambda) = -1 + O(\lambda^{2/3})$$

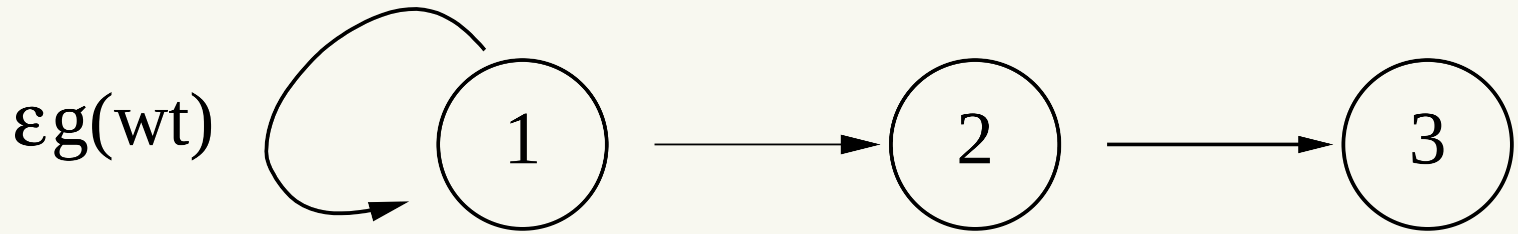
Thus $x_3(t)$ is periodic with same period as $x_2(t)$

$$x_3(t) = y(t)e^{it} = \lambda^{1/6}u(t)e^{it} \to \lambda^{1/6}u_0(\lambda)e^{it} = -\lambda^{1/6}e^{it} + O(\lambda^{5/6})$$

Feed Forward Simulation



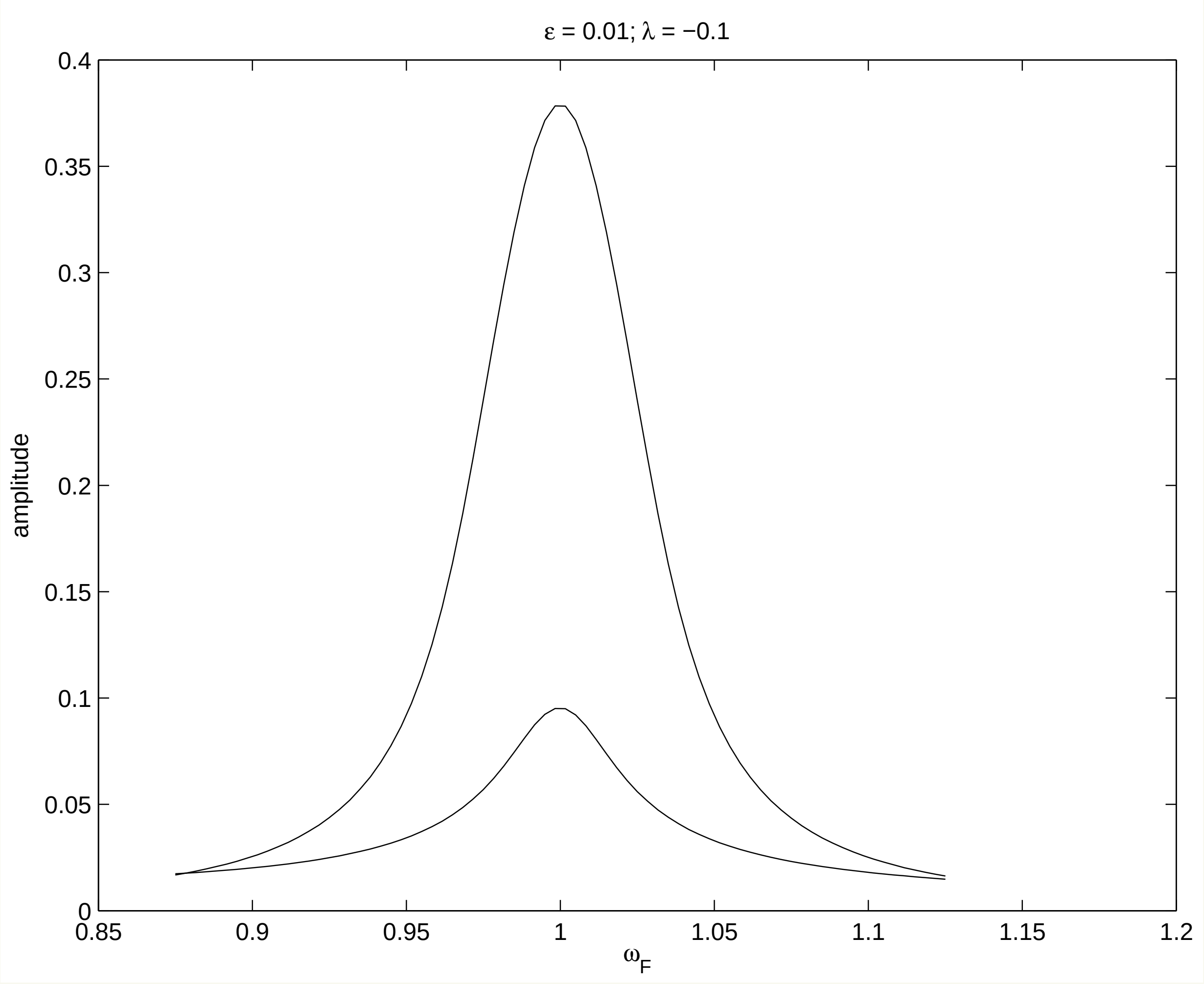
Forced Feed Forward Network



- forcing at frequency ω_f and amplitude ε
- network tuned near Hopf bifurcation with frequency ω_h
- $\lambda < 0$ so that equilibrium is stable
- Three parameters: λ, ϵ, $\omega_f - \omega_h$

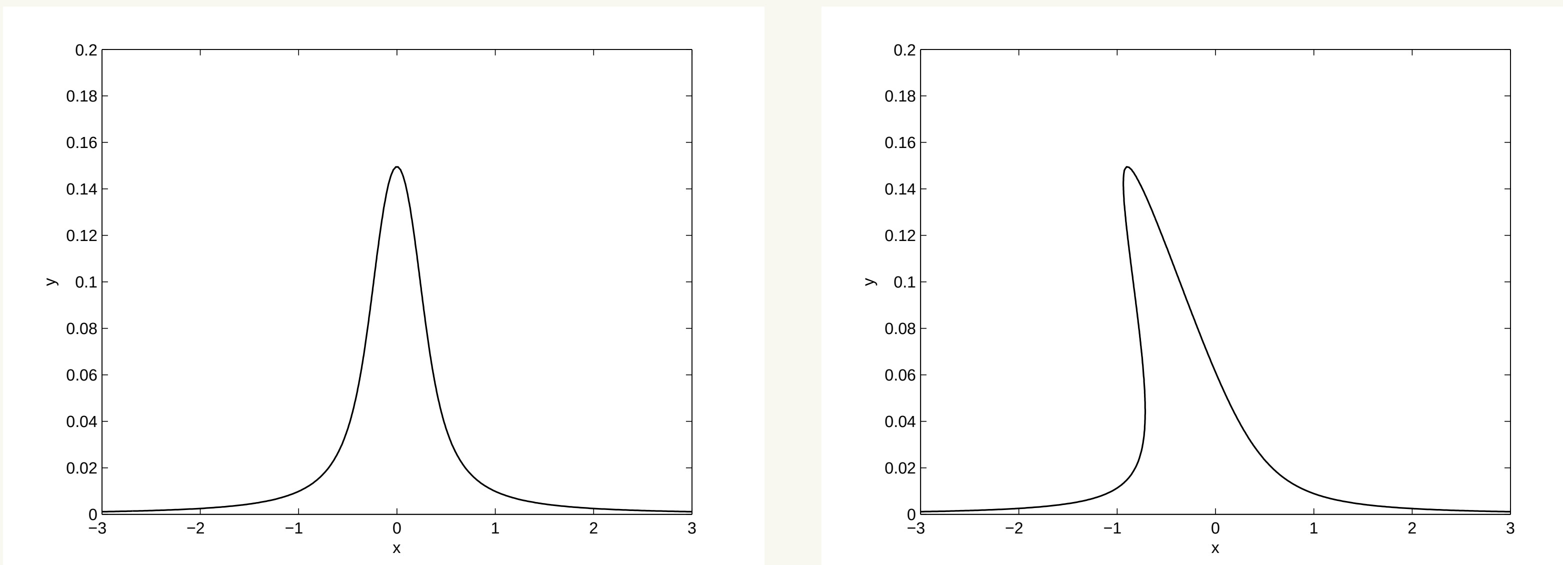
Numerics with Aronson

- $\dot{z} = (-0.1 + i - |z|^2)z + 0.01(e^{i\omega_F t} + 2e^{2i\omega_F t} - 0.5e^{3i\omega_F t})$



Periodic Forcing of Hopf

- $\dot{z} = (\lambda + \omega_H i - (1 + i\gamma)|z|^2)z + \varepsilon e^{2\pi i \omega_f t}$
- $\omega = \omega_f - \omega_H$, $\lambda = -0.0218$, $\varepsilon = 0.02$

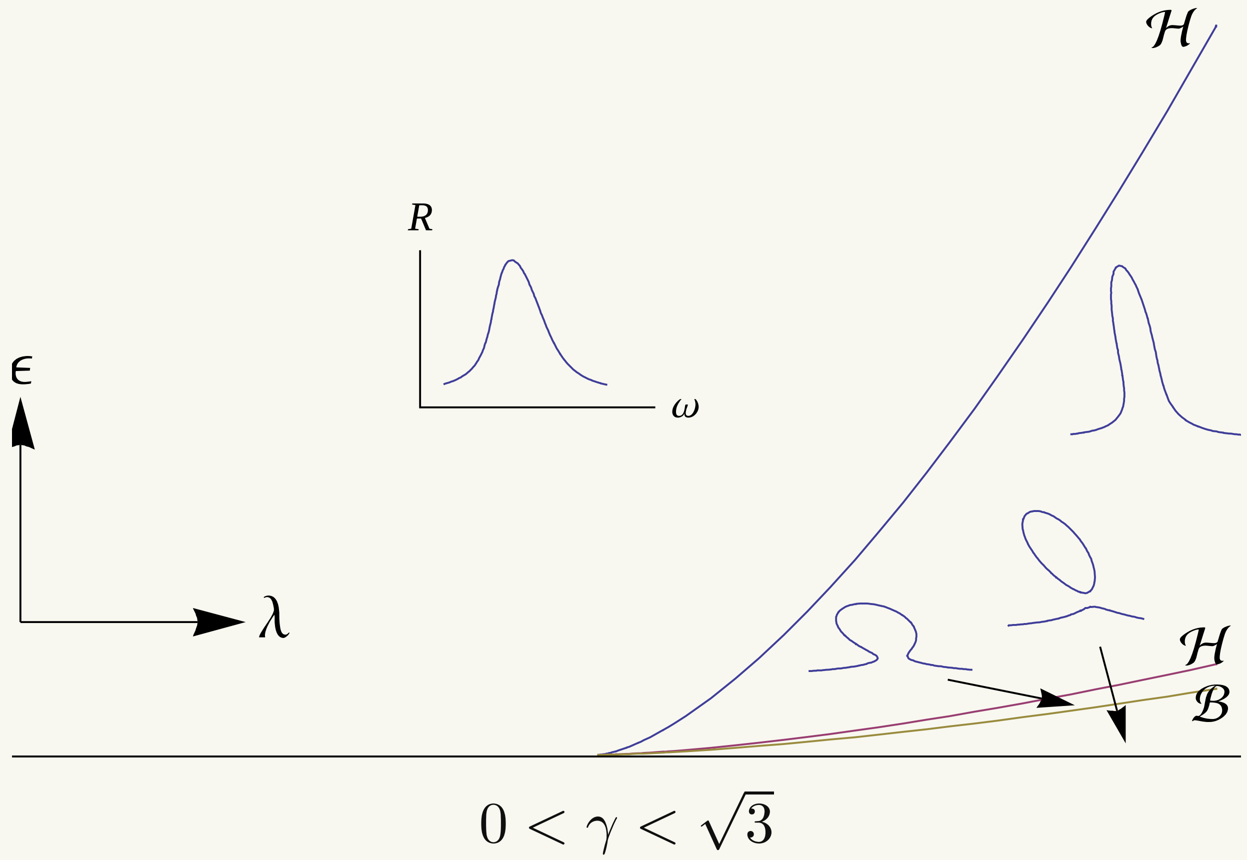


$\gamma = 0$ $\gamma = 3$ $\gamma = 6$

G., Postlethwaite, Shiau, and Zhang (2009)

Bifurcation Diagrams: $\gamma < \sqrt{3}$

For fixed ε and λ and bifurcation parameter ω, the bifurcation diagrams are



Zhang and G. (2011)

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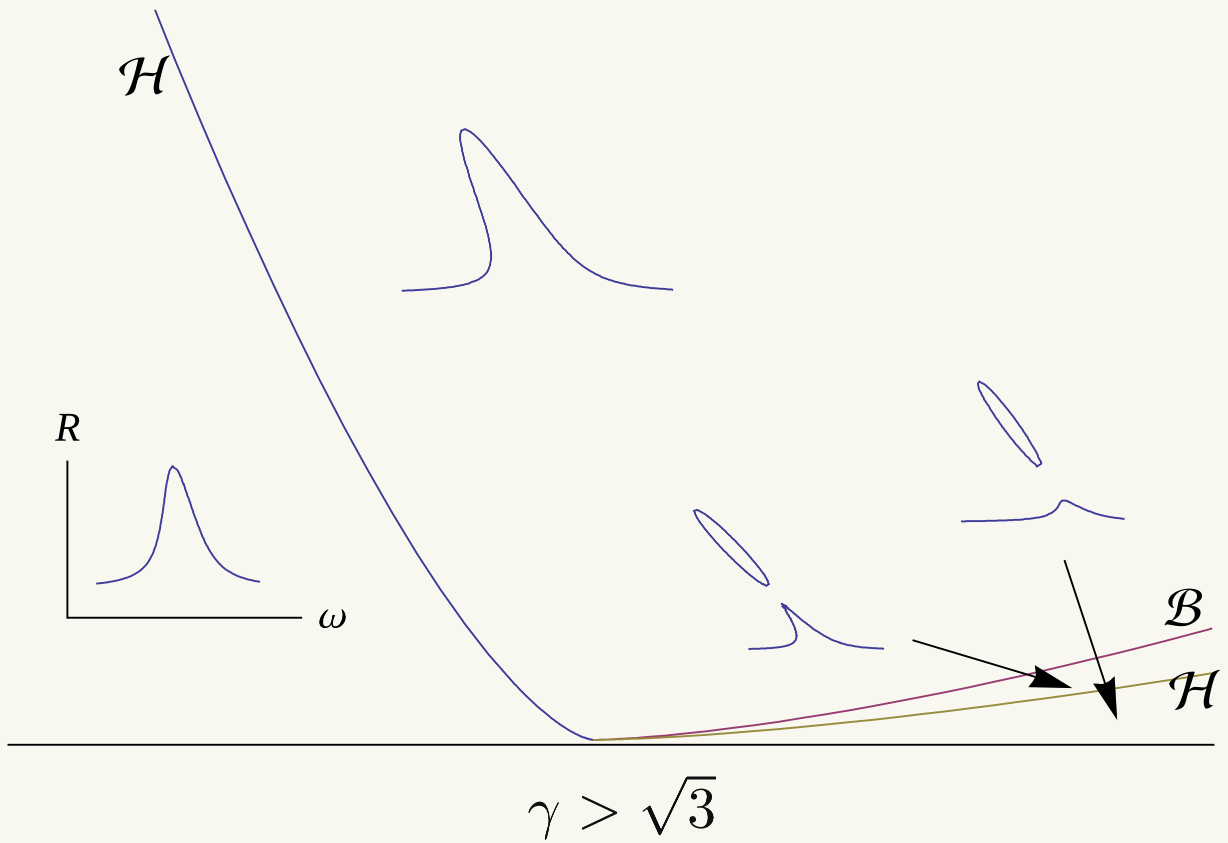
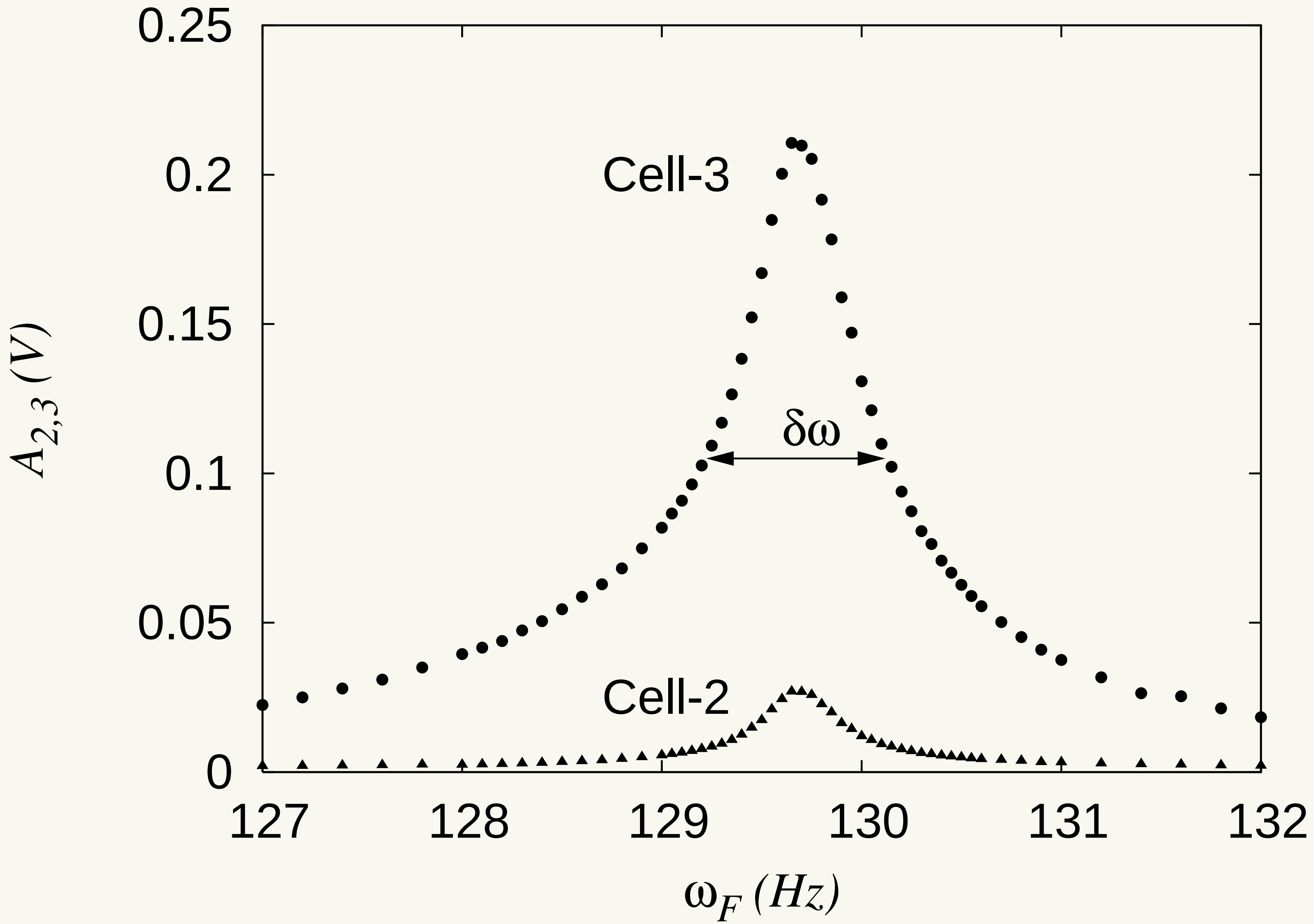


McCullen-Mullin Experiment



McCullen, Mullin, and G. (2007)

Guckenheimer-Holmes Heteroclinic Cycle

Γ acts on $\mathbf{R}^3$ generated by

$$\begin{array}{rcl} (x,y,z) & \mapsto & (\pm x, \pm y, \pm z) \\ (x,y,z) & \mapsto & (y,z,x) \end{array}$$

$|\Gamma| = 24$ and $\Gamma =$ symmetry group of cube

- $F(0,0,0) = 0$ since $\mathrm{Fix}(-x,-y,-z) = \{0\}$
- Coordinate axes flow-invariant since $\mathrm{Fix}(-x,-y,z) = \mathbf{R}\{(0,0,1)\}$
- Generic pitchfork bifurcation leads to equilibrium on z-axis
- Symmetry: equilibria on x- and y-axes
- Coordinate planes are flow-invariant since $\mathrm{Fix}(-x,y,z) = \{(0,y,z)\}$

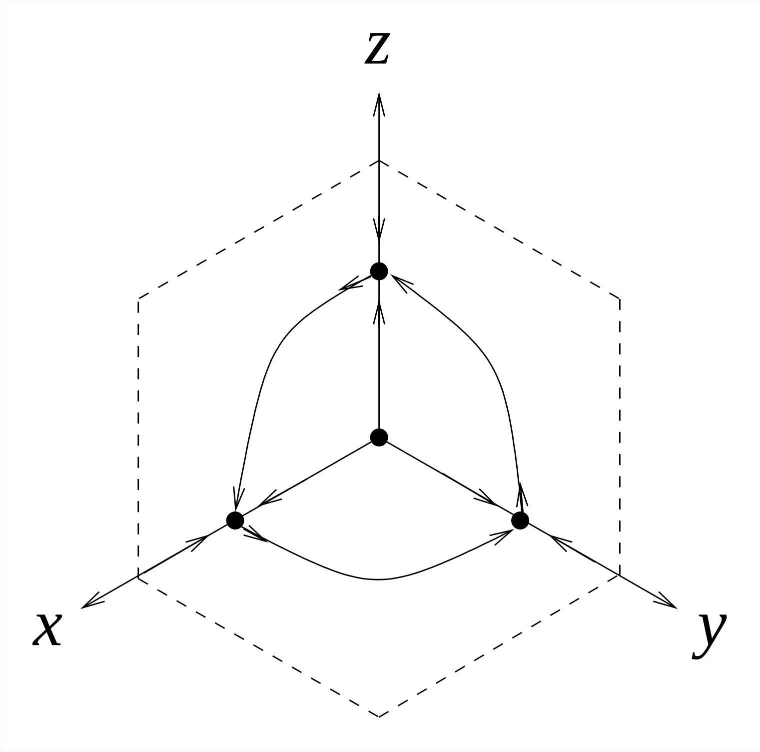
Guckenheimer and Holmes (1988)

Construction of Cycle

Suppose

- There are no other equilibria in coordinate planes
- Two remaining eigenvalues of equilibria on axes have opposite sign
- Infinity is a source

Phase portrait is

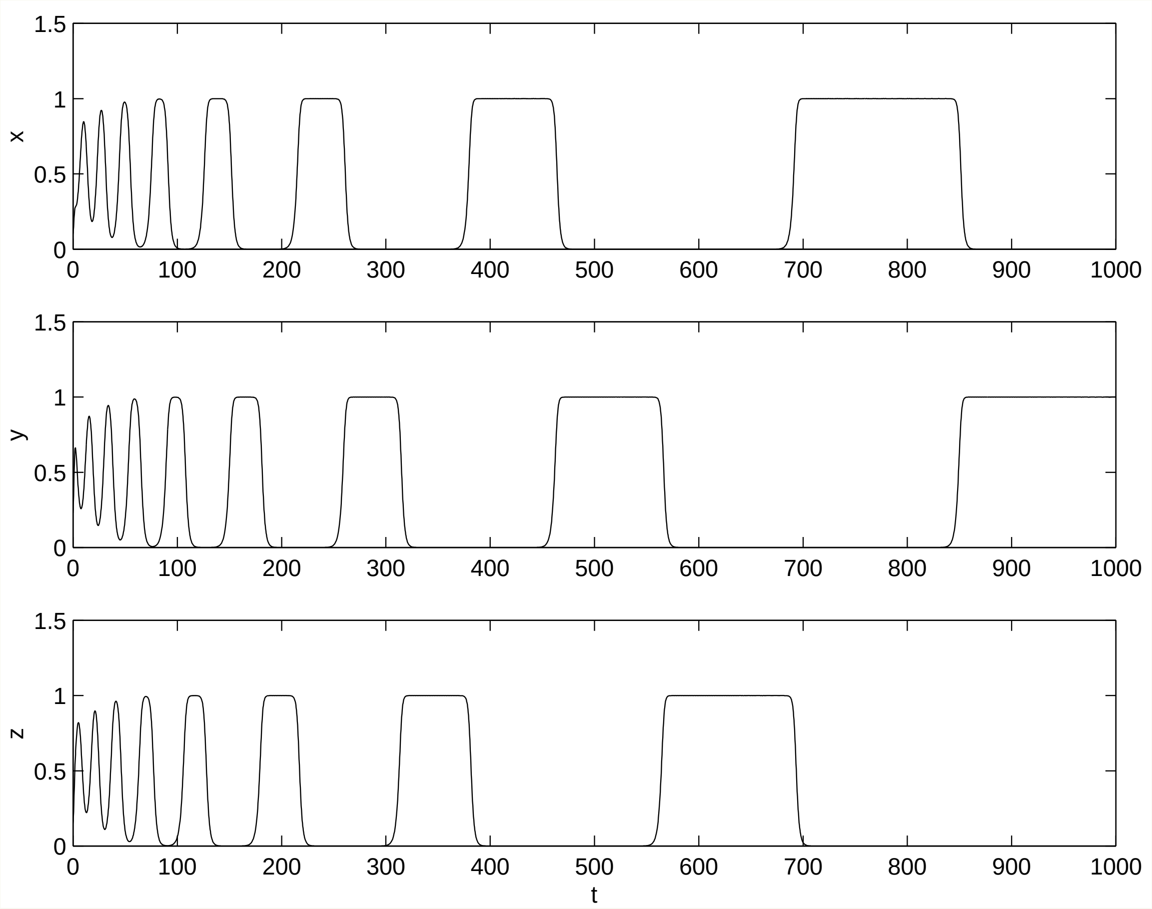


Integration of Cycle: $\lambda = 1.0$, $A = 1.0$, $B = 1.5$, $C = 0.6$

Consider third order truncation of Γ-equivariant system

$$F(x,y,z) = (f_1(x,y,z), f_2(x,y,z), f_3(x,y,z))$$

$$\begin{aligned} f_1(x,y,z) &= \lambda x + (Ax^2 + By^2 + Cz^2)x \\ f_2(x,y,z) &= \lambda y + (Cx^2 + Ay^2 + Bz^2)y \\ f_3(x,y,z) &= \lambda z + (Bx^2 + Cy^2 + Az^2)z \end{aligned}$$

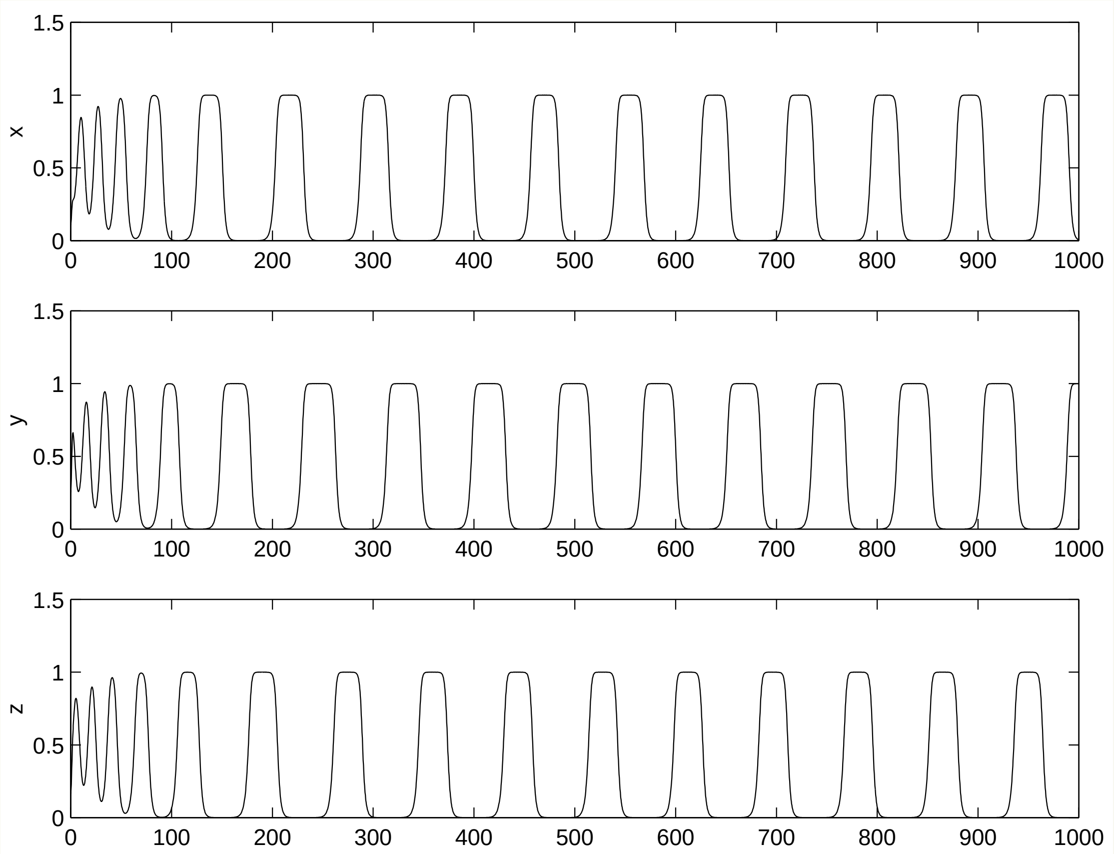


Breaking Symmetry of Guckenheimer-Holmes Heteroclinic Cycle

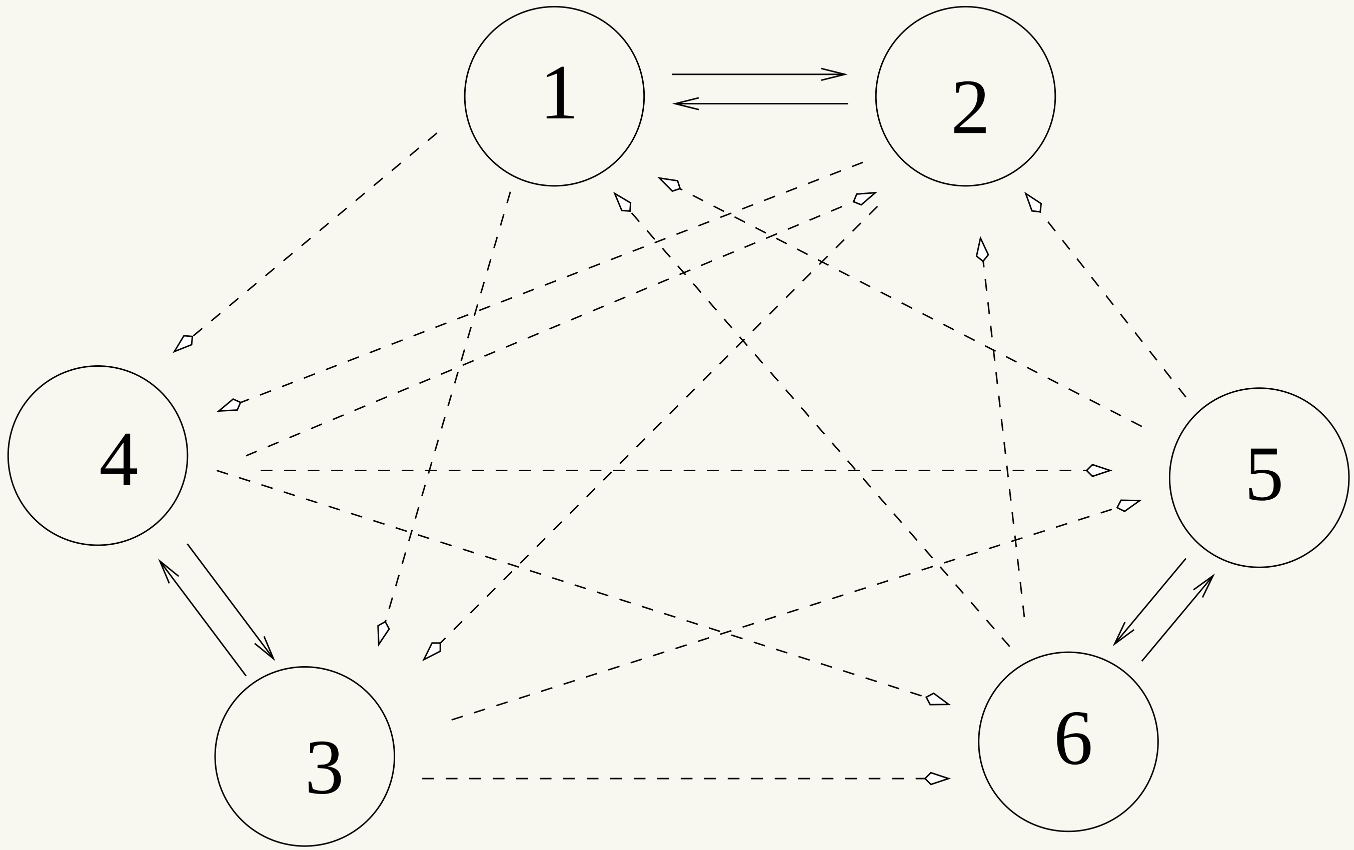
Breaking symmetry perturbs cycle to periodic solution. For example:

$$\begin{aligned}\dot{x} &= x - (Ax^2 + By^2 + Cz^2)x + \epsilon y\\ \dot{y} &= y - (Cx^2 + Ay^2 + Bz^2)y + \epsilon x\\ \dot{z} &= z - (Bx^2 + Cy^2 + Az^2)z + \epsilon z\end{aligned}$$

where $A = 1.0$, $B = 1.5$, $C = 0.6$, $\epsilon = 0.00001$

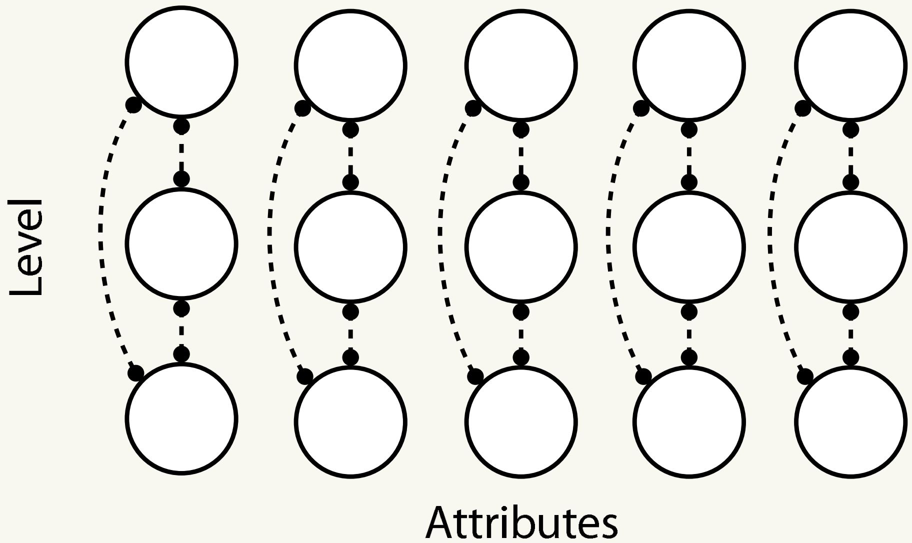


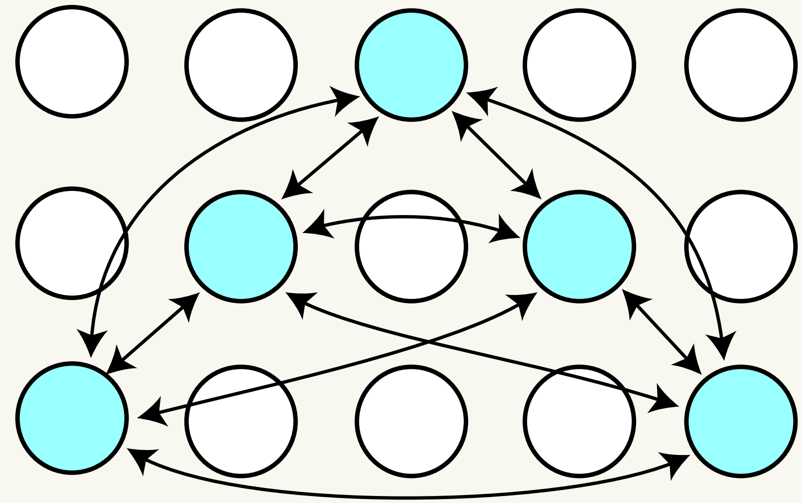
Coupled Cell Version of Guckenheimer-Holmes Cycle



Dionne, G. and Stewart (1994, 1996); Field et al. (2006-12)

Wilson’s Generalized Rivalry Model





- Column represent attributes; rows represent level of attribute
- (L) Dashed lines: reciprocal inhibition between cells in column
- (R) Solid lines: reciprocal excitation between cells in learned pattern

Wilson (2008, 2009); Diekman, G., McMillen, and Wang (2012)

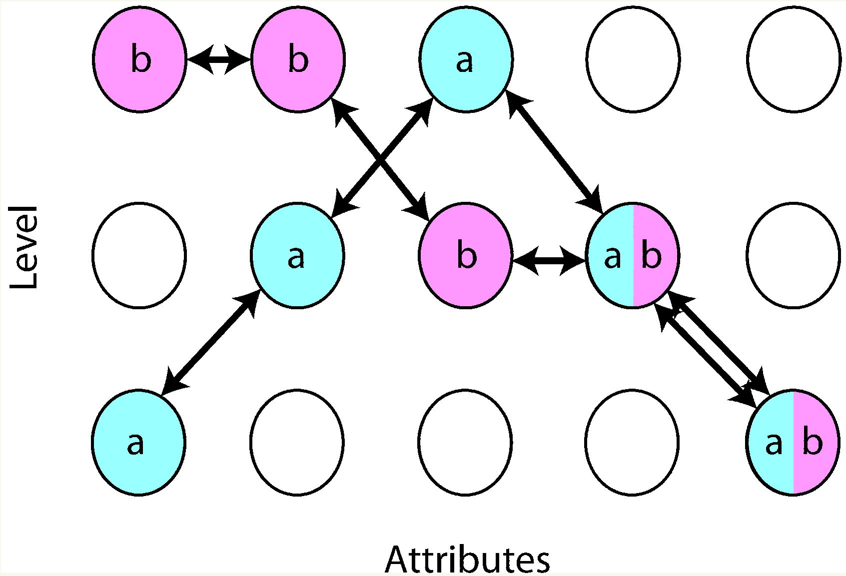
Simplest Rivalry Equations Between Competing Units a and b

- Units represent perception of images presented to eyes
- Unit a consists of an activity variable a^E representing a firing rate, and a fatigue variable a^H that reduces activity on long time scale

$$\begin{aligned} \varepsilon\, \dot{a}^E &= -a^E + \mathcal{G}\left(I - \beta b^E - g a^H\right) \\ \dot{a}^H &= a^E - a^H \\ \varepsilon\, \dot{b}^E &= -b^E + \mathcal{G}\left(I - \beta a^E - g b^H\right) \\ \dot{b}^H &= b^E - b^H \end{aligned}$$

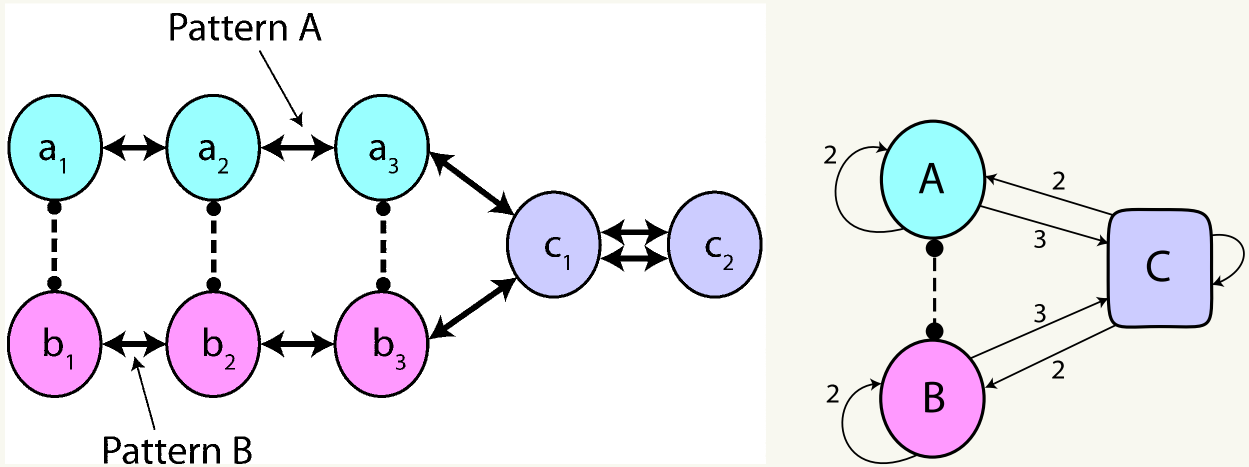
- β is reciprocal inhibition between units
- I is external signal strength to units
- a^H reduces the activity in unit a with strength g
- $\mathcal{G}$ is gain: nonnegative, nondecreasing, and $\mathcal{G}(z) = 0$ for $z \leq 0$
- $\varepsilon \ll 1$ is ratio of time scales on which $*^E$ and $*^H$ evolve

Two Learned Patterns a and b



- Network: n attribute columns with m cells representing attribute levels; two equations in each cell
- Learned pattern = one cell from each attribute column
- Reciprocal excitatory connections between these cells
- Cells in learned pattern are all-to-all connected (not indicated)
- Inhibitory connections in columns not indicated

Inactive Cells



(L) Two learned patterns with 2 cells in common (inactive cells deleted).
(R) Quotient network. Integers indicate multi-arrow couplings

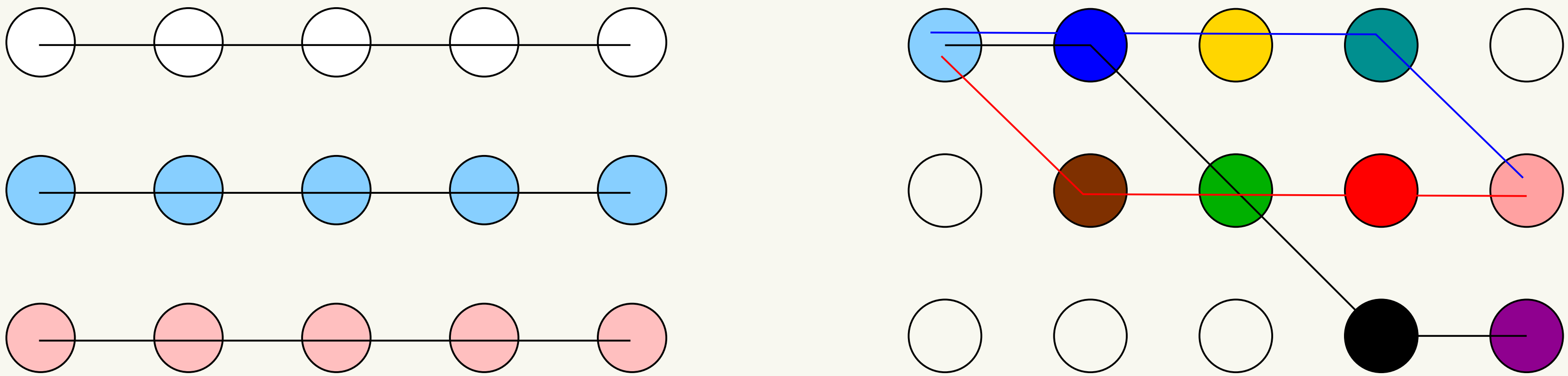
- Inactive cells may be ignored, thus reducing network to $2n - k$ cells, where k is number of active cells in common in two-patterns
- Understand dynamics using quotient network: 2-cell network if no cells in common or 3-cell network if cells in common
- Quotient network corresponds to subspace Δ. For many parameters Δ is locally attracting. So, reduction to quotient captures dynamics

Solution Types

Three types of states:

- Fusion = equilibria in which patterns have equal values
- Winner-Take-All = equilibria with different activity levels
- Rivalry = two or more patterns oscillate in periods of dominance
- States: synchronous equilibria; asynchronous equilibria; oscillations
- Rivalry could stem from Hopf bifurcation or heteroclinic cycle

Many Learned Patterns



(L) Three patterns with no cells pairwise in common; (R) Each pair of patterns has common active cells

- Wilson: Rivalry predominates in 5 attribute 3 intensity level system when there are four or five learned patterns
- n attribute m intensity level system can learn m^n patterns (243 in working example)
- Extreme case (all learned patterns) may be tractable: wreath product $S_m \wr S_n$ with $(m!)^n n!$ elements (933120 in working example)
- Wreath product symmetric coupled systems can lead to heteroclinic cycles. Guckenheimer-Holmes cycle has wreath product group