Monomial summability and doubly singular differential equations

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Abstract

In this work, we consider systems of differential equations that are *doubly singular*, i.e. that are both singularly perturbed and exhibit an irregular singular point. If the irregular singular point is at the origin, they have the form

$$\varepsilon^\sigma x^{r+1}\frac{d\mathbf{y}}{dx} = f(x,\varepsilon,\mathbf{y}),\ f(0,0,\mathbf{0})=\mathbf{0}$$

with f analytic in some neighborhood of $(0,0,\mathbf{0})$. If the Jacobian $\frac{df}{d\mathbf{y}}(0,0,\mathbf{0})$ is invertible, we show that the unique bivariate formal solution is *monomially summable*, i.e. summable with respect to the monomial $t=\varepsilon^\sigma x^r$ in a (new) sense that will be defined. As a preparation, Poincaré asymptotics and Gevrey asymptotics in a monomial are studied.

1 Introduction

The study of systems of holomorphic differential equations with singularities has been the subject of a large number of articles during the last decades. A typical example of this class of equations is the following:

$$x^{r+1}\mathbf{y}' = F(x,\mathbf{y}),$$

where F is holomorphic in a neighbourhood of $(0,\mathbf{0})$ in $\mathbb{C}^{n+1}$, and $r\geq 0$. If $r=0$, the singularity is of regular type, which implies that formal power series solutions are convergent. If $r>0$ then the point $x=0$ is an irregular singularity. In order to give a meaning to a formal, in general non-convergent solutions, (multi)-summability theory has been developed by different authors ([R1, BBRS, MaRa, Br, B1, E1],...).

Another type of well studied differential equations are singularly perturbed differential equations. These equations depend on a parameter such that, for some values of it, the order of the equation suddenly changes. One of the best studied equations of this kind is the forced van der Pol equation

$$\varepsilon x'' + (x^2 - 1)x' + x = a.$$

This equation has been carefully investigated from different points of view, including non-standard analysis [BCDD], Gevrey character [C], geometry [DR] and asymptotic analysis (including summability)[FS]. In general, formal solutions of singularly perturbed differential equations of the form $\varepsilon^{\sigma}\mathbf{y}' = F(x, \varepsilon, \mathbf{y})$, F analytic near $(0, 0, \mathbf{0})$, $\sigma \geq 1$ are *not* summable.

In this work we shall analyze systems of holomorphic differential equations exhibiting both kinds of singularities: they are singularly perturbed and have a singularity with respect to x as well. For the case of a singular perturbation and a regular singularity at the same time, Russell and Sibuya [RS1, RS2] study the block-diagonalization, and block-triangularization of linear systems using asymptotic expansions only with respect to ε. In their work, they encounter problems related to the nature of the difference of the eigenvalues at the origin, so they have to exclude some directions from the sectors they consider. We know of no work studying summability in this case.

Our purpose is the investigation of systems that are singularly perturbed and *irregular* singular as well; surprisingly, the results are nicer than in the previous case. If the irregular singularity is at $x = x_0 \in \mathbb{C}$, then such a system adopts the form

$$(S) \qquad \varepsilon^{\sigma}(x - x_0)^{r+1}\mathbf{y}' = f(x, \varepsilon, \mathbf{y}), \ \sigma, r \geq 1,$$

where $f : D_{x_0} \times D_0 \times D_{\mathbf{y}_0} \to \mathbb{C}^n$ is holomorphic and D_{x_0}, D_0, $D_{\mathbf{y}_0}$ are neighbourhoods of x_0, 0, $\mathbf{y}_0$ respectively in $\mathbb{C}$, $\mathbb{C}$, $\mathbb{C}^n$. If the irregular singularity is at $x = \infty$, then its form is

$$(\tilde{S}) \qquad \varepsilon^{\sigma}\mathbf{y}' = x^{r-1}f(1/x, \varepsilon, \mathbf{y}), \ \sigma, r \geq 1,$$

with $f : D_0 \times D_0 \times D_{\mathbf{y}_0} \to \mathbb{C}^n$ holomorphic.

A change of variables allows us to suppose that $x_0 = 0$, $\mathbf{y}_0 = \mathbf{0}$, and we will suppose $f(0, 0, \mathbf{0}) = \mathbf{0}$. We shall restrict ourselves to systems whose linear part at the origin is invertible, i.e. $\dfrac{\partial f}{\partial \mathbf{y}}(0, 0, \mathbf{0})$ is an invertible linear map. Under these conditions, such a system has a unique formal power series solution. The main objective of our work is to establish properties of this solution.

The interest for such systems is not new, and has occupied different authors. In [Si1], Y. Sibuya considers linear systems of the form $(\tilde{S})$ that also depend regularly upon a second parameter μ. Asymptotic expansions in x and in ε are constructed, and the author combines them in a certain way, but they are not really bivariate asymptotic expansions.

In the book [M2], after introducing the notion of strong asymptotic expansion, H. Majima considers systems of differential equations

$$(1) \qquad \mathbf{x}^{\mathbf{p}}x_1\frac{d\mathbf{u}}{dx_1} + A_0\mathbf{u} = a(\mathbf{x}, \mathbf{u}),$$

where $\mathbf{x} = (x_1, \ldots, x_n)$, $\mathbf{p} = (p_1, \ldots, p_n)$, $p_1 > 0$, A_0 an invertible constant matrix, $a(0, \mathbf{0}) = \mathbf{0}$ and $\frac{\partial a}{\partial \mathbf{u}}(0, \mathbf{0}) = \mathbf{0}$. He shows the existence of solutions, with strong asymptotic expansion, in polysectors $V = V_1 \times \cdots \times V_n$, contained in a *strictly proper domain*. More precisely, if $\lambda_i \neq 0$ are the eigenvalues of A_0, V is strictly proper with respect to λ_i if

$$\bar{V} \cap \{\mathbf{x} \mid \cos(\arg \lambda_i / \mathbf{x}^{\mathbf{p}}) < 0\}$$

has only one connected component. V is a strictly proper domain if it is strictly proper with respect to all eigenvalues of A_0. This result shows that the sets

$$\{\mathbf{x} \in (\mathbb{C} \setminus \{0\})^n \mid \alpha < \arg \mathbf{x}^{\mathbf{p}} < \beta\},$$

for certain $\alpha < \beta$, are important when treating this kind of problems.

The doubly singular differential equations we are considering fall into the category studied by Majima. Conversely, it seems possible to carry over our results to (1) by allowing additional regular parameter dependence. First, consider $\varepsilon x_1^{p_1+1} \frac{d\mathbf{u}}{dx_1} = f(\mathbf{x}, u)$, then replace $\varepsilon = x_2^{p_2} \cdot \ldots \cdot x_n^{p_n}$.

We consider equations of the form (S), as explained above for $x_0 = 0$ only, and investigate solutions in sets of the type

$$\{(x, \varepsilon) \in (\mathbb{C}^2 \setminus \{0\})^2 \mid \alpha < \arg(x^r \varepsilon^\sigma) < \beta\},$$

which we call "sectors in the monomial $x^r \varepsilon^\sigma$". This seems the appropriate setting for treating doubly singular differential equations, and in future works, more general ones.

In this work, we introduce a new type of asymptotic expansions in a monomial $x_1^p x_2^q$, that appears in a natural way in our discussions. We develop the theory of Gevrey asymptotic expansions and summability with respect to the monomial. Then the formal series solutions of (S) are shown to be monomially summable in the newly defined sense.

Expansions with respect to a monomial were already used by Martinet and Ramis in [MR], when studying the analytic classification of resonant foliations. In their work, nevertheless, it seems not clear that their definition is independent of the representation of a power series as a series in $x_1^p x_2^q$; it was not their objective to give a detailed discussion of this notion. Existence of solutions in sectors in a monomial and different types of summability (but not monomial summability) were also shown in the article [BM]. There only the linear case was considered. At the end of section 3, we compare our definitions with the above. Let us also mention that J. Écalle, in [E2], considers this type of equations, that own what he calls equational and coequational resurgence.

As a consequence of our results, we obtain a block-decomposition theorem for general linear systems $x^{r+1} \varepsilon^\sigma \mathbf{y}' = A(x, \varepsilon)\mathbf{y}$. In future works we shall extend this result in order to study "monomial multisummability" of formal solutions. The notion of classical multisummability will be replaced by "summability with respect to $(\mathbf{m}_1, \ldots, \mathbf{m}_r)$", $\mathbf{m}_i$ being monomials in x, ε. There we will prove that a divergent series cannot be summable with respect to two different monomials. This will show in particular that the choice of $x^r \varepsilon^\sigma$ in the present work is appropriate.

More generally, systems of equations $P(x, \varepsilon)\mathbf{y}' = F(x, \varepsilon, \mathbf{y})$ may be considered, $P(x, \varepsilon)$ being a polynomial. For these, a notion of summability with respect to a polynomial should be defined, and this can be done combining the techniques we develop in the present work with desingularization of plane curves. This will also be the subject of future articles.

The plan of the article is as follows.

In section 2 we shall present some examples that lead to systems of differential equations of the type we are considering here. In section 3 we shall develop the notion of asymptotic expansion we shall use through the rest of our work. In section 4 we shall show how to perform a rank reduction to our system. After this simplification, we shall have an equivalent problem, but with $r = \sigma = 1$. This will also simplify notation through the rest of this work without loosing generality. Section 5 shall be devoted to the study of the Gevrey character of the unique formal solution of the equation, and to the search of an actual holomorphic solution having the formal series as a bivariate asymptotic expansion, according to the definitions given in section 3. As main technique, we shall perform a change of variables in order to obtain an almost regularly perturbed system of ordinary differential equations, where the perturbation parameter has its values in an annulus depending upon the main variable. Well known methods of asymptotic analysis can be carried over and we shall see that the formal power series solution is summable in the newly defined sense. In section 6, as an illustration, we shall discuss an example and obtain monomial summability for it in a more elementary way.

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2 Examples of equations

In this section, we shall present some examples of systems or dynamical systems that can be brought, by a convenient change of variables, to the form we shall study throughout this work.

Example 2.1. Consider the singularly perturbed dynamical system:

$$\left\{ \begin{array}{rcl} \varepsilon \dot{x} &=& -x^p + y \\ \dot{y} &=& -x^q \, \phi(x) \end{array} \right. \tag{2}$$

where $\dot{}$ denotes the derivative with respect to t, $p,\ q \in \mathbb{N}^*$ and $\phi(x)$ is analytic near $x_0 = 0$ with $\phi(0) \neq 0$. Suppose $q + 1 \geq 2p$.

Let $v = \frac{-x^p+y}{\varepsilon}$. In the new variables (x, v), we obtain

$$\left\{ \begin{array}{rcl} \dot{x} &=& v \\ \varepsilon \dot{v} &=& -px^{p-1}(v + \frac{\phi(x)}{p} x^{q-p+1}) \end{array} \right. \tag{3}$$

We are only looking for solutions $x(t), v(t)$ such that $v(t)$ tends to 0 at least as fast as $x(t)$ when $t \to \infty$. So let $v = -\frac{\phi(x)}{p}\ x^{q-p+1}\ z$. In the variables (x, z), the above system has the form:

$$(4)\quad \left\{\begin{array}{rcl} \dot{x} & = & -\frac{\phi(x)}{p}\; x^{q-p+1}\; z \\ \varepsilon x^{q-p+1}\phi(x)\; \dot{z} & = & -px^q\phi(x)(z-1)+\varepsilon x^{2q-2p+1}\phi(x)z^2(\frac{(q-p+1)}{p}\phi(x)+\frac{x}{p}\phi'(x))\ . \end{array}\right.$$

We are less interested in the t-dependence than in the orbits in the (x,z)-plane, so we eliminate t and regard z as a function of x. (If $z = z(\varepsilon, x)$ with $z(0,0) \neq 0$ has been found and the t-dependence is wanted, it can be retrieved from the first equation in (4) integrating $\frac{dt}{dx} = -\frac{x^{p-q-1}}{z(\varepsilon,x)\phi(x)}$ and inverting the function thus obtained.) This leads to the equation

$$\varepsilon\ x^{2q-2p+2}\phi^2(x)z\frac{dz}{dx} = p^2x^q\phi(x)(z-1) - \varepsilon x^{2q-2p+1}\phi(x)z^2((q-p+1)\phi(x)+x\phi'(x))$$

As $\phi(0) \neq 0$, $1/\phi(x)$ is analytic in the neighbourhood of $x_0 = 0$. Dividing by $x^q\phi^2(x)$, we obtain:

$$\varepsilon\ x^{q-2p+2}z\frac{dz}{dx} = \frac{p^2}{\phi(x)}(z-1) - \varepsilon x^{q-2p+1}z^2((q-p+1)+x\frac{\phi'(x)}{\phi(x)})$$

Now, let $z = 1+\varepsilon U$ and $r := q-2p+1 \geq 0$. If $r \geq 1$, the equation becomes

$$\varepsilon\ x^{r+1}\frac{dU}{dx} = \frac{p^2}{(1+\varepsilon U)\phi(x)}\ U - x^r(1+\varepsilon U)((p+r)+x\frac{\phi'(x)}{\phi(x)})$$

and the linear part at the origin, i.e. $\frac{\partial f}{\partial U}(0,0,0) = \frac{p^2}{\phi(0)}$ is not zero. Therefore the equation satisfies the assumptions of our work. If $r = 0$, on the other hand, the equation is of a type that we do not consider in this work.

Example 2.2. Now consider the singularly perturbed dynamical system:

$$(5)\qquad \left\{\begin{array}{rcl} \varepsilon\dot{x} & = & y-(x^3/3-x) \\ \dot{y} & = & A-1-(x-1)^{k+1} \end{array}\right.$$

where $\dot{}$ denotes $\frac{d}{dt}$, k a nonnegative integer and $A \in \mathbb{R}$.
If $k=0$, we recognize the classical forced van der Pol equation. Let $X = x-1,\ V = \dot{X}$. In the variables (X,V), system (5) becomes:

$$(6)\qquad \left\{\begin{array}{rcl} \dot{X} & = & V \\ \varepsilon\dot{V} & = & -X(2+X)(V+\frac{X^k}{2+X})+A-1 \end{array}\right.$$

Let $V = -\frac{X^k}{2+X}Z$. In the variables (X,Z), after eliminating the t-dependence again, the system has the form:

$$\varepsilon\frac{X^{2k}}{(2+X)^2}Z\frac{dZ}{dX} = X^{k+1}(Z-1)+A-1-\varepsilon X^{2k-1}\ \frac{2k+(k-1)X}{(2+X)^3}Z^2$$

Letting $Z = 1+\varepsilon U$, the equation becomes

$$\varepsilon X^{2k}(1+\varepsilon U)\frac{dU}{dX} = X^{k+1}(2+X)^2U + \alpha(2+X)^2 - \frac{X^{2k-1}}{2+X}\Big(2k+(k-1)X\Big)(1+\varepsilon U)^2$$

where $\alpha = \frac{A-1}{\varepsilon}$. In the case of $\alpha = 0$, we obtain:

$$\varepsilon X^{k-1}\frac{dU}{dX} = \frac{(2+X)^2}{1+\varepsilon U}U - \frac{X^{k-2}}{2+X}\Big(2k+(k-1)X\Big)(1+\varepsilon U)$$

If $k \geq 3$, we have an equation of the type considered in the present work at $X = 0$. For $k = 1,\ 2$, it is of a different type, not studied here at $X = 0$. In the classical case $k = 0$ and also in the cases $k = 1, 2, 3$, it is again of the type studied here, even if $\alpha \neq 0$, but now at $X = \infty$.

Example 2.3. In the case $n = 2$, a last example is given by the system:

$$\left\{\begin{array}{rcl} \dot{x} & = & \varepsilon x^k \\ \dot{y}_1 & = & xy_1 - y_2 + x \\ \dot{y}_2 & = & y_1 + xy_2 \end{array}\right. \tag{7}$$

where $x, y_1, y_2 \in \mathbb{C}$ and $k \in \mathbb{N} \setminus \{0, 1\}$. Let $\mathbf{y} = \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$ and eliminate t again. Then, the system becomes

$$\varepsilon\ x^k \frac{d\mathbf{y}}{dx} = A(x)\ \mathbf{y} + \mathbf{b}(x)$$

where $A(x) = \begin{pmatrix} x & -1 \\ 1 & x \end{pmatrix}$ is invertible at the origin and $\mathbf{b}(x) = \begin{pmatrix} x \\ 0 \end{pmatrix}$.

3 Asymptotic expansions

3.1 Asymptotic expansions in one variable

We recall here some basic notions of asymptotic expansions and Gevrey asymptotic expansions; as this will be important for us, the values of the functions are allowed to be in any complex Banach space. For further details, see for instance [B2].

Let E be a complex Banach space, and $\hat{f}(x) = \sum a_n x^n \in E[[x]]$. A (open) sector in $\mathbb{C}$ is a set $V(a, b; r) = \{x \in \mathbb{C} \mid a < \arg x < b,\ 0 < |x| < r\}$. We will omit frequently a, b, r, and speak of a sector V. Let us denote $\mathcal{O}(V; E)$ the set of holomorphic functions in V with values in E. If $f \in \mathcal{O}(V; E)$, f is said to have $\hat{f}$ as an asymptotic expansion at the origin if for each proper subsector $V' = V'(a', b'; r')$ ($a < a' < b' < b$, $0 < r < r'$) and each $N \in \mathbb{N}$, there exists $C(V', N) > 0$ such that

$$\left\|f(x) - \sum_{n=0}^{N-1} a_n x^n\right\| \leq C(V', N) \cdot |x|^N \text{ in } V'.$$

The asymptotic expansion is s-Gevrey if, moreover, $C(V', N)$ can be chosen as $C(V', N) = C(V') \cdot A(V')^N \cdot N!^s$, with constants $C(V')$, $A(V')$ depending only on V'. We will write $f \sim \hat{f}$ and $f \sim_s \hat{f}$ in the s-Gevrey case, respectively. Observe that $f \sim_s \hat{f}$ implies that the formal series $\hat{f}$ is s-Gevrey, i.e. there exist $C, A > 0$ such that $|a_n| \leq CA^n n!^s$ for all $n \in \mathbb{N}$. The set of all such formal series will be denoted by $E[[x]]_s$.

Asymptotic expansions are unique, and respect algebraic operations and differentiation. The so called Borel-Ritt-Gevrey theorem and Watson's lemma are of great importance. The following result collects them.

Theorem 3.1. *Let $V = V(a, b; r)$, $\hat{f} \in E[[x]]_s$, and $s > 0$. Then:*

1. *If $b - a \leq s\pi$, there exists $f \in \mathcal{O}(V; E)$ such that $f \sim_s \hat{f}$.*

2. *If $f \in \mathcal{O}(V; E)$ is such that $f \sim_s 0$, then, for each proper subsector V' of V, there are positive constants such that*

$$||f(x)|| \leq C(V') \cdot \exp\left(-A(V')/|x|^{1/s}\right).$$

3. *If $b - a > s\pi$ and $f_1, f_2 \in \mathcal{O}(V; E)$ have $\hat{f}$ as their s-Gevrey asymptotic expansion, then $f_1 = f_2$.*

Because of the above theorem, a function $f \in \mathcal{O}(V; E)$ is uniquely determined by its s-Gevrey asymptotic expansion $\hat{f}$, provided that the opening of V is larger than $s\pi$. If such a function exists for a formal series $\hat{f}$, then it is said to be k-summable in V with $k = 1/s$ and f is called the k-sum of $\hat{f}$ on V. More precisely

Definition 3.2. Let $s > 0$, $k = 1/s$ and $\hat{f} \in E[[x]]_s$.

1. The formal series $\hat{f}$ is called k-summable on $V = V(a, b; r)$, if $b - a > s\pi$ and there exists a function $f \in \mathcal{O}(V; E)$ such that $f \sim_s \hat{f}$.

2. The formal series $\hat{f}$ is called k-summable in the direction $\theta \in \mathbb{R}$, if there exist $\delta, r > 0$ such that $\hat{f}$ is k-summable on the sector $V(\theta - s\frac{\pi}{2} - \delta, \theta + s\frac{\pi}{2} + \delta; r)$.

3. The formal series $\hat{f}$ is simply called k-summable, if it is k-summable in every direction $\theta \in \mathbb{R}$ with finitely many exceptions mod 2π.

The above notion of k-summability in a direction θ does not indicate how to obtain a sum from a given series; it can be shown to be equivalent to the following statement: Given $\hat{f}(x) = \sum a_n x^n$, its Borel transform $g(t) = \sum a_n t^n / \Gamma(1 + n/k)$ is analytic in a neighborhood of the origin, can be continued analytically in some infinite sector containing the ray $\arg t = \theta$, it has exponential growth there and hence the Laplace integral $f(x) = k\, x^{-k} \int_{\arg t = \tilde{\theta}} e^{-t^k/x^k}\ g(t)\, t^{k-1} dt$ defining the sum of $\hat{f}$ converges for x in a certain sector $V = V(\theta - \frac{\pi}{2k} - \delta, \theta + \frac{\pi}{2k} + \delta; r)$ and suitably chosen $\tilde{\theta}$ close to θ. It satisfies $f \sim_s \hat{f}$ on V.

We recall also the very useful characterization of functions having an s-Gevrey asymptotic expansion due to J.P. Ramis and Y. Sibuya ([Si3], [RaSi]).

Theorem 3.3. *Suppose that the sectors $V_j = V(a_j, b_j; r), 1 \leq j \leq m$, form a covering of the punctured disk $D(0; r)$. Given $f_j : V_j \to E$ bounded and analytic, assume that for every proper subsector V' of $V_{j_1} \cap V_{j_2}$ (if $V_{j_1} \cap V_{j_2} \neq \emptyset$) there is a constant $\gamma(V') > 0$ such that*

$$||f_{j_1}(x) - f_{j_2}(x)|| = O(\exp(-\gamma(V')/|x|^{1/s})) \tag{8}$$

for $x \in V'$. Then the functions f_j have common s-Gevrey asymptotic expansions.

Conversely, if a function $f : V \to E$ having an s-Gevrey asymptotic expansion is given, then a covering $V_j, 1 \leq j \leq m$ and functions $f_j : V_j \to E$ can be found that satisfy estimates like (8) and $f = f_1$, $V = V_1$.

Such a family $f_1, \ldots, f_m$ is sometimes called a *k-precise quasi-function.*

An important case that can be expressed in terms of asymptotic expansions in a Banach space is the following. Consider a holomorphic function $f \in \mathcal{O}(D \times V; E)$, D being an open subset in $\mathbb{C}^n$, V a sector in $\mathbb{C}$. If $K \subseteq D$ is a compact, the mapping $x \mapsto f(.,x)$, where $f(.,x)$ denotes the function mapping $\mathbf{y}$ to $f(\mathbf{y},x)$, induces a holomorphic function $\tilde{f}_K : V \longrightarrow \mathcal{C}(K; E)$ (space of continuous functions). If, for every such compact K, $\tilde{f}_K$ has an asymptotic expansion $\hat{f}_K$, f is said to have an asymptotic expansion in V, uniformly on each compact in D. The right hand sides of these asymptotic expansions are

$$\hat{f}_K(x) = \sum_{n=0}^{\infty} a_n|_K \cdot x^n$$

with certain $a_n \in \mathcal{O}(D; E)$, $n = 0, 1, \ldots$. Analogously, the notion of s-Gevrey uniform asymptotic expansion can be defined.

3.2 Asymptotic expansions in $x_1 x_2$

Let us briefly recall the well-known notion of a Gevrey series in x_1, x_2. Let $(E, \|\cdot\|)$ be a complex Banach space and

$$\hat{g}(x_1, x_2) = \sum_{n,m} a_{nm} x_1^n x_2^m \in E[[x_1, x_2]].$$

We say that $\hat{g}$ is a (s_1, s_2)-Gevrey series if one of the following equivalent conditions is satisfied:

1. The series $\sum \frac{a_{nm}}{n!^{s_1} m!^{s_2}} x_1^n x_2^m$ converges near $x_1 = x_2 = 0$.

2. The series $\sum \frac{a_{nm}}{\Gamma(1 + ns_1 + ms_2)} x_1^n x_2^m$ converges near $x_1 = x_2 = 0$.

Let $E[[x_1, x_2]]_{(s_1,s_2)}$ denote the set of (s_1, s_2)-Gevrey series. Some properties follow easily from the definition:

1. $E[[x_1, x_2]]_{(0,0)} = E\{x_1, x_2\}$ is the set of convergent series.

2. If $s_1 \leq s_1'$ and $s_2 \leq s_2'$ then $E[[x_1, x_2]]_{(s_1,s_2)} \subseteq E[[x_1, x_2]]_{(s_1',s_2')}$. Moreover the inclusion is strict if and only if $(s_1' + s_2') - (s_1 + s_2) > 0$.

3. $E[[x_1, x_2]]_{(0,s)}$ is the set of series that can be written as

$$\sum_{m=0}^{\infty} a_{*m}(x_1) x_2^m,$$

with $a_{*m} \in \mathcal{O}_b(D_R; E)$ for some $R > 0$, and

$$\sum_{m=0}^{\infty} ||a_{*m}|| \, x_2^m \in \mathbb{C}[[x_2]]_s,$$

where, for any $a \in \mathcal{O}_b(D_R; E)$, $||a|| := \sup\{||a(x_1)|| \mid x_1 \in D_R\}$. So, there is a natural isomorphism $E[[x_1, x_2]]_{(0,s)} \cong \bigcup_{R>0}(\mathcal{O}_R[[x_2]]_s)$, where $\mathcal{O}_R$ denotes the Banach space $\mathcal{O}_b(D_R; E)$, with the norm given by the supremum. This is an isomorphism of topological vector spaces in a somewhat more general framework than the Banach spaces we are dealing with (see [**R2**, **Mo**]).

Analogously, $E[[x_1, x_2]]_{(s,0)}$ is the set of series that can be written as $\sum_{m=0}^{\infty} a_{m*}(x_2) x_1^m$, with $a_{m*} \in \mathcal{O}_b(D_R; E)$ for some $R > 0$, and $\sum_{m=0}^{\infty} ||a_{m*}|| \, x_1^m \in \mathbb{C}[[x_1]]_s$,

4. $E[[x_1, x_2]]_{(s_1,s_2)} \cap E[[x_1, x_2]]_{(s_1',s_2')} \subseteq E[[x_1, x_2]]_{(s_1'',s_2'')}$ if $(s_1'', s_2'') \in \mathbb{R}^2$ is on the segment between (s_1, s_2) and (s_1', s_2').

In our work, the intersection of $E[[x_1, x_2]]_{(0,s)}$ and $E[[x_1, x_2]]_{(s,0)}$ will play an important role.

Definition/Proposition 3.4. *Consider a formal power series $\hat{g}(x_1, x_2) = \sum_{n,m} a_{nm} x_1^n x_2^m \in E[[x_1, x_2]]$ and $s > 0$. The following statements are equivalent:*

1. *The series $\hat{g}$ belongs to $E[[x_1, x_2]]_{(0,s)} \cap E[[x_1, x_2]]_{(s,0)}$ (and hence to all $E[[x_1, x_2]]_{(\sigma,s-\sigma)}$ for $0 \leq \sigma \leq s$).*

2. *There exist $M, R > 0$ such that $|a_{mn}| \leq M R^{-(n+m)} \cdot \min(n!, m!)^s$.*

3. *If we write $\hat{g}$ (uniquely) in powers of $x_1 x_2$ as follows:*

$$\hat{g}(x_1, x_2) = \sum_{k=0}^{\infty} (b_k(x_1) + c_k(x_2))(x_1 x_2)^k, \tag{9}$$

where $b_k(x_1) \in E[[x_1]]$, $c_k(x_2) \in E[[x_2]]$, $c_k(0) = 0$ then $b_k, c_k \in \mathcal{O}(D_R; E)$ for some $R > 0$ and for $0 < r < R$ the series

$$(T\hat{g})(t) = \sum_{k=0}^{\infty} g_k t^k, \quad g_k(x_1, x_2) = b_k(x_1) + c_k(x_2) \tag{10}$$

satisfies $(T\hat{g})(t) \in \mathcal{E}_r[[t]]_s$, where $\mathcal{E}_r$ denotes the Banach space of all sums g defined by $g(x_1, x_2) = b(x_1) + c(x_2)$ with $b, c \in \mathcal{O}_b(D_r; E)$ and $c(0) = 0$.

4. *If we write $\hat{g}$ (uniquely) as follows*

$$\hat{g}(x_1,x_2)=\sum_{n=0}^{\infty}x_1^n u_n(x_1x_2)+\sum_{n=1}^{\infty}x_2^n v_n(x_1x_2)\ ,$$

with $u_n, v_n \in E[[t]]$, then all u_n, v_n are s-Gevrey in t and for r sufficiently small the series $\sum_{n=0}^{\infty}\left|\left|\hat{\mathcal{B}}_s u_n\right|\right|_r x_1^n$ and $\sum_{n=1}^{\infty}\left|\left|\hat{\mathcal{B}}_s v_n\right|\right|_r x_2^n$ converge.
Here $\hat{\mathcal{B}}_s\left(\sum_{l=0}^{\infty}d_l t^l\right)=\sum_{l=0}^{\infty}d_l(l!)^{-s}\tau^l$ and $\left|\left|\sum_{l=0}^{\infty}d_l\tau^l\right|\right|_r=\sum_{l=0}^{\infty}|d_l|r^l$.

If $\hat{g}$ satisfies one (and hence all) of the above conditions, we say that $\hat{g}$ is s-Gevrey in the monomial x_1x_2. The set of all formal power series satisfying the above conditions will be denoted by $\mathcal{G}_s^{x_1x_2}$.

The proof of the equivalence of these conditions to condition 2. is straightforward (using Cauchy's estimates for the coefficients of a Taylor series etc.) and is left to the reader.

The isomorphism $T : E[[x_1,x_2]] \to (E[[x_1]]+x_2E[[x_2]])\,[[t]]$ defined in (10) will be used later again. Observe that according to the above proposition, T maps $\mathcal{G}_s^{x_1x_2}$ onto $\bigcup_{r>0}\mathcal{E}_r[[t]]_s$; here $\mathcal{E}_r[[t]]_s$ are considered as subsets of $(E[[x_1]]+x_2E[[x_2]])\,[[t]]$ in the canonical way.

As for the formal series, we want to define and discuss asymptotic expansions in the product x_1x_2.

We call "sector in x_1x_2" a set $\Pi=\Pi(a,b;R)\subseteq(\mathbb{C}\setminus\{0\})^2$,

$$\Pi=\{(x_1,x_2)\in\mathbb{C}^2 \mid a<\arg(x_1x_2)<b,\ 0<|x_1|<R,\ 0<|x_2|<R\}\ \ ;$$

here any convenient branch of arg may be used. This is the inverse image of $V(a,b;R^2)\times D(0,R)^2$ by the map

$$(x_1,x_2)\longmapsto(x_1x_2,x_1,x_2).$$

Using the map

$$\pi:\left\{\begin{array}{ccc}\mathbb{C}^2 & \longrightarrow & \mathbb{C}^2\\ (x_1,x_2) & \longmapsto & (x_1x_2,x_2),\end{array}\right.$$

we see that its image is

$$\pi(\Pi)=\{(t,x_2)\in(\mathbb{C}\setminus\{0\})^2 \mid a<\arg t<b,\ \tfrac{|t|}{R}<|x_2|<R\}.$$

A proper subsector in x_1x_2 is, naturally, given by $\tilde{\Pi}=\Pi(a',b';R')$ with $a<a'<b'<b$, $0<R'<R$.

Consider now a sector in x_1x_2, say $\Pi=\Pi(a,b;R)$, and $f\in\mathcal{O}(\Pi;E)$. The function $(t,x_2)\mapsto f\left(\frac{t}{x_2},x_2\right)$ is defined on $\pi(\Pi)$ and hence, for fixed t with $0<|t|<R^2$ and $a<\arg t<b$, the function $x_2\mapsto f(\frac{t}{x_2},x_2)$ is holomorphic and single valued in the annulus $\frac{|t|}{R}<|x_2|<R$. Thus we can write down its Laurent series

$$f\left(\frac{t}{x_2},x_2\right)=\sum_{n\in\mathbb{Z}}f_n(t)x_2^n\ ;$$

clearly $f_n\in\mathcal{O}(V(a,b;R^2);E)$. Under these conditions:

Proposition 3.5. *Suppose that* $|f(x_1,x_2)| \leq K(|x_1x_2|)$ *on some subsector* $\tilde{\Pi} = \Pi(a',b';R')$ *of* Π *where* $K:]0,R'^2] \to \mathbb{R}_+$*, and* $R' < R$*. Then, for* $t \in V(a',b';R'^2)$*:*

- *If* $n \in \mathbb{N}$*,* $|f_n(t)| \leq K(|t|)/R'^n$*.*
- *If* $n \geq 1$*,* $|f_{-n}(t)| \leq K(|t|)\,|t|^n/R'^n$*.*

Proof. If suffices to apply Cauchy estimates for the coefficients of a Laurent series. □

If f is bounded on some subsector $\tilde{\Pi}$, then the proposition allows to define $v_n(t) = f_n(t)$ for $n \geq 0$ and (in view of $x_2 = t/x_1$) $u_n(t) = f_{-n}(t)t^{-n}$. By abuse of notation, we set

$$Tf(t)(x_1,x_2) = \sum_{n=0}^{\infty} u_n(t)x_1^n + \sum_{n=1}^{\infty} v_n(t)x_2^n$$

for $t \in V(a',b';R'^2)$ and $|x_2|,|x_1| < R'$. Then, by construction, $Tf(x_1x_2)(x_1,x_2) = f(x_1,x_2)$ for $(x_1,x_2) \in \tilde{\Pi}$. For $t \in V(a',b';R'^2)$, $Tf(t)$ defines an element of the Banach space $\mathcal{E}_r$ introduced in Definition/Proposition 3.4 for any $r < R'$. This element will be denoted by $Tf(t)\,|_{\mathcal{E}_r}$. Clearly $Tf\,|_{\mathcal{E}_r}\colon V(a',b';R'^2) \to \mathcal{E}_r$ is holomorphic.

The above considerations suggest the definition for an asymptotic expansion in x_1x_2 appropriate for our purpose; in the second part we express it more directly in terms of x_1 and x_2.

Definition/Proposition 3.6. *Let* f *be a holomorphic function on* $\Pi = \Pi(a,b;R)$ *with values in a complex Banach space* E *and* $\hat{f} \in E[[x_1,x_2]]$*. We will say that* f *has* $\hat{f}$ *as asymptotic expansion at the origin in* x_1x_2 *if there exists* $0 < \tilde{R} \leq R$ *such that* $T\hat{f}(t) = \sum_{n=0}^{\infty} g_n t^n \in \mathcal{E}_{\tilde{R}}[[t]]$ *(cf. Definition/Proposition 3.4 for the definition of* $T\hat{f}$*) and one of the following equivalent conditions is satisfied:*

1. *For every* $r \in]0,\tilde{R}[$ *one has* $Tf(t)\,|_{\mathcal{E}_r} \sim T\hat{f}(t)\,|_{\mathcal{E}_r}$ *as* $V(a,b;r^2) \ni t \to 0$ *in the sense of section 3.1.*
2. *For every* $\tilde{\Pi} = \Pi(a',b';r)$ *subsector of* Π *with* $0 < r < \tilde{R}$ *and every* N*, there exists* $C(N,\tilde{\Pi})$ *such that for* $(x_1,x_2) \in \tilde{\Pi}$

$$\left\| f(x_1,x_2) - \sum_{n=0}^{N-1} g_n(x_1,x_2)(x_1x_2)^n \right\| \leq C(N,\tilde{\Pi}) \cdot |x_1x_2|^N \ .$$

Analogously, we define the notion of s*-Gevrey asymptotic expansion if* $T\hat{f} \in \mathcal{G}_s^{x_1x_2}$ *and* $Tf \sim_s T\hat{f}$ *or, equivalently,* $C(N,\tilde{\Pi})$ *can be chosen as* $K(\tilde{\Pi})A(\tilde{\Pi})^N N!^s$*, respectively.*

Proof. It suffices to prove that the second condition implies the first; the converse is trivial. Consider again $T\hat{f}(t)(x_1,x_2) = \sum_{n=0}^{\infty}(b_n(x_1) + c_n(x_2))t^n$, and write $b_n(x_1) = \sum_{k=0}^{\infty} b_{nk}x_1^k$, $c_n(x_2) = \sum_{k=1}^{\infty} c_{nk}x_2^k$ for $|x_1|,|x_2| < \tilde{R}$ and $g_n \in \mathcal{E}_{\tilde{R}}$, $g_n(x_1,x_2) = b_n(x_1) + c_n(x_2)$.

Put $A_N(t)=\sum_{n=0}^{N-1} g_n t^n$ so that $A_N(t)(x_1,x_2)=\sum_{n=0}^{N-1}(b_n(x_1)+c_n(x_2))t^n$. Using the coefficients of b_n, c_n, we rewrite A_N for $t\in V(a,b;\tilde{R}^2)$, $|x_1|,|x_2|<\tilde{R}$:

$$A_N(t)(x_1,x_2)=\sum_{k=0}^{\infty}P_{Nk}(t)x_1^k+\sum_{k=1}^{\infty}Q_{Nk}(t)x_2^k,\quad \text{where}$$
$$P_{Nk}(t)=\sum_{n=0}^{N-1}b_{nk}t^n,\quad Q_{Nk}(t)=\sum_{n=0}^{N-1}c_{nk}t^n\ .$$

By the hypothesis, for every subsector $\tilde{\Pi}=\Pi(a',b';r)$ of Π with $0<r<\tilde{R}$ and every $N\in\mathbb{N}$ there is a $C(N,\tilde{\Pi})>0$ such that for $(x_1,x_2)\in\tilde{\Pi}$

$$||f(x_1,x_2)-A_N(x_1x_2)(x_1,x_2)||\le C(N,\tilde{\Pi})\,|x_1x_2|^N\quad .$$

Applying proposition 3.5 to this inequality and using the Laurent series of $A_N(t)(\frac{t}{x_2},x_2)$ we obtain that

$$||f_k(t)-Q_{Nk}(t)||\le C_N\,|t|^N\,r^{-k}\ \text{and}\ \left|\left|f_{-k}(t)-t^kP_{Nk}(t)\right|\right|\le C_N\,|t|^{N+k}\,r^{-k}$$

for $k\ge 0$ resp. $k>0$ and $t\in V(a',b';r^2)$. This easily yields

$$||Tf(t)(x_1,x_2)-A_N(t)(x_1,x_2)||\le 2C_N\left(1-\tfrac{\tilde{r}}{r}\right)^{-1}\,|t|^N$$

for $t\in V(a',b';\tilde{r}^2)$, $|x_1|,|x_2|\le\tilde{r}$ and any $\tilde{r}<r$.

For the statement concerning s-Gevrey estimates, simply replace C_N by $C(\tilde{\Pi})\,A(\tilde{\Pi})^N\,N!^s$. □

In the sequel, we discuss some properties of asymptotic expansions in x_1x_2.

Proposition 3.7. *If $f\in\mathcal{O}(\Pi;E)$ has a s-Gevrey asymptotic expansion in x_1x_2 where $\hat{f}=0$, then, for all $\tilde{\Pi}=\Pi(a',b';R')$ ($a<a'<b'<b$, $0<R'<\tilde{R}$) there exist $C,\ B>0$ such that on $\tilde{\Pi}$*

$$||f(x_1,x_2)||\le C\cdot\exp\left(-\frac{B}{|x_1x_2|^{1/s}}\right)\ .$$

Proof. As in the classical case, we choose N close to the optimal value $(A|x_1x_2|)^{-1/s}$ in the definition of an s-Gevrey asymptotic expansion in x_1x_2. Stirling's formula yields the statement. □

Using the first condition in the definition of an s-Gevrey asymptotic expansion in x_1x_2 and using proposition 3.5 with $K(z)=\exp(-\gamma/z^{1/s})$, the theorem of Ramis-Sibuya (theorem 3.3) immediately implies

Theorem 3.8. *Suppose that the sectors $\Pi_j=\Pi(a_j,b_j;r), 1\le j\le m$ in x_1x_2, form a covering of $D(0,r)\times D(0,r)\setminus\{(0,0)\}$. Given $f_j:\Pi_j\to E$ bounded and analytic, assume that for every subsector Π' of $\Pi_{j_1}\cap\Pi_{j_2}$ (provided that $\Pi_{j_1}\cap\Pi_{j_2}\neq\emptyset$) there is a constant $\gamma(\Pi')>0$ such that*

$$||f_{j_1}(x_1,x_2)-f_{j_2}(x_1,x_2)||=O(\exp(-\gamma(\Pi')/|x_1x_2|^{1/s}))\tag{11}$$

for $(x_1, x_2) \in \Pi'$. Then the functions f_j have asymptotic expansions in x_1x_2 with a common right hand side and the expansions are s-Gevrey.

Conversely, if a function $f : \Pi \to E$ having an s-Gevrey asymptotic expansion in x_1x_2 is given, then a covering $\Pi_j, 1 \leq j \leq m$ and functions $f_j : \Pi_j \to E$ can be found that satisfy estimates like (11) and $f = f_1$.

An immediate consequence of the above theorem is the compatibility of Gevrey asymptotic expansions in a x_1, x_2 and the elementary operations: if Π is a sector in x_1, x_2 and $f, g : \Pi \to \mathbb{C}$ have s-Gevrey asymptotic expansions in x_1x_2, then so do $f+g$, $f \cdot g$, $\partial f/\partial x_1$ and $\partial f/\partial x_2$. This is not obvious from definition 3.6.

Before introducing and discussing summability in x_1x_2, we state a version of Watson's lemma for asymptotic expansions in x_1x_2.

Theorem 3.9. *Let $\Pi = \Pi(a, b; R)$ be a sector in x_1x_2, with $b - a > s\pi$, and suppose that $f \in \mathcal{O}(\Pi; E)$ has a $\hat{f} = 0$ as its s-Gevrey asymptotic expansion. Then $f \equiv 0$.*

Proof. By proposition 3.7, we have that, in a subsector $\tilde{\Pi}$, with opening larger than $s\pi$,

$$||f(x_1, x_2)|| \leq C \cdot \exp\left(-\frac{B}{|x_1x_2|^{1/s}}\right) .$$

Applying proposition 3.5 we obtain, by the classical Watson's lemma, that $f_n(t) \equiv 0$ for all integers n, so $f \equiv 0$, as desired. $\square$

We now introduce and discuss *summability in x_1x_2*.

Definition 3.10. Let $s > 0, k = 1/s$ and a formal series $\hat{f}(x_1, x_2) = \sum_{l,m=0}^{\infty} a_{l,m} x_1^l x_2^m$ be given.

1. We say that $\hat{f}$ *is k-summable in x_1x_2 on* $\Pi = \Pi(a, b; R)$ if $b - a > s\pi$ and there exists a holomorphic bounded function $f : \Pi \to E$ such that f has $\hat{f}$ as its s-Gevrey asymptotic expansion in x_1x_2 on Π in the sense of Definition/Proposition 3.6. Then f is called the s-sum of $\hat{f}$ on Π. If it exists, it is unique, by Theorem 3.9.

2. The formal series $\hat{f}$ is called k-summable in x_1x_2 in the direction $\theta \in \mathbb{R}$, if there exist $\delta, r > 0$ such that $\hat{f}$ is k-summable in x_1x_2 on the sector $\Pi(\theta - s\frac{\pi}{2} - \delta, \theta + s\frac{\pi}{2} + \delta; r)$ in x_1x_2.

3. The formal series $\hat{f}$ is simply called k-summable, if it is k-summable in every direction $\theta \in \mathbb{R}$ with finitely many exceptions mod 2π.

Observe that r in the definition of the k-summability in direction θ might depend upon θ and (in the case of k-summability) might tend to 0 as θ approches one of the finitely many exceptions (called *singular directions*). This is the case in our exemple in section 6.

The first condition in Definition/Proposition 3.6 shows that $\hat{f}$ is k-summable in x_1x_2 on $\Pi(a, b; R)$ if and only if the formal series $T\hat{f} = \sum_{n=0}^{\infty} d_n t^n$ with coefficients in $\mathcal{E}_r$ is k-summable on $V(a, b; r^2)$. Thus k-summability in x_1x_2 can be expressed as summability of certain formal series in one variable with coefficients in some Banach space. This allows to apply known theorems to k-summability in x_1x_2.

Laurent series expansion of $f(\frac{t}{x_2}, x_2)$ shows that for $\hat{f}(x_1,x_2) = \sum_{l,m=0}^{\infty} a_{l,m}x_1^l x_2^m$ k-summable with sum f on $\Pi(a,b;R)$, the series $\hat{u}_n(t) = \sum_{l=0}^{\infty} a_{l+n,l}t^l$ and $\hat{v}_n(t) = \sum_{l=0}^{\infty} a_{l,l+n}t^l$ are k-summable in $V(a,b;R^2)$ with sums we call $u_n(t)$,$v_n(t)$, say, and the series $\sum_{n=0}^{\infty} x_1^n u_n(t) + \sum_{n=1}^{\infty} x_2^n v_n(t)$ converge for all $|x_2|, |x_1| < R$ and have $Tf(t)(x_1,x_2)$ as their sum.

As for s-Gevrey asymptotic expansions, the sum, product and partial derivatives of funtions k-summable in x_1x_2 are k-summable, too.

It is a natural question whether asymptotics in x_1x_2 imply asymptotics in *one* variable while the other one is fixed. Here we have

Proposition 3.11. *Let $\Pi = \Pi(a,b;R)$ and $f \in \mathcal{O}(\Pi;E)$ be a function with $\hat{f} \in E[[x_1,x_2]]$ as an asymptotic expansion in x_1x_2 on Π. Then there exists $\tilde{R} \in]0,R]$ such that for every $x_{2,0}$ with $|x_{2,0}| < \tilde{R}$, the function $f_{x_{2,0}}(x_1) = f(x_1,x_{2,0})$ has an asymptotic expansion in x_1 on $V(a-\arg(x_{2,0}), b-\arg(x_{2,0}); R)$. If the asymptotic expansion of f is s-Gevrey then that of $f_{x_{2,0}}$ is s-Gevrey, too. If $\hat{f}$ is k-summable in x_1x_2 in some direction θ then $\hat{f}_{x_{2,0}}$ is k-summable in the direction $\theta - \arg(x_{2,0})$ for sufficiently small $|x_{2,0}|$.*

Proof. According to Definition/Proposition 3.6, there exists $\tilde{R} \in]0,R]$ such that $T\hat{f}(t)$ has coefficients having radii of convergence not smaller than $\tilde{R}$ and such that, for any strict subsector $\tilde{\Pi} = \Pi(a',b';r)$ of $\Pi(a,b;\tilde{R})$, there exist constants $C(N,\tilde{\Pi})$ such that on $\tilde{\Pi}$ we have

$$\left|\left|f(x_1,x_2) - \sum_{n=0}^{N-1}(b_n(x_1)+c_n(x_2))(x_1x_2)^n\right|\right| \le C(N,\tilde{\Pi}) \cdot |x_1x_2|^N .$$

Fixing any $x_{2,0}$ with $0 < |x_{2,0}| < r$ yields:

$$\left|\left|f(x_1,x_{2,0}) - f_{N,x_{2,0}}(x_1)\right|\right| \le C(N,\tilde{\Pi}) \cdot |x_{2,0}|^N \cdot |x_1|^N ,$$

for x_1 with $0 < |x_1| < r$, $a' - \arg(x_{2,0}) < \arg(x_1) < b' - \arg(x_{2,0})$; here

$$f_{N,x_{2,0}}(x_1) = \sum_{n=0}^{N-1}(b_n(x_1)+c_n(x_{2,0}))(x_1x_{2,0})^n \in E[[x_1]]$$

is a series of radius of convergence at least $\tilde{R}$. The above estimates are sufficient to conclude that $f_{x_{2,0}}$ has an asymptotic expansion as $x_1 \to 0$ in $V(a,b;R)$ and that its right hand side is $\hat{f}(x_1,x_{2,0}) = (T\hat{f})(x_1x_{2,0})(x_1,x_{2,0})$. The Gevrey character of such an asymptotic expansion is easily carried over using theorem 3.8. Using definition 3.10, this implies the statement on k-summability; here $|x_{2,0}|$ has to be smaller than r used there. ∎

All the definitions and results in this section can be extended to asymptotic expansions and summability in a monomial $x_1^p x_2^q$, p, q being positive integers. This will be done below for some statements only.

A formal series $\hat{g}(x_1,x_2) = \sum a_{nm}x_1^n x_2^m \in E[[x_1,x_2]]$ is s-Gevrey in $x_1^p x_2^q$ if and only if the following equivalent conditions are satisfied:

1. $\hat{g} \in E[[x_1, x_2]]_{(s/p,0)} \cap E[[x_1, x_2]]_{(0,s/q)}$, and hence,

$$\hat{g} \in \bigcap_{ps_1+qs_2=s} E[[x_1, x_2]]_{(s_1,s_2)}.$$

2. There exists $M,\ R > 0$ such that

$$||a_{mn}|| \leq M \cdot R^{-(m+n)} \cdot \min(n!^{1/p}, m!^{1/q})^s.$$

The operator T of Definition/Proposition 10 is modified in the following way: write $\hat{g}(x_1, x_2) = \sum_{k=0}^{\infty} g_k(x_1, x_2)(x_1^p x_2^q)^k$ according to the filtration of $\mathbb{C}[[x_1, x_2]]$ given by the sequence of ideals $\mathfrak{a}^k$, $\mathfrak{a} = (x_1^p x_2^q)$. Then $(T\hat{g})(t) := \sum_{k=0}^{\infty} g_k \cdot t^k$. As in the case $p = q = 1$, $\hat{g}$ is s-Gevrey in $x_1^p x_2^q$ if and only if the g_k converge in a common neighbourhood of the origin, say D_R, and for each $0 < r < R$, $(T\hat{g})(t) \in \mathcal{E}_r^{p,q}[[t]]$ is s-Gevrey, where $\mathcal{E}_r^{p,q}$ denotes the Banach space of all functions of the same form as the g_k ($\mathcal{E}_r^{1,1} = \mathcal{E}_r$) that are analytic and bounded on $|x|_1, |x|_2 < r$.

A "sector in $x_1^p x_2^q$" is a set $\Pi_{p,q}(a, b; R) \subseteq (\mathbb{C} \setminus \{0\})^2$,

$$\Pi_{p,q}(a, b; R) = \{(x_1, x_2) \in (\mathbb{C} \setminus \{0\})^2 \mid a < \arg(x_1^p x_2^q) < b,\ 0 < |x_1|^p, |x_2|^q < R\} \quad .$$

In order to define Tg, consider the following diagram:

$$\begin{tikzcd} \Pi_{p,q}(a,b;R) \arrow[r, "g"] \arrow[d, "\sigma"'] & \mathbb{C} \\ \Pi_{1,1}(a,b;R) & \end{tikzcd}$$

where $\sigma(x_1, x_2) = (x_1^p, x_2^q)$. We exploit the observation that $(x_1, x_2) \in \Pi_{p,q}(a, b; R)$ if and only if $(\xi x_1, \eta x_2)$ is in the same set, where ξ, η denote primitive p^{th} and q^{th} roots of unity, respectively. This permits to write

$$g(x_1, x_2) = \sum_{\substack{0\leq i<p \\ 0\leq j<q}} x_1^i x_2^j g_{ij}(x_1^p, x_2^q), \tag{12}$$

with functions g_{ij} analytic on $\Pi_{1,1}(a, b; R)$. In other words, g is written as a combination of functions that close the preceding diagram. In order to find these functions g_{ij}, it suffices to solve the van der Monde type system

$$g(\xi^m x_1, \eta^l x_2) = \sum_{\substack{0\leq i<p \\ 0\leq j<q}} \xi^{mi} \eta^{lj} x_1^i x_2^j g_{ij}(x_1^p, x_2^q),\ m = 0, \ldots, p-1,\ l = 0, \ldots, q-1.$$

The definition of Tg can now easily be carried over from the (old) definition (above 3.6) of the Tg_{ij}. Asymptotic expansions in the monomial $x_1^p x_2^q$ are then defined analogously to Definition/Proposition 3.6. Functions with vanishing s-Gevrey asymptotic expansion at the origin turn out to be $f \in \mathcal{O}(\Pi; E)$ such that

$$||f(x_1, x_2)|| \leq C \cdot \exp\left(-\frac{B}{|x_1^p x_2^q|^{1/s}}\right).$$

Formula (12), which also works for series, even allows to reduce all considerations for the monomial $x_1^p x_2^q$ to the simpler monomial $t_1 t_2$ discussed in detail in this section. The idea to go over to a simpler monomial by using a formula of the form (12) will be applied to the differential equation (S) in the next section.

Let us compare our notions with [**MR**, section 4]. There, the set $\hat{B}_\rho$ is defined as $\rho^*\mathbb{C}\{x_1, x_2\}[[t]]$, and it is independent of p, q. The set $\hat{B}_{\rho,s} = \rho^*\mathbb{C}\{x_1, x_2\}[[t]]_s$ is precisely our set of s-Gevrey series in $x_1^p x_2^q$. The functions with asymptotic expansion are introduced there using $\mathcal{C}^\infty$-Whitney functions. This can also be done with s-Gevrey asymptotic expansions (see [**Mo**]), so the notion introduced by Martinet and Ramis agrees with ours. More precisely, take an element $\hat{f} \in \mathbb{C}\{x_1, x_2\}[[t]]_s$, and apply Borel transform and truncated Laplace transform in t, obtaining a family of functions $\{f_i\}$ in a covering, having $\hat{f}$ as s-Gevrey asymptotic expansion at the origin. The differences are exponentially small of order $1/s$ in t, so, the differences between the functions $g_i(x_1, x_2) = f_i(x_1, x_2, x_1^p x_2^q)$ are exponentially small of order $1/s$ in $x_1^p x_2^q$. Applying theorem 3.8, we obtain in particular that $\hat{g}(x_1, x_2) = \hat{f}(x_1, x_2, x_1^p x_2^q)$ is s-Gevrey in $x_1^p x_2^q$ in the sense defined in Definition/Proposition 3.4.

In order to compare to [**BM**], observe first that proposition 3.11 can easily be generalized as follows: if f has an asymptotic expansion in $x_1 x_2$ on some sector Π in $x_1 x_2$, then, for fixed $x_{1,0}, x_{2,0}$ and nonnegative integers n_1, n_2 with $n_1 + n_2 > 0$, the function $g : t \mapsto f(t^{n_1} x_{1,0}, t^{n_2} x_{2,0})$ has an asymptotic expansion in a corresponding sector V; if the asymptotic expansion of f is s-Gevrey, then that of g is $s/(n_1+n_2)$-Gevrey. Thus, if a formal series $\hat{f}(x_1, x_2)$ is 1-summable, say, in $x_1 x_2$ in some direction, then the corresponding $\hat{g}(t)$ is $(n_1 + n_2)$-summable in a certain direction. This is equivalent to the $(s, 1-s)$-summability of $\hat{f}$ in the sense of [**BM**] for $s = n_1/(n_1+n_2)$; observe that [**BM**] allow also irrational s. The precise relations between $(s, 1-s)$-summability *for all* $s \in [0, 1]$ and summability in $x_1 x_2$ will be established in a future work.

4 Rank Reduction

As we said in the introduction, we are going to study systems of n differential equations

$$(S) \qquad x^{r+1}\varepsilon^\sigma \mathbf{y}' = f(x, \varepsilon, \mathbf{y}),$$

with $f(0, 0, \mathbf{0}) = \mathbf{0}$, whose linear part is invertible at the origin; it is given by a linear map $A = D_\mathbf{y} f(0, 0, \mathbf{0}) : \mathbb{C}^n \longrightarrow \mathbb{C}^n$. Here, $\mathbf{y}$ denotes $\mathbf{y}(x, \varepsilon)$, and the derivative $\mathbf{y}' = \mathbf{y}'(x, \varepsilon)$ means $\dfrac{\partial \mathbf{y}}{\partial x}(x, \varepsilon)$.

In this section we explain how to reduce this system to a simpler one, where $r = \sigma = 1$. Our method is an adaptation of the well known rank reduction (cf. [**T**], and [**L**] or [**B2**] for a modern version). It leads to $r = \sigma = 1$ and maintains the invertibility hypothesis, but increases the dimension of the system.

Formally, write f uniquely as

$$f(x, \varepsilon, \mathbf{y}) = \sum_{\substack{0 \le i \le r-1 \\ 0 \le j \le \sigma-1}} f_{ij}(x^r, \varepsilon^\sigma, \mathbf{y}) \cdot x^i \varepsilon^j,$$

and similarly the unknown $\mathbf{y}$:

$$(*) \qquad \mathbf{y}(x, \varepsilon) = \sum_{\substack{0 \le i \le r-1 \\ 0 \le j \le \sigma-1}} \mathbf{z}_{ij}(x^r, \varepsilon^\sigma) \cdot x^i \varepsilon^j.$$

Call $\eta = x^r$, $\mu = \varepsilon^\sigma$. We shall transform (S) in a new system of dimension $N = nr\sigma$, with unknowns $\mathbf{z}_{ij}(\eta, \mu)$. The linear part of the original equation being invertible, so will be the linear part of the new one. Inserting the preceding formulas into (S), we find

$$\sum_{\substack{0 \le i \le r-1 \\ 0 \le j \le \sigma-1}} r\eta^2 \mu \mathbf{z}'_{ij}(\eta, \mu) x^i \varepsilon^j + \sum_{i,j} i\eta\mu \mathbf{z}_{ij}(\eta, \mu) x^i \varepsilon^j =$$

$$= \sum_{\substack{0 \le i \le r-1 \\ 0 \le j \le \sigma-1}} f_{ij} \left(\eta, \mu, \sum_{\substack{0 \le k \le r-1 \\ 0 \le l \le \sigma-1}} \mathbf{z}_{kl}(\eta, \mu) x^k \varepsilon^l \right) x^i \varepsilon^j,$$

where $\mathbf{z}'_{ij}(\eta, \mu)$ now means $\dfrac{\partial \mathbf{z}_{ij}}{\partial \eta}(\eta, \mu)$.

The coefficient of $x^i \varepsilon^j$ in the last sum can be written as a polynomial in x, ε, identifying x^r with η and ε^σ with μ where they appear. We obtain

$$\sum_{\substack{0 \le i' \le r-1 \\ 0 \le j' \le \sigma-1}} F^{ij}_{i'j'}(\eta, \mu, \mathbf{z}) x^{i'} \varepsilon^{j'},$$

where $\mathbf{z}$ denotes the vector of all $\mathbf{z}_{ij}$. Observe that $F^{ij}_{i'j'}(0, 0, \mathbf{z})$ only depends upon $\mathbf{z}_{kl}$ with $k \le i'$, $l \le j'$. Identifying coefficients of $x^i \varepsilon^j$, we have the new equations

$$\begin{aligned} & r\eta^2 \mu \mathbf{z}'_{ij}(\eta, \mu) + i\eta\mu \mathbf{z}_{ij}(\eta, \mu) = \\ & = \sum_{\substack{0 \le k \le i \\ 0 \le l \le j}} F^{kl}_{i-k,j-l}(\eta, \mu, \mathbf{z}) + \sum_{\substack{i < k \le r-1 \\ 0 \le l \le j}} F^{kl}_{r+i-k,j-l}(\eta, \mu, \mathbf{z})\eta + \\ & + \sum_{\substack{0 \le k \le i \\ j < l \le \sigma-1}} F^{kl}_{i-k,\sigma+j-l}(\eta, \mu, \mathbf{z})\mu + \sum_{\substack{i < k \le r-1 \\ j < l \le \sigma-1}} F^{kl}_{r+i-k,\sigma+j-l}(\eta, \mu, \mathbf{z})\eta\mu. \end{aligned}$$

We want to show that its linearization at the origin, denoted by $L : \mathbb{C}^N \longrightarrow \mathbb{C}^N$, is invertible. Here, $\mathbb{C}^N$ is considered as a direct sum $\bigoplus_{ij} V_{ij}$, with coordinates $\mathbf{z}_{ij}$ in V_{ij}

($\dim_{\mathbb{C}} V_{ij} = n$). Call $L_{ij} := L\mid_{V_{ij}}$. In order to visualize the behaviour of L_{ij}, consider first the case $(i,j) = (0,0)$. The system becomes

$$r\eta^2\mu\mathbf{z}'_{00}(t,\mu) = G_{00}(\eta,\mu,\mathbf{z}),$$

where $G_{00}(0,0,\mathbf{z}) = F^{00}_{00}(0,0,\mathbf{z}) = f_{00}(0,0,\mathbf{z}_{00})$ and thus

$$L_{00}(\mathbf{z}_{00}) = D_{\mathbf{z}_{00}}f_{00}(0,0,\mathbf{0})(\mathbf{z}_{00}) = A(\mathbf{z}_{00}).$$

Now, order lexicographically the set of pairs (i,j) and consider some $(i,j) \neq (0,0)$. As above, it is easily seen that

$$L_{ij}(\mathbf{z}_{ij}) = A(\mathbf{z}_{ij}) + \tilde{L}_{ij}(\mathbf{z}),$$

where $\tilde{L}_{ij}(\mathbf{z})$ depends only upon $\mathbf{z}_{kl}$ with $(k,l) < (i,j)$. Therefore, the linear operator L is a lower block-triangular operator whose $r\sigma$ diagonal blocks are all equal to A. In particular, A being invertible, so is L.

In this new system of equations we have $r = \sigma = 1$. If $(\mathbf{z}_{ij}(\eta,\mu))$ is a vector solution of this system, solutions of the original one are obtained. Properties such as analyticity, formalness, Gevrey character or summability of the solutions can be translated from one system to another, as stated at the end of the preceding section.

So, in the sequel, we consider systems of N singularly perturbed differential equations

$$(RS) \qquad x^2\varepsilon\mathbf{y}' = f(x,\varepsilon,\mathbf{y}),$$

where, in order to simplify notation, we use again x, ε as variables instead of η, μ.

5 Gevrey character of the formal solution and summability

Consider the system (RS) with $f(0,0,\mathbf{0}) = \mathbf{0}$, and invertible linear part. Under these conditions, we shall see first that (RS) has a unique formal power series solution in the two variables x,ε. Such a formal power series solution $\hat{\mathbf{y}}(x,\varepsilon) \in \mathbb{C}[[x,\varepsilon]]^N$, if it exists, can be written as

$$\hat{\mathbf{y}}(x,\varepsilon) = \sum_{n=0}^{\infty} \mathbf{y}_{n*}(\varepsilon)x^n,$$

with $\mathbf{y}_{n*}(\varepsilon) \in \mathbb{C}[[\varepsilon]]^N$. Inserting $\hat{\mathbf{y}}(x,\varepsilon)$ in (RS) we obtain first that

$$\mathbf{0} = f(0,\varepsilon,\mathbf{y}_{0*}(\varepsilon)).$$

The invertibility of $A = \frac{\partial f}{\partial \mathbf{y}}(0,0,\mathbf{0})$ and the holomorphic implicit function theorem show that it has a unique solution $\mathbf{y}_{0*}(\varepsilon)$, which is holomorphic, with $\mathbf{y}_{0*}(0) = \mathbf{0}$ and defined in a neighbourhood of 0. By the change of variables $y = \tilde{y} + y_{0*}(\varepsilon)$, we can assume in the sequel without loss of generality that $f(0,\varepsilon,\mathbf{0}) = \mathbf{0}$ for all ε.

In order to obtain the remaining coefficients, we write

$$f(x,\varepsilon,\mathbf{y}) = \sum_{\substack{\mathbf{i}\in\mathbb{N}^n\\ j\in\mathbb{N}}} f_{\mathbf{i},j}(\varepsilon)x^j\mathbf{y}^{\mathbf{i}} \quad \text{where now} \quad f_{\mathbf{0},0}=0.$$

Inserting this series in the differential equation, comparison of the coefficient of x^{n+1} yields

$$\varepsilon n\mathbf{y}_{n*}(\varepsilon) = f_{0,n+1}(\varepsilon) + f_{\mathbf{1},0}(\varepsilon)\mathbf{y}_{n+1,*}(\varepsilon) + \text{known terms.}$$

As $f_{\mathbf{1},0}(0) = A$ is invertible, there exist a disk D_R in the ε-plane such that $f_{\mathbf{1},0}(\varepsilon)$ is invertible, so $\mathbf{y}_{n+1,*}$ can be defined as a holomorphic function $\mathbf{y}_{n+1,*}\in\mathcal{O}(D_R)^N$ (assuming that R is small enough so that $\mathbf{y}_{0,*}\in\mathcal{O}(D_R)^N$).

Proposition 5.1. *We have*

$$\hat{\mathbf{y}}(x,\varepsilon)\in\mathbb{C}[[x,\varepsilon]]^N_{(s,1-s)},$$

for all $s\in[0,1]$ and thus, $\hat{\mathbf{y}}$ is a 1-Gevrey series in $x\varepsilon$.

Proof. Similar computations as for the classical irregular singularities show that $\hat{\mathbf{y}}$ is a 1-Gevrey series with respect to x, i.e. $\hat{\mathbf{y}}(x,\varepsilon)\in\mathbb{C}[[x,\varepsilon]]^N_{(1,0)}$ (see [B2] for details). In order to study the Gevrey order with respect to ε, write analogously

$$\hat{\mathbf{y}}(x,\varepsilon) = \sum_{m\geq 0}\mathbf{y}_{*m}(x)\varepsilon^m,$$

with $\mathbf{y}_{*m}(x)\in\mathbb{C}[[x]]^N$. As above, it is easily seen that all the series $\mathbf{y}_{*m}(x)$ are convergent and have a common radius of convergence.

The use of Nagumo norms as in [S1] or [CDRSS] shows that, in fact, $\hat{\mathbf{y}}(x,\varepsilon)\in\mathbb{C}[[x,\varepsilon]]^N_{(0,1)}$. See also [BM] for some details of these computations. Thus by Definition/Proposition 3.4, we obtain that

$$\hat{\mathbf{y}}(x,\varepsilon)\in\mathbb{C}[[x,\varepsilon]]^N_{(s,1-s)},$$

for all $s\in[0,1]$ and that $\hat{\mathbf{y}}$ is a 1-Gevrey series in $x\varepsilon$. □

This result will also be obtained from stronger results that will be proved later on.

Consider again the system (RS), with $f(0,0,\mathbf{0})=\mathbf{0}$, $\dfrac{\partial f}{\partial x}(0,0,\mathbf{0}) = A_0$ invertible. If the formal solution is $\displaystyle\sum_{m=0}^{\infty}\mathbf{y}_{*m}(x)\varepsilon^m$, then we can make the change of variables

$$\mathbf{y} = \mathbf{y}_{*0}(x) + \varepsilon\mathbf{y}_{*1}(x) + \varepsilon\tilde{\mathbf{y}},$$

and obtain a prepared form of our equation:

$$x^2\varepsilon\frac{d\tilde{\mathbf{y}}}{dx} = A(x,\varepsilon)\tilde{\mathbf{y}} + \varepsilon G(x,\varepsilon,\tilde{\mathbf{y}}),$$

with $A(x,\varepsilon)$, $G(x,\varepsilon,\tilde{\mathbf{y}})$ holomorphic in a neighbourhood of the origin, and $A(0,0)=A_0$.

In this section our objective is to study the summability of the unique formal solution, with respect to $t=x\varepsilon$. So, as we did in the last section, take $t=x\varepsilon$ as a new variable, and let $\mathbf{z}(t,\varepsilon):=\tilde{\mathbf{y}}\left(\frac{t}{\varepsilon},\varepsilon\right)$, $a(t,\varepsilon)=A\left(\frac{t}{\varepsilon},\varepsilon\right)$, $g(t,\varepsilon,\mathbf{z})=G\left(\frac{t}{\varepsilon},\varepsilon,\mathbf{z}\right)$ with $a(t,\varepsilon)$ holomorphic in

$$\{(t,\varepsilon)\in(\mathbb{C}\setminus\{0\})^2 \mid 0<|t|<R^2,\ \tfrac{|t|}{R}<|\varepsilon|<R\},$$

and $g(t,\varepsilon,\mathbf{z})$ in

$$\{(t,\varepsilon,\mathbf{z})\in(\mathbb{C}\setminus\{0\})^2\times\mathbb{C}^N \mid 0<|t|<R^2,\ \tfrac{|t|}{R}<|\varepsilon|<R,\ |\mathbf{z}|<\rho\}.$$

The new equation is then

$$(E)\qquad t^2\frac{d\mathbf{z}}{dt}=a(t,\varepsilon)\mathbf{z}+\varepsilon g(t,\varepsilon,\mathbf{z}).$$

This looks like a regularly perturbed differential equation, but it is not, because the domain of the variable ε depends on t. The main objective of this section is to study the solutions of (E) in this domain.

We can suppose that $|a(t,\varepsilon)-A_0|<\mu$, and $|g(t,\varepsilon,\mathbf{z})|<M$ and that μ is as small as needed, if R, ρ are small enough. Let $\lambda_1,\ldots,\lambda_N$ be the eigenvalues of A_0 ($\lambda_i\neq 0$), $\theta_j=\arg\lambda_j$ ($j=1,\ldots,N$), ordered in such a way that $\theta_1\leq\ldots\leq\theta_N\leq\theta_1+2\pi=:\theta_{N+1}$. Fix any $j\in\{1,\ldots,N\}$ such that $\theta_j<\theta_{j+1}$, fix $\delta>0$ sufficiently small and consider $V_R=V(\theta_j-\frac{\pi}{2}+2\delta,\theta_{j+1}+\frac{\pi}{2}-2\delta;R^2)$. The choice of a small enough δ guarantees that V_R is a *large* sector.

Consider, for $l=1,\ldots,N$, the operators

$$\Lambda_l : y \quad\longmapsto\quad t^2y'-\lambda_l y.$$

Classical results on meromorphic ordinary differential equations show that Λ_l has a continuous right inverse $T_l:\mathcal{O}_b(V_R)\to\mathcal{O}_b(V_R)$ (Recall that $\mathcal{O}_b(V_R)$ denotes the Banach space of functions holomorphic and bounded on V_R) constructed by variation of constants:

$$T_lh(t)=\int_0^t e^{-\lambda_l\left(\frac{1}{t}-\frac{1}{\tau}\right)}\tau^{-2}h(\tau)\,d\tau\quad,$$

where integration is taken over suitable paths in V_R (depending upon l and t) combining arcs on which $\mathrm{Re}(\frac{1}{\tau}e^{i\theta_l})$ tends to $-\infty$ as $\tau\to 0$, more precisely $\arg(\frac{1}{\tau}-\frac{1}{\tau_0})\in\{-\theta_l-\frac{\pi}{2}-\delta,-\theta_l+\frac{\pi}{2}+\delta\}$ mod 2π for some τ_0, or (for t in the "shadow zones") on which $|\tau|$ is constant. The paths are best described in the u-plane, $u=1/\tau$ (cf figure 1; here we used $\theta_j=\pi/7,\theta_{j+1}=\pi$, $\delta=0.2$ and $l=j$).

There is a positive constant K such that for all t in V_R and all l, the above paths from 0 to t can be chosen such that the maximum of $\mathrm{Re}\left(\frac{\lambda_l}{\tau}-\frac{\lambda_l}{t}\right)$ on it is less than K. This implies the continuity of T_l. Unfortunately, the paths cannot be chosen such that additionally $|\tau|$ is increasing on them. For the sequel let $m(t)$ denote the maximum of $|\tau|$ on the paths of integration chosen for t and $l=1,\ldots,N$; we will need later that there exists a positive constant C such that for all $t\in V_R$, one has

$$(13)\qquad |t|\leq m(t)\leq\min(R^2,C\ |t|).$$

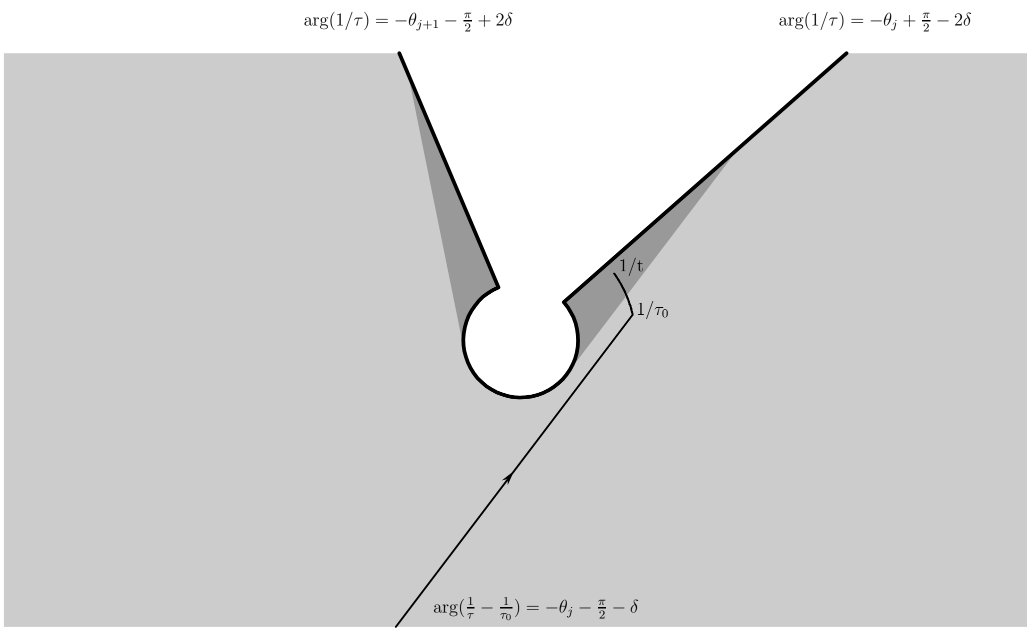

Figure 1: The image of the sector V_R (in light and dark grey) in the u-plane, $u = 1/\tau$, with "shadow zones" (dark gray) and a corresponding path of integration

Let $T : \mathcal{O}_b(V_R)^N \to \mathcal{O}_b(V_R)^N$ be the operator $(T_1, \ldots, T_N)$, and consider

$$\mathcal{F}(\mathbf{z}) := T(N_0\mathbf{z} + (a(t,\varepsilon) - A_0)\mathbf{z} + \varepsilon g(t, \varepsilon, \mathbf{z})),$$

where $N_0 = A_0 - \mathrm{diag}(\lambda_1, \ldots, \lambda_N)$ which can also be assumed to have a norm smaller than μ. Observe that, by the construction of the T_l, we cannot define $\mathcal{F}$ for all ε with $\frac{|t|}{R} < |\varepsilon| < R$; we need that all (τ, ε), τ on the paths chosen for t and T_l, are in the domain. Thus we define $\mathcal{F}$ on the set of all functions $\mathbf{z}(t,\varepsilon)$ that are holomorphic for $t \in V_R$ and ε with $\frac{m(t)}{R} < |\varepsilon| < R$ and that are bounded by ρ and its images are functions analytic and bounded on the same domain. If R, μ and ρ are small enough, $\mathcal{F}$ is clearly a contraction. Hence (E) admits a solution on the domain described above which contains the image $\pi(\Pi)$ of the sector $\Pi(\theta_j - \frac{\pi}{2} + 2\delta, \theta_{j+1} + \frac{\pi}{2} - 2\delta; \tilde{R})$ in $x\varepsilon$ for $\tilde{R} = R/C$, where C is the constant in (13). We omit the ˜ in the sequel.

Let now $\tilde{V}_R = V(\theta_j + \frac{\pi}{2} + 2\delta, \theta_{j+1} + \frac{3\pi}{2} - 2\delta; R^2)$. Both sectors V_R and $\tilde{V}_R$ cover the punctured disk and $V_R \cap \tilde{V}_R = V_R^+ \cup V_R^-$, where in V_R^+,

$$\arg t \in \left(\theta_j + \tfrac{\pi}{2} + 2\delta, \theta_{j+1} + \tfrac{\pi}{2} - 2\delta\right),$$

and in V_R^-,

$$\arg t \in \left(\theta_j - \tfrac{\pi}{2} + 2\delta, \theta_{j+1} - \tfrac{\pi}{2} - 2\delta\right).$$

As before, there is a solution $\tilde{\mathbf{z}}$ on $\pi(\tilde{\Pi}) = \{(t,\varepsilon) \mid t \in \tilde{V}_R,\ \frac{|t|}{R} < |\varepsilon| < R\}$, holomorphic and bounded if R is small enough. Our goal is to use the theorem of Ramis-Sibuya in order to study summability, so we must study the differences

$$\begin{aligned} d^+(t,\varepsilon) &= \tilde{\mathbf{z}}(t,\varepsilon) - \mathbf{z}(t,\varepsilon) \text{ on } V_R^+ \\ d^-(t,\varepsilon) &= \mathbf{z}(t,\varepsilon) - \tilde{\mathbf{z}}(t,\varepsilon) \text{ on } V_R^-. \end{aligned}$$

It is sufficient to treat $d := d^+$ on V_R^+ ; analogous considerations will apply to d^-. We have, then, on V_R^+,

$$\begin{aligned} t^2\tilde{\mathbf{z}}' &= a(t,\varepsilon)\tilde{\mathbf{z}} + \varepsilon g(t,\varepsilon,\tilde{\mathbf{z}}) \\ t^2\mathbf{z}' &= a(t,\varepsilon)\mathbf{z} + \varepsilon g(t,\varepsilon,\mathbf{z}). \end{aligned}$$

Subtracting these two equations we obtain

$$t^2 d' = a(t,\varepsilon)d + \varepsilon(g(t,\varepsilon,\tilde{\mathbf{z}}) - g(t,\varepsilon,\mathbf{z})).$$

If we write

$$g(t,\varepsilon,\tilde{\mathbf{z}}) - g(t,\varepsilon,\mathbf{z}) = h(t,\varepsilon,\mathbf{z},\tilde{\mathbf{z}})(\tilde{\mathbf{z}} - \mathbf{z})$$

with some analytic matrix valued function h, then we have a linear equation

$$(\mathcal{L}) \qquad t^2 d' = (a(t,\varepsilon) + \varepsilon h(t,\varepsilon,\mathbf{z}(t,\varepsilon),\tilde{\mathbf{z}}(t,\varepsilon))) \cdot d = \tilde{a}(t,\varepsilon)d,$$

and $|\tilde{a}(t,\varepsilon) - A_0| \leq \mu$ if the radius R is small enough. We want to show that d is exponentially small on V_R^+ as t tends to zero, uniformly for ε satisfying $\frac{|t|}{\tilde{R}} < |\varepsilon| < \tilde{R}$ for some positive $\tilde{R} < R$.

Observe that a scalar equation $t^2u' = \lambda u + h(t)$ with some function h holomorphic and bounded on V_R^+ has a *unique* bounded solution (given by variation of constants) if $\mathrm{Re}(\lambda e^{-i\theta}) < 0$ for some $\theta \in I_{V^+} := \left]\theta_j + \frac{\pi}{2} + \delta, \theta_{j+1} + \frac{\pi}{2} - \delta\right[$, whereas all solutions of $t^2u' = \lambda u$ are exponentially small on V_R^+ if $\mathrm{Re}(\lambda e^{-i\theta}) > 0$ for all $\theta \in I_{V^+}$. This suggests that we divide the set of eigenvalues of A_0 in two classes. Reordering the λ_l if necessary, we can assume

1. If $l = 1, \ldots, L$, then $\mathrm{Re}(\lambda_l e^{-i\theta}) > 0$ for every $\theta \in I_{V^+}$.
2. If $l = L+1, \ldots, N$, then there exists a $\theta \in I_{V^+}$ such that $\mathrm{Re}(\lambda_l e^{-i\theta}) < 0$.

In the cases that all eigenvalues fall into one of the above two classes, the subsequent considerations can be simplified. Without loss of generality, we can assume that A_0 is block diagonal $A_0 = \begin{pmatrix} A_{01} & 0 \\ 0 & A_{02} \end{pmatrix}$. The system $(\mathcal{L})$ is divided accordingly into blocks:

$$\begin{aligned} t^2 d_1' &= \tilde{a}_1(t,\varepsilon)d_1 + \tilde{a}_2(t,\varepsilon)d_2 \\ t^2 d_2' &= \tilde{a}_3(t,\varepsilon)d_1 + \tilde{a}_4(t,\varepsilon)d_2, \end{aligned}$$

where $\tilde{a}_1(t,\varepsilon) \to A_{01}$, $\tilde{a}_4(t,\varepsilon) \to A_{02}$, $\tilde{a}_2(t,\varepsilon) \to 0$, $\tilde{a}_3(t,\varepsilon) \to 0$, as t, ε tend to the origin. We are first looking for a matrix $p(t,\varepsilon)$ such that the change of variables $d_1 = \tilde{d}_1, d_2 = \tilde{d}_2 + p(t,\varepsilon)\tilde{d}_1$ block-triangularizes the system. The new system is

$$(14) \qquad \begin{aligned} t^2 \tilde{d}_1' &= \check{a}_1(t,\varepsilon)\tilde{d}_1 + \check{a}_2(t,\varepsilon)\tilde{d}_2 \\ t^2 \tilde{d}_2' &= \check{a}_3(t,\varepsilon)\tilde{d}_1 + \check{a}_4(t,\varepsilon)\tilde{d}_2, \end{aligned}$$

where $\check{a}_1 = \tilde{a}_1 + \tilde{a}_2\, p$, $\check{a}_2 = \tilde{a}_2$, $\check{a}_3 = \tilde{a}_3 - t^2 p' - p\,\tilde{a}_1 - p\,\tilde{a}_2\, p + \tilde{a}_4\, p$ and $\check{a}_4 = \tilde{a}_4 - p\,\tilde{a}_2$. As we want that $\check{a}_3 = 0$, we have to solve the matrix Riccati equation

$$t^2 p' = \tilde{a}_3(t,\varepsilon) + \tilde{a}_4(t,\varepsilon)\, p - p\,\tilde{a}_1(t,\varepsilon) - p\,\tilde{a}_2(t,\varepsilon)\, p \tag{15}$$

on V_R^+. Its linear part at $t=0$ is the linear mapping $P \mapsto A_{02}\, P - P\, A_{01}$ and has the eigenvalues $\lambda_l - \lambda_k$, $l = L+1, \ldots, N$; $k = 1, \ldots, L$. By assumption, for every such couple (l,k), there exists a direction $\theta \in I_{V^+}$ such that $\operatorname{Re}((\lambda_l - \lambda_k)e^{-i\theta}) < 0$. Hence the same method as above can be applied to (15) and this yields the existence of a holomorphic bounded matrix-valued function $p = p(t,\varepsilon)$ block-triangularizing the system (14). Thus it remains to prove that (14) with $\check{a}_3 = 0$ has only exponentially small solutions on V_R^+.

As above, Λ_l is invertible for $l > L$ and thus, maybe after a further reduction of R, the second equation has the unique solution $\tilde{d}_2 = 0$ and it remains to prove that $\tilde{d}_1$ satisfying

$$t^2 \tilde{d}_1'(t,\varepsilon) = \check{a}_1(t,\varepsilon)\tilde{d}_1(t,\varepsilon) \tag{16}$$

is exponentially small on V_R^+.

For that purpose, fix $\theta \in I_{V^+}$ and put

$$\Delta(s) := \left|\tilde{d}_1\left(\tfrac{e^{i\theta}}{s}, \varepsilon\right)\right|^2, \quad s > R^{-2},\ \frac{1}{sR} < |\varepsilon| < R,\ \theta \in I_{V^+}.$$

As $\lim_{(t,\varepsilon)\to 0} \check{a}_1(t,\varepsilon) = A_{01}$ and as there is some α satisfying

$$0 < \alpha < \min\{\operatorname{Re}(\lambda_l e^{-i\theta}) \mid l = 1, \ldots, L,\ \theta \in \bar{I}_{V^+}\},$$

we find

$$\Delta'(s) \;=\; -2\operatorname{Re}\left\langle \tilde{d}_1\left(e^{i\theta}s^{-1}, \varepsilon\right), e^{i\theta}s^{-2}\tilde{d}_1'\left(e^{i\theta}s^{-1}, \varepsilon\right)\right\rangle \le (-2\alpha + \mu_2)\Delta(s),$$

where $\mu_2 < 2\alpha$ if R is sufficiently small. With some positive β, this yields

$$\Delta(s + s\beta) \le e^{-(2\alpha-\mu_2)\beta s}\Delta(s) \le e^{-(2\alpha-\mu_2)\beta s}\left|\left|\tilde{d}_1\right|\right|_R^2,$$

where we use $||d||_R = \sup\{|d(t,\varepsilon)| \mid t \in V_R^+, \frac{|t|}{R} < |\varepsilon| < R\}$. Substituting $s = \sigma/(1+\beta)$ and using $\gamma = (2\alpha - \mu_2)\beta/(2+2\beta)$, we obtain $\Delta(\sigma) \le e^{-2\gamma\sigma}\left|\left|\tilde{d}_1\right|\right|_R^2$ for $\sigma > (1+\beta)R^{-2}$, $\frac{1+\beta}{\sigma R} < |\varepsilon| < R$. Taking the maximum over all $\theta \in I_{V^+}$, this yields with $\tilde{R} = R/(1+\beta)$

$$\sup_{\frac{|t|}{\tilde{R}} < |\varepsilon| < \tilde{R}} \left|\tilde{d}_1(t,\varepsilon)\right| \le e^{-\gamma/|t|}\left|\left|\tilde{d}_1\right|\right|_R \quad \text{for } 0 < |t| < \tilde{R}^2 \ .$$

The following theorem uses the preceding conclusions, and contains the main result of our work.

Theorem 5.2. *The unique formal solution of equation (RS) is 1-summable in $x\varepsilon$.*

Proof. With preceding notations, we have proved that the functions d^+ and d^- that were the differences between $\tilde{\mathbf{z}}$ and $\mathbf{z}$ in V^+, V^-, are exponentially small with respect to t, uniformly for ε with $\frac{|t|}{\tilde{R}} \leq |\varepsilon| \leq \tilde{R}$. Going back to the original equation (RS) using $\mathbf{y}(x,\varepsilon) = \mathbf{y}_{*0}(x) + \varepsilon\mathbf{y}_{*1}(x) + \varepsilon\mathbf{z}(x\varepsilon,\varepsilon)$ and $\tilde{\mathbf{y}}(x,\varepsilon) = \mathbf{y}_{*0}(x) + \varepsilon\mathbf{y}_{*1}(x) + \varepsilon\tilde{\mathbf{z}}(x\varepsilon,\varepsilon)$, we have, for every j and every $\delta > 0$, found $r > 0$ and solutions $\mathbf{y} : \Pi_1 \to \mathbb{C}^N, \tilde{\mathbf{y}} : \Pi_2 \to \mathbb{C}^N$ of (RS), where Π_1, Π_2 are the sectors $\Pi_1 = \Pi(\theta_j - \frac{\pi}{2} + \delta, \theta_{j+1} + \frac{\pi}{2} - \delta; r)$, $\Pi_2 = \Pi(\theta_j + \frac{\pi}{2} + \delta, \theta_{j+1} + 3\frac{\pi}{2} - \delta; r)$ in $x\varepsilon$. Their differences are $\varepsilon d^+(x\varepsilon,\varepsilon)$ on $\Pi(\theta_j + \frac{\pi}{2} + \delta, \theta_{j+1} + \frac{\pi}{2} - \delta; r)$ and $\varepsilon d^-(x\varepsilon,\varepsilon)$ on $\Pi(\theta_j - \frac{\pi}{2} + \delta, \theta_{j+1} - \frac{\pi}{2} - \delta; r)$ and are exponentially small in the sense that there exist constants C, B such that they are smaller that $C\,e^{-B/|x\varepsilon|}$. Thus theorem 3.8 can be applied and yields that $\mathbf{y}$ and $\tilde{\mathbf{y}}$ have 1-Gevrey asymptotic expansions in $x\varepsilon$ in the sense of Definition/Proposition 3.6. As the opening angles of the sectors are larger than π, we obtain by Definition 3.10, that they are 1-summable in $x\varepsilon$ in their sectors. This gives 1-summability of the formal solution in all directions in $]\theta_j, \theta_{j+1}[$. As $j \in \{1, \ldots, N\}$ was arbitrary, the theorem is proved. □

For use in future works we note

Corollary 5.3. *Consider a linear system of doubly singular differential equations* $x^{r+1}\varepsilon^\sigma\mathbf{y}' = A(x,\varepsilon)\mathbf{y}$*, with* $r, \sigma \geq 1$*, such that* $A(0,0)$ *is block diagonal* $A(0,0) = diag(A_1, A_2)$*, where* A_1*,* A_2 *have no eigenvalues in common. Then, the above system is formally equivalent to a (formal) block diagonal system* $x^{r+1}\varepsilon^\sigma\mathbf{y}' = B(x,\varepsilon)\mathbf{y}$*,* $B(x,\varepsilon) = diag(B_1(x,\varepsilon), B_2(x,\varepsilon))$ *with* $B_i(0,0) = A_i$ *(*$i = 1,2$*). Moreover,* B *and the equivalence are* $x^r\varepsilon^\sigma$*-summable.*

Proof. Analogous to the classical proof (cf. [W]), applying theorem 5.2 to conclude. □

6 An example of a monomial sum

In this section we shall develop a simple example in an elementary way. Consider the differential equation

$$\varepsilon x^2 y' = (1+x)y - \varepsilon x.$$

We know that it has a formal solution $\hat{y} = \hat{y}(x,\varepsilon)$, but the monomial summability is not easily seen directly from the coefficients. Let us make the change of variables $t = \varepsilon x$, that yields

$$t^2\frac{dy}{dt} = \left(1 + \frac{t}{\varepsilon}\right)y - t,$$

which, for *fixed* $\varepsilon \neq 0$ can be treated classically. The sum of the formal solution in the direction θ, rewritten in the variables x, ε is given by

$$y_\theta(\varepsilon, x) = \int_0^{\infty e^{i\theta}} e^{-s/(\varepsilon x)}(1-s)^{1/\varepsilon - 1}ds.$$

We want to write this as the 1-sum in $x\varepsilon$ of $\hat{y}$ despite the fraction $1/\varepsilon$ in the above formula. Thus, we introduce

$$z_\theta(\varepsilon, x, t) = \int_0^{\infty e^{i\theta}} \exp\left(\frac{1}{t}(-s + x\log(1-s))\right)\frac{1}{1-s}ds,$$

defined for $|\arg t-\theta|<\pi/2$, which has the property $y_\theta(\varepsilon,x)=z_\theta(\varepsilon,x,\varepsilon x)$ and does not contain x or ε in a denominator. This can be written as an ordinary Laplace integral making the change of variable $s-x\log(1-s)=\sigma$, and $s=\varphi(\sigma,x)$, that can be defined as the analytic continuation of the local inverse of the expression for σ. We find that

$$z_\theta(\varepsilon,x,t)=\int_0^{\infty e^{i\theta}} e^{-\sigma/t}\frac{\frac{\partial\varphi}{\partial\sigma}(\sigma,x)}{1-\varphi(\sigma,x)}d\sigma,$$

for any direction $\theta\neq 0$. As a Laplace integral, z_θ is a 1-sum of its asymptotic expansion $\hat{z}\in\mathbb{C}\{x\}[[t]]$. Observe that the x-neighborhood of 0, where the Laplace integral has this asymptotic expansion, reduces to 0 as θ approches the singular direction $\theta=0$, because $s\mapsto s-x\log(1-s)$ is not invertible near $s=1+x$.

To complete the example, we now discuss the difference

$$d(\varepsilon,x):=y_\delta(\varepsilon,x)-y_{2\pi-\delta}(\varepsilon,x),$$

for δ small, and defined for $\arg(x\varepsilon)\in\left]-\frac{\pi}{2}+\delta,\frac{\pi}{2}-\delta\right[$. The difference d must be a multiple of a solution of the homogenized original equation, i.e. $e^{-1/(\varepsilon x)}x^{1/\varepsilon}$. Applying Hankel's formula for the reciprocal Gamma function, we obtain

$$d(\varepsilon,x)=2\pi i\,e^{-1/(\varepsilon x)}(\varepsilon x)^{1/\varepsilon}\frac{1}{\Gamma\left(1-\frac{1}{\varepsilon}\right)}.$$

Observe that for some values of ε, namely $\varepsilon=\frac{1}{n}$, with integer $n\geq 1$, we have $d(\varepsilon,x)=0$, so there is no Stokes phenomenon.

This difference can be written as $d(\varepsilon,x)=D(\varepsilon,x,\varepsilon x)$, where

$$D(\varepsilon,x,t):=-2\pi i\varepsilon\,e^{-1/t}t^{x/t}\frac{1}{\Gamma(-x/t)}.$$

This is not exactly the form of the difference used throughout this work, but in this form computations are easier. Now, suppose that $|\arg(x/t)|>\eta>0$. Using Stirling's formula for the complex Gamma function, we obtain a bound

$$\left|\frac{t^{x/t}}{\Gamma(-x/t)}\right|\leq\exp(K\,|x|\log|x|\,/\,|t|),\ \ |x|\leq r,\ \ |\arg t|\leq\pi/2$$

with some positive contants r,K. From this it is easily seen that, if $|\arg t|\leq\pi/2-\delta$, and $|\varepsilon|$, $|x|$ are small enough, there is a bound

$$|D(\varepsilon,x,t)|\leq C\cdot\exp(-\alpha/\,|t|),$$

with a certain $\alpha>0$, as stated in the proof of our main result. If $|\arg(x/t)|$ is small, the inversion formula

$$\frac{1}{\Gamma(-x/t)}=\frac{\Gamma(1+x/t)\cdot\sin(-\pi x/t)}{\pi}$$

has to be applied first and the rest of the computations remain without changes.

7 Conclusion

We started with an equation (S), doubly singular, that is a singular perturbation of an irregular singularity. After a rank reduction, we can suppose $r = \sigma = 1$ (equation (RS)). Under the conditions of our work (invertibility of the linear part), we show that the unique formal solution $\hat{\mathbf{y}}(x, \varepsilon)$ of (RS) is $(s, 1-s)$-Gevrey for every $s \in [0, 1]$, and moreover, that it is 1-summable in $x\varepsilon$.

Returning to the original equation (S), let $\hat{\mathbf{y}}(x, \varepsilon)$ be the unique formal solution. It can be written as

$$\hat{\mathbf{y}}(x, \varepsilon) = \sum_{\substack{0 \le i \le r-1 \\ 0 \le j \le \sigma - 1}} \hat{\mathbf{y}}_{ij}(x^r, \varepsilon^\sigma) \cdot x^i \varepsilon^j,$$

and the above considerations show that $\hat{\mathbf{y}}_{ij}(u, \eta)$ are 1-summable in $u\eta$, hence $\hat{\mathbf{y}}_{ij}(x^r, \varepsilon^\sigma)$ are 1-summable in $x^r \varepsilon^\sigma$, and so is $\hat{\mathbf{y}}(x, \varepsilon)$ as a finite linear combination of them. The notion of k-summability (one variable case) is replaced here by the notion of summability with respect to a monomial (k-summability means summability with respect to x^k). This seems to be the appropriate setting in order to treat summability or multisummability problems in several variables, problems that will be treated in a future work.

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