Attractors and limit cycles

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Part I. ATTRACTORS

1 Maximal attractors and their Hausdorff dimension

There are various non-equivalent definitions of attractors of dynamical systems. The first (chronologically) and the simplest one is that of maximal attractors. In early 80's, Hausdorff dimension of maximal attractors was estimated from above for so-called k-contracting dynamical systems by Douadi, Osterle and others, see for example [6]. These ideas were applied to the attractors of evolutionary systems in the infinite-dimensional phase spaces, [6]; see also the monograph of Babin–Vishik and the reference therein.

    1. Definition of $A_{\max}$
    2. Lyapunov stability.
    3. k-contracting dynamical systems.
    4. Hausdorff dimension.
    5. Attractors of k-contracting systems.
    6. Navier-Stokes equation in dimension 2.
    7. Difference with dimension 3. Millennium problem.

2 Hyperbolic attractors and structural stability

In 1970, Ruelle and Takens defined strange attractors as those that are different from equilibrium points and periodic orbits. Groundbreaking example of the Smale-Williams solenoid started a new vision of attractors. For a short time there was a conjecture (Arnold, Smale) that these attractors are generic.

It became clears soon that this is not the fact. Yet this is an important class of attractors, very well understood. In particular, these attractors support a "physically meaningful" so called SRB measure. Hyperbolic attractors are structurally stable.

On the other hand, a problem "what may be said about attractors of generic dynamical systems?" is one of the the central questions in the theory of dynamical systems. Related conjectures by Palis will be presented in the next lecture.

1. Concept of the structural stability.
2. Anosov diffeomorphisms and their structural stability.
3. Solenoid.
4. Hyperbolic invariant sets.
5. SRB measure.
6. Necessary and sufficient condition for structural stability.
7. Palis conjecture.

3 Various definitions of attractors. Palis paradigm. Boundary preserving maps

Maximal attractors have a lot of excessive points. This was stressed by Milnor [11] who suggested a new definition of attractor. A little bit later other nonequivalent definitions of attractor were suggested by Ilyashenko [1]. One of the major problems in the field is: whether the noncoincidence of various types of attractors is generic? This problem may be considered both for diffeomorphisms of closed manifolds and of manifolds with boundary onto themselves. Some exotic properties of attractors only conjectured in the first class are proved in the second one.

1. Milnor attractor.
2. Statistical and minimal attractors.
3. Non-coincidence: is it generic?
4. Palis conjecture on the finitude of attractors.
5. A different world: boundary preserving map.
6. Lyapunov unstable (hairy) Milnor attractors.

4 Attractors of boundary preserving maps

Attractors mentioned in the title have various strange properties. For an open set in the space of diffeomorphisms they may be Lyapunov unstable, have intermingled basins of attraction and positive, but not full Lebesgue measure.

1. Relations between random and classical dynamical systems [3].
2. Attractors with intermingled basins (I. Kan example, [10], [7]).
3. Thick attractors.
4. Bony attractors.
5. Open problems.

5 Skew products and their perturbations

Theory of normal hyperbolicity (Fenichel-Hirsh-Pugh-Shub, [5]) describes persistence of smooth invariant manifolds under small perturbations. Some perturbations exhibit a counterintuitive property: they have an invariant foliation such that there exist a set of the full measure that intersects each fiber by a finite number of points. This effect is called "Fubini nightmare", see [12], [14], [13] . Yet the perturbed skew products may have invariant foliations whose fibers are Hölder continuous with respect to the parameter [4]. Together with so called "special ergodic theorems" and "Falkoner lemma" this allows sometimes to overcome the Fubini nightmare [8].

1. Normal hyperbolicity.
2. Persistence and smoothness of normally hyperbolic invariant manifolds, by Fenichel.
3. Perturbations of skew products: Hirsh-Pugh-Shub theory.
4. Fubini nightmare.
5. Hölder property of perturbed skew products.
6. Special ergodic theorems.
7. Falconer lemma.
8. Fubini regained .

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6 Center-focus problem and small amplitude limit cycles

This is a classical subject initiated by Poincaré and intensively developed until the end of 90's, [18], Sections 12, 13. The unsolved cyclicity problem for centers of the polynomial vector fields of degree higher than 2 is one of the simplified versions of the Hilbert's 16th problem.

1. Complexification of the Poincaré map.
2. Bautin ideal.
3. Index of the Bautin ideal.
4. Center-focus problem for polynomials.
5. Quadratic vector fields. Dulac's center conditions.
6. Radicality of the Dulac's ideal and small amplitude limit cycles.
7. Cyclicity problem for centers of polynomial vector fields
8. Mourtada theorem.

7 Non-accumulation Theorem for hyperbolic polycycles

This is one of the simplest results about nonaccumulation of limit cycles. It claims that limit cycles of an analytic vector field can not accumulate to a separatrix polygone whose vertexes are hyperbolic saddles, [18], Section 24.

1. Dulac's strategy.
2. Extendabilty and expendability of correspondence maps for hyperbolic saddles.
3. Almost regular germs.
4. Phragmen-Lindelof type theorems
5. Non-accumulation

8 Generation of limit cycles through the perturbation of Hamiltonian vector fields

Perturbation problem mentioned in the title is closely related to counting of zeros of Abelian integrals. Finding an upper bound for these zeros is called Infinitesimal Hilbert problem. A survey of results on this problem starting with late sixties up to nowadays will be given, [18], Section 25; [17], [15].

1. Poincaré - Pontryagin criterion.
2. Infinitesimal Hilbert problem.
3. Solution for quadatic vector fields.
4. Exactness theorem.
5. Freedom in the location of limit cycles for polynomial vector fields.
6. Solution of the Infinitesimal Hilbert problem.

9 Complex foliations

Complex foliations of projective plane have algebraic origin. Their topology is drastically different from that of real plane. Typical are foliations with dense leaves, countable number of non-simply connected leaves and with various rigidity properties [18], Section 28. Complex foliations of the affine planes have drastically different properties [16].

1. Extension of a polynomial vector fields to the complex plane.
2. Density of leaves.
3. Complex saddle connections.
4. Countable number of limit cycles.
5. Absolute rigidity.
6. Total rigidity.
7. Foliations on Stein manifolds.

10 Growth-and-Zeros theorem and applications

Growth-and-Zeros theorem estimates the number of zeros of a holomorphic function in a smaller domain K through the growth rate of this function from K to a larger domain U. This theorem is well adjusted to the study of the Poincaré map of a complexified polynomial different equation, because

the growth rate may be estimated through the right hand side. This approach provides some partial estimates for Abel and Lienard equations and for quadratic vector fields. Results of [19], [20], together with some results yet unpublished will be discussed.

1. Growth-and-Zeros theorem.
2. Applications to Abel, Lienard and generalized Lienard equations.
3. δ-tame limit cycles of quadratic vector fields (joint work with J. Llibre).
4. Perturbations of singular quadratic vector fields.

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