Asymptotique Gevrey

Mireille Canalis-Durand

January 21, 2010

Contents

Introduction

In this course, our purpose is to recall and precise basic definitions and principal results about Gevrey estimates and Gevrey asymptotic expansions. The course is inspired of many talks, papers and books of B. Candelpergher [Can89], W. Balser [Bal94], B. Malgrange [Mal95], J. Martinet and J.-P. Ramis [MR88], J.-P. Ramis [Ram93], J.-P. Ramis and R. Schäfke [RS96] (chap. 6, pp.362-366), Y. Sibuya [Sib90-1] and J.C. Tougeron [Tou89]. The readers interested in results about Gevrey asymptotic expansions will find more details with the references listed above.
In the first section, we define the Gevrey series, we present Gevrey asymptotic expansions, the Borel and truncated Laplace transforms and we relate Gevrey asymptotic expansions with the exponential precision of two approximate summation of Gevrey formal power series: the incomplete Laplace transform and a "least term cut-off". In the third section, we introduce the k-summability. In the final part, we shall apply the Gevrey asymptotic theory to a singularly perturbed differential equation.

1 Gevrey asymptotics theory

1.1 Gevrey series

Let $\varepsilon \in \mathbb{C}$. We consider $\mathbb{C}[[\varepsilon]]$ in order to study the asymptotic solutions of singularly perturbed differential equations (see section 3), the formal series being power series into the variable ε.

1.1.1 Definitions

Definition 1.1 *Let k, A two positive numbers. A formal power series $\widehat{a}(\varepsilon) = \sum_{m\geq 0} a_m\, \varepsilon^m \in \mathbb{C}[[\varepsilon]]$ is said to be* Gevrey of order $1/k$ and type A, *if there exist two nonnegative numbers C and α such that*

$$\forall m,\ m \geq 0,\quad |\, a_m \,| \leq C\ A^{m/k}\ \Gamma(\alpha + m/k) \tag{1.1}$$

Remark 1.1 *Let k and α two positive numbers, we have*[1]*:*
There exist $K_1 > 0$ and $K'_1 > 0$ such that, for all $m \in \mathbb{N}^*$:

$$K'_1\ (\alpha + m/k)^{\alpha+m/k-1/2}\ e^{-\alpha-m/k} \leq \Gamma(\alpha + m/k) \leq K_1\ (\alpha + m/k)^{\alpha+m/k-1/2}\ e^{-\alpha-m/k}$$

There exist $K_2 > 0$ and $K'_2 > 0$ such that, for all $m \in \mathbb{N}^*$:

$$K'_2\ m^{\alpha-1/(2k)-1/2}\ (1/k)^{m/k}\ (m!)^{1/k} \leq \Gamma(\alpha + m/k) \leq K_2\ m^{\alpha-1/(2k)-1/2}\ (1/k)^{m/k}\ (m!)^{1/k}$$

Proof: For $x > 0$, the Stirling formula

$$\Gamma(x) = \exp(-x)\ x^x\ (\frac{2\pi}{x})^{1/2}\ (1 + e(x))$$

where $e(x) \to 0$ as $x \to +\infty$ proves the lemma. □

[1] If $x > 0$, $\Gamma(x) = \int_0^\infty e^{-u} u^{x-1} du$ and $\Gamma(n) = (n-1)!$ for $n \in \mathbb{N}^*$.

Remark 1.2 *The property above is equivalent to: there exist $K > 0$, $\beta \geq 0$ such that*

$$\forall m,\ m > 0,\quad |\ a_m\ | \leq K\ m^{\beta}\ (\frac{A}{k})^{m/k}\ (m!)^{1/k} \tag{1.2}$$

So, the inequality (1.1) leads the inequality (1.2) with

$$K = C\ K_2, \beta = Max\{\alpha - 1/2 - 1/k, 0\}$$

Remark 1.3 *The type is linked to the radius of convergence of the series $\widehat{\mathcal{B}}(\widehat{a})$ where $\widehat{\mathcal{B}}$ is the formal Borel transform (see subsection 1.3).*

Remark 1.4 *In the particular case $k = 1$, the property (1.2) becomes: there exist $K > 0$ and $\beta \geq 0$ such that*

$$\forall m,\ m > 0 \quad |\ a_m\ | \leq K\ m^{\beta}\ A^m\ m!$$

Example 1.1 *The series $\sum_{m \geq 0} m!\ \varepsilon^m$ is Gevrey of order 1 and type 1.*
The series $\sum_{m \geq 0} 2^m\ m!\ \varepsilon^m$ is Gevrey of order 1 and type 2.
The series $\sum_{m \geq 0} \sqrt{m!}\ \varepsilon^m$ is Gevrey of order 1/2 and type 2.
The series $\sum_{m \geq 0} 2\ (m!)^3\ \varepsilon^m$ is Gevrey of order 3 and type 1/3.

Remark 1.5 *The original Gevrey order 2 later becomes Gevrey order 1: in the first definition given by E. Gevrey at the begining of 20th century [Gev18], the series $\sum_{m \geq 0} a_m\ x^m$ where $|\ a_m\ | \leq C\ A^m\ m!$ was called a Gevrey series of order 2 (because $a_m = \frac{f^m(0)}{m!}$ and $|\ f^m(0)\ | \leq C\ A^m\ (m!)^2$). Now, with the definition given by J.-P. Ramis and R. Schäfke [RS96], we say that this series is Gevrey of order 1.*

Definition 1.2 *We denote by $\mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$ the algebra of the formal series (in $\mathbb{C}[[\varepsilon]]$) Gevrey of order $1/k$ and type $A > 0$ and we denote[2] $\mathbb{C}[[\varepsilon]]_{\frac{1}{k}} = \cup_{A>0}\ \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$.*
For $k = +\infty$, we have $\mathbb{C}[[\varepsilon]]_{\frac{1}{k}} = \mathbb{C}\{\varepsilon\}$ ($\frac{1}{\infty} = 0$) the algebra of convergent series. These convergent series define holomorphic functions on a neighbourhood of the origine.

Remark 1.6 *1) Let $k, A > 0$. If $\widehat{a}(\varepsilon)$ is a Gevrey series of order $1/k$ and type A, then $\widehat{a}(\varepsilon)$ is a Gevrey series of order $1/k$ and type B, $B > A$: $\mathbb{C}[[\varepsilon]]_{\frac{1}{k},A} \subset \mathbb{C}[[\varepsilon]]_{\frac{1}{k},B}$.*
2) Let $0 < k_1 < k_2$ and $A > 0$. If $\widehat{a}(\varepsilon)$ is a Gevrey series of order $1/k_2$ and type A, then $\widehat{a}(\varepsilon)$ is a Gevrey series of order $1/k_1$ and type $A^{\frac{k_1}{k_2}}$.

So we have the new definition:

Definition 1.3 *The series $\widehat{a}(\varepsilon) = \sum_{m \geq 0} a_m\ \varepsilon^m$ is* Gevrey of order exactly $1/k$ *if it is Gevrey of order $1/k$ and there exists no $k' > k$ such that it is Gevrey of order $1/k'$.*

In this course, we need formal series whose coefficients are holomorphic functions in a variable x on a neighbourhood of $0 \in \mathbb{C}^n$, $n \geq 1$.

[2] This last set was denoted by $\mathbb{C}[[\varepsilon]]_k$ or $\mathbb{C}\{\varepsilon\}_{\frac{1}{k}}$ in some older papers.

Definition 1.4 *Let $k > 0$, $A > 0$ and let $(f_m(x))_{m\geq 0}$ a sequence of holomorphic functions on a domain $D \subset \mathbb{C}^n$. The formal series $\widehat{f}(x,\varepsilon) = \sum_{m\geq 0} f_m(x)\ \varepsilon^m \in \mathbb{C}\{x\}[[\varepsilon]]$ is* Gevrey of order $1/k$ and type A, uniformly in x, *if there exist $C > 0$ and $\alpha > 0$ such that*

$$\forall m,\ m \geq 0,\ \forall x \in D,\quad |\ f_m(x)\ | \leq C\ A^{m/k}\ \Gamma(\alpha + m/k) \tag{1.3}$$

Thus,

$$\|\ f_m\ \|_D \leq C\ A^{m/k}\ \Gamma(\alpha + m/k)$$

where $\|\ \|_D$ is the supremum of $\|\ f_m\ \|$ when $x \in D$.

We denote $\mathbb{C}\{x\}[[\varepsilon]]_{\frac{1}{k},A}$ the algebra of these formal series and $\mathbb{C}\{x\}[[\varepsilon]]_{\frac{1}{k}} = \cup_{A>0}\ \mathbb{C}\{x\}[[\varepsilon]]_{\frac{1}{k},A}$.

1.1.2 Properties of $\mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$

Proposition 1.1 $\left(\mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}, +, ., \times, '\right)$ *is a commutative differential sub-algebra*[3] *of $\mathbb{C}[[\varepsilon]]$.*

Proof: The operations + and . are stable. In order to prove the stability of the product of two Gevrey series of order $1/k$ and type A, we have to show that:
if $\widehat{a}(\varepsilon) = \sum_{m\geq 0} a_m\ \varepsilon^m \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$ and $\widehat{b}(\varepsilon) = \sum_{m\geq 0} b_m\ \varepsilon^m \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$ then

$$\widehat{a}(\varepsilon) \times \widehat{b}(\varepsilon) = \sum_{m\geq 0} (\sum_{p=0}^{m} a_p b_{m-p})\ \varepsilon^m \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$$

and the next lemma (due to G.N. Watson [Wat12]) proves the result.

Lemma 1.2 *Let $k > 0$ and $l \geq 2$, l integer. Then*

$$\sum_{p_1+\ldots+p_l=m} (p_1!)^{\frac{1}{k}} (p_2!)^{\frac{1}{k}} \ldots (p_l!)^{\frac{1}{k}} \leq \gamma(k)^{l-1} (m!)^{\frac{1}{k}} \quad \forall m \textit{ integer}$$

where $\gamma(k)$ is a positive real only depending on k.

□
Finally, we have to prove that the map

$$\begin{array}{rcl} d : \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A} & \longrightarrow & \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A} \\ \widehat{a}(\varepsilon) & \mapsto & \widehat{a}'(\varepsilon) \end{array}$$

is $\mathbb{C}$-linear and verifies $(\widehat{a}(\varepsilon) \times \widehat{b}(\varepsilon))' = \widehat{a}'(\varepsilon) \times \widehat{b}(\varepsilon) + \widehat{a}(\varepsilon) \times \widehat{b}'(\varepsilon)$.
It is true with the Stirling formula: let $\widehat{a}(\varepsilon) = \sum_{m\geq 0} a_m \varepsilon^m \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$.
Then $\widehat{a}'(\varepsilon) = \sum_{m\geq 0} (m+1)\ a_{m+1}\varepsilon^m$ and the Stirling formula implies that $\frac{\Gamma(1+\frac{m+1}{k})}{\Gamma(1+m/k)\ (m/k)^{\frac{1}{k}}} \longrightarrow 1$ as $m \longrightarrow +\infty$. Therefore, $\widehat{a}'(\varepsilon) \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$. □

Remark 1.7 *The proposition (1.1) is true if $k = +\infty$.*

---

[3] We denote by $'$ the derivative with respect to ε.

Proposition 1.3 *The series* $\sum_{m\geq 0} a_m\varepsilon^m \in \mathbb{C}[[\varepsilon]]$ *is Gevrey of order* $1/k$ *and type* A *if and only if* $\sum_{m\geq 0} a_m\varepsilon^{pm}$ *is Gevrey of order* $\frac{1}{pk}$ *with the same type* A *for* $p > 0$, p *integer.*

Proof: It is straightforward with the definition (1.1). □

Example 1.2 *The Euler series* $\sum_{m\geq 0}(-1)^m m!\ x^{m+1} \in \mathbb{C}[[x]]$ *is the formal solution of the Euler equation*

$$x^2\ y' + y = x, \quad y(0) = 0$$

where $x = 0$ *is an irregular singular point*[4]. *This series is Gevrey of order* 1 *and type* 1. *Therefore the Leroy series* $\sum_{m\geq 0}(-1)^m m!\ x^{2(m+1)}$ *that is a formal solution of the Leroy equation*

$$\frac{x^3}{2}\ y' + y = x^2, \quad y(0) = 0$$

is Gevrey of order $1/2$ *and type* 1.

Proposition 1.4 *Let* $\Phi(u,v)$ *an analytic function in the neighbourhood of* $0 \in \mathbb{C}^2$ *and let* $\widehat{u}$, $\widehat{v} \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$ *such that* $\widehat{u}(0) = 0$, $\widehat{v}(0) = 0$. *Show that* $\Phi(\widehat{u},\widehat{v}) \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$.

Proof: See exercises I.
Other properties, as an implicit functions theorem in $\mathbb{C}[[\varepsilon]]_{\frac{1}{k}}$, are treated in [Mal95, Sib90-1].

1.1.3 Formal Borel transform and Formal Laplace transform

This paragraphe is widely inspired by [LR95] and [Ram93]. We introduce two transformations, the *formal Borel transform* ([Bor99]) that allows us to recognize Gevrey series and its inverse: the *formal Laplace transform.*

Formal Borel transform

Definition 1.5 *Let* $\widehat{a}(\varepsilon) = \sum_{m\geq 0} a_m\varepsilon^m$ *an entire series with a radius of convergence equal to* $R \geq 0$. *We call* formal Borel transform of order $1/k$ *of* $\widehat{a}$ *, the series, denotes by* $\widehat{\mathcal{B}}_k(\widehat{a})$

$$\widehat{\mathcal{B}}_k(\widehat{a})(\lambda) = a_0\delta + \sum_{m\geq 0} \frac{a_{m+1}}{\Gamma(1+m/k)}\lambda^m$$

where δ *is the Dirac measure at* 0.

Remark 1.8 *In the particular case where* $k = 1$,

$$\widehat{\mathcal{B}}_1(\widehat{a})(\lambda) = a_0\delta + \sum_{m\geq 0} \frac{a_{m+1}}{m!}\lambda^m$$

Remark 1.9 *Let* $m \geq 0$. *We replace* ε^{m+1} *by* $\frac{\lambda^m}{\Gamma(1+m/k)}$ *in order to obtain the formal Borel transform of order* $1/k$ *of a series.*

[4] See the algebraic differential equations [Mal91].

Hypothesis: Now we suppose $a_0 = 0$ (that is to say we study $(\widehat{a}(\varepsilon) - a_0)$).
The formal Borel transform $\widehat{\mathcal{B}}_k$ link Gevrey series of order $1/k$ and convergent series:

Proposition 1.5 *Let $\widehat{a}(\varepsilon) = \sum_{m\geq 0} a_m\varepsilon^m$ an entire series with a radius of convergence $R \geq 0$. Its formal Borel transform of order $1/k$, $\widehat{\mathcal{B}}_k(\widehat{a})(\lambda)$ is a convergent series in the λ-plane if and only if $\widehat{a}(\varepsilon)$ is a Gevrey series of order $1/k$.*

Proof: We suppose that the series $\widehat{\mathcal{B}}_k(\widehat{a})(\lambda)$ has a radius of convergence $r \neq 0$. Then, $\forall\, r_1,\ 0 < r_1 < r$, there exists B_{r_1} such that

$$\mid \frac{a_{m+1}}{\Gamma(1+m/k)} \mid \leq B_{r_1} {r_1}^{-m}$$

$$i.e. \quad \mid a_{m+1} \mid \leq B_{r_1} {r_1}^{-m}\, \Gamma(1+m/k) \leq \bar{B}_{r_1} A^{\frac{m+1}{k}}\, \Gamma(1+(m+1)/k)$$

where $A > r_1^{-k}$.
Conversely, if $\widehat{a}(\varepsilon)$ is a Gevrey series of order $1/k$ and type A, there exist $C > 0, \alpha > 0$ such that:

$$\mid a_m \mid \leq C A^{m/k} \Gamma(\alpha + m/k) \quad \textit{forall } m \geq 1$$

and $\mid \frac{a_{m+1}}{\Gamma(1+m/k)} \mid$ is upperbounded by $\frac{CA^{(m+1)/k}\Gamma(\alpha+(m+1)/k)}{\Gamma(1+m/k)}$.
Moreover $\frac{CA^{(m+1)/k}\Gamma(\alpha+(m+1)/k)}{\Gamma(1+m/k)} \leq \tilde{C}\rho^{-m}$ where $A\rho^k < 1$. So $\widehat{\mathcal{B}}_k(\widehat{a})(\lambda)$ is a convergent series with a radius of convergence $r \geq \rho$. $\square$

Remark 1.10 *Let $\widehat{a}$ in $\mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$. We can define another formal Borel transform such that $\widehat{\mathcal{B}}_k(\widehat{a})(\lambda)$ is a convergent series on the* closed *disc $\bar{D}_R(0)$ where $R = (1/A)^{1/k}$ (see Appendix A of the course).*

Remark 1.11 *If the series $\widehat{a}(\varepsilon)$ is a convergent series, the series $\widehat{\mathcal{B}}_1(\widehat{a})(\lambda)$ has a radius of convergence equal to $+\infty$.*

Properties of the Formal Borel transform. In the table 1.1, we have gathered the main properties of $\widehat{\mathcal{B}}_1$. Let

$$\widehat{a}(\varepsilon) = \sum_{m\geq 1} a_m\varepsilon^m \ \ \textit{and} \ \ \widehat{\mathcal{B}}_1(\widehat{a})(\lambda) = \sum_{m\geq 0} \frac{a_{m+1}}{m!}\lambda^m$$

$\widehat{a}(\varepsilon)$	$\widehat{\mathcal{B}}_1(\widehat{a})(\lambda)$
ε^{p+1}, (p integer)	$\frac{\lambda^p}{p!}$
1	δ (Dirac measure)
ε^α, ($\alpha \in \mathbb{C}$, $-\alpha$ non integer)	$\frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}$
$\varepsilon\ \widehat{a}(\varepsilon)$	$\int_0^\lambda \widehat{\mathcal{B}}_1(\widehat{a})(u)du$
$(\widehat{a} \times \widehat{b})(\varepsilon)$	$\widehat{\mathcal{B}}_1(\widehat{a}) * \widehat{\mathcal{B}}_1(\widehat{b})(\lambda) := \int_0^\lambda \widehat{\mathcal{B}}_1(\widehat{a})(u)\ \widehat{\mathcal{B}}_1(\widehat{b})(\lambda - u)du$
$\frac{\widehat{a}(\varepsilon)}{\varepsilon}$ where $\widehat{a}(\varepsilon) = \sum_{m\geq 1} a_m \varepsilon^m$	$a_1\delta + \frac{d}{d\lambda}[\widehat{\mathcal{B}}_1(\widehat{a})(\lambda)]$
$\varepsilon^2 \frac{d}{d\varepsilon}(\widehat{a}(\varepsilon))$	$\lambda\widehat{\mathcal{B}}_1(\widehat{a})(\lambda)$

table 1.1: Main properties of $\widehat{\mathcal{B}}_1$.

Formal Laplace transform. The formal Laplace transform is the inverse operator of the formal Borel transform.

Definition 1.6 *We call* formal Laplace transform *of order* $1/k$ *of a series* $\widehat{b}(\lambda) = \sum_{m\geq 0} b_m \lambda^m$, *the series denoted by* $\widehat{\mathcal{L}}_k(\widehat{b})$

$$\widehat{\mathcal{L}}_k(\widehat{b})(\varepsilon) = \sum_{m\geq 0} b_m \Gamma(1 + m/k)\ \varepsilon^{m+1}$$

Proposition 1.6 *Let* $\widehat{b}(\lambda) = \sum_{m\geq 0} b_m \lambda^m$ *and* $\widehat{a}(\varepsilon) = \sum_{m\geq 1} a_m \varepsilon^m$. *Then we have*

$$\widehat{\mathcal{B}}_k(\widehat{\mathcal{L}}_k)(\widehat{b}) = \widehat{b}$$

$$\widehat{\mathcal{L}}_k(\widehat{\mathcal{B}}_k)(\widehat{a}) = \widehat{a}$$

Proof: Let $\widehat{b}(\lambda) = \sum_{m\geq 0} b_m \lambda^m$

$$\widehat{\mathcal{B}}_k(\widehat{\mathcal{L}}_k)(\widehat{b})(\lambda) = \widehat{\mathcal{B}}_k(\sum_{m\geq 0} b_m \Gamma(1 + m/k)\ \varepsilon^{m+1}) = \sum_{m\geq 0} b_m \lambda^m$$

Let $\widehat{a}(\varepsilon) = \sum_{m\geq 1} a_m \varepsilon^m$

$$\widehat{\mathcal{L}}_k(\widehat{\mathcal{B}}_k)(\widehat{a})(\varepsilon) = \widehat{\mathcal{L}}_k(\sum_{m\geq 0} \frac{a_{m+1}}{\Gamma(1 + m/k)} \lambda^m) = \sum_{m\geq 0} a_{m+1} \varepsilon^{m+1}$$

□

The main properties of $\widehat{\mathcal{L}}_1$ are gathered in the table 1.2. Let

$$\widehat{b}(\lambda) = \sum_{m\geq 0} b_m \lambda^m \text{ and } \widehat{\mathcal{L}}_1(\widehat{b})(\varepsilon) = \sum_{m\geq 0} b_m m!\ \varepsilon^{m+1}$$

$\widehat{b}(\lambda)$	$\widehat{\mathcal{L}}_1(\widehat{b})(\varepsilon)$
1	ε
λ^p, (p entier)	$p!\ \varepsilon^{p+1}$
$\int_0^\lambda \widehat{b}(u)du = \sum_{m\geq 0} \frac{b_m}{(m+1)}\lambda^{m+1}$	$\varepsilon\ \widehat{\mathcal{L}}_1(\widehat{b})(\varepsilon)$
$\frac{d}{d\lambda}\widehat{b}(\lambda) = \sum_{m\geq 1} m\ b_m\ \lambda^{m-1}$	$\frac{1}{\varepsilon}\widehat{\mathcal{L}}_1(\widehat{b})(\varepsilon) - b_0$
$\lambda \times \widehat{b}(\lambda)$	$\varepsilon^2 \frac{d}{d\varepsilon}(\widehat{\mathcal{L}}_1(\widehat{b})(\varepsilon))$

table 1.2: Main properties of $\widehat{\mathcal{L}}_1$.

1.2 Gevrey asymptotic expansions

It is well known that if a complex-valued function $\phi(z)$ is holomorphic and bounded on a domain $0 <| z |< r$, where r is a positive number, then ϕ is *represented* by a convergent powers series in z.
We shall hereafter consider a similar but slightly more general situation with divergent powers series and functions holomorphic on some sectors.

1.2.1 History

The classical theory of Asymptotic Expansions, due to H. Poincaré [Poi81] is partially solving the problem of the representation of a divergent powers series.
H. Poincaré wanted to apply his theory to the analytic differential equations: his motivation was to give a sense to a divergent power series solution of a differential equation, i.e. to "represent" this formal solution in a true solution.
Let $\widehat{f}$ be a series. We can associate an analytic function f to this series ($\widehat{f}$ being the asymptotic expansion of f), but this function is not unique. To reduce this non-unicity, G. N. Watson [Wat12] and F. Nevanlinna [Nev19] introduced the concept of *Gevrey Asymptotic Expansion*. More recently, in the late 1970s, J.-P. Ramis reintroduced and developed systematically Gevrey asymptotic expansion in relation with analytic ordinary differential equations in the complex domain.

1.2.2 Definitions

The sectors We denote by $S_{r,\alpha,\beta}$, an open sector whose vertex is at the origin, $S_{r,\alpha,\beta} = \{\varepsilon /\ \alpha < arg\ \varepsilon < \beta,\ 0 <| \varepsilon |< r\}$. We denote by $S_{\alpha,\beta}$ if $r = +\infty$.

Definition 1.7 *Let* $S_{r,\alpha,\beta} = \{\varepsilon\ /\ 0 <| \varepsilon |< r,\ \alpha < arg\ \varepsilon < \beta\}$ *where* $r, \alpha, \beta > 0$ *an open sector on XXXX surface de Riemann du LogarithmeXXXX. A* subsector $S_{r',\alpha',\beta'}$ *of* $S_{r,\alpha,\beta}$ *is defined by* $S_{r',\alpha',\beta'} = \{\varepsilon\ /\ 0 <| \varepsilon |< r',\ \alpha' < arg\ \varepsilon < \beta'\}$ *where* $0 < r' < r,\ \alpha < \alpha' < \beta' < \beta$

and we denote $S_{r',\alpha',\beta'} \prec S_{r,\alpha,\beta}$. *Moreover, we denote by* $\mid S_{r,\alpha,\beta} \mid = \beta - \alpha$ *the opening of* $S_{r,\alpha,\beta}$.

Definition 1.8 *If* N *sectorial domains* $S_l = S_{r,\alpha_l,\beta_l}$ $(l = 1, ..., N)$ *satisfy the condition* $\bigcup_{l=1}^{N} S_l = \{\varepsilon,\ 0 <\mid \varepsilon \mid< r\}$, *we call* $\{S_1, ..., S_N\}$ a covering at $\varepsilon = 0$.

Definition 1.9 *More precisely,* $\{S_1, ..., S_N\}$ *is a* good covering at $\varepsilon = 0$ *if*
(i) $\alpha_l < \alpha_{l+1}$ *for* $l = 1...N$ *where* $\alpha_{N+1} = \alpha_1 + 2\pi$
(ii) $\beta_l - \alpha_l < \pi$ *for* $l = 1...N$
(iii) $S_l \cap S_{l+1} \neq \emptyset$ *for* $l = 1...N$ *and* $S_l \cap S_k = \emptyset$ *otherwise if* $l \notin \{k \pm 1, k\}$, *where* $S_{N+1} = S_1$.

Asymptotic expansion in the Poincaré sense

Definition 1.10 *Let* S *be an open sector of the complex plane whose vertex is at the origin. Let* $\widehat{f}(\varepsilon) = \sum_{m\geq 0} b_m\ \varepsilon^m \in \mathbb{C}[[\varepsilon]]$ *be a formal power series. Let* f *be a function analytic on the sector* S. *We will say that* f *is asymptotic to* $\widehat{f}(\varepsilon) = \sum_{m\geq 0} b_m\ \varepsilon^m$ *on the sector* S *in the Poincaré sense if for every closed subsector* S' *of* $S \cup \{0\}$ *and every positive integer* $N \in \mathbb{N}^*$, *there exists a positive constant* $C_{S',N}$ *such that:*

$$\forall\ \varepsilon \in S', \quad \mid f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m \mid \leq C_{S',N} \quad \mid \varepsilon \mid^N$$

Remark 1.12 *An* analytic *function on a open disk*[5] $D_r(0)$, $r > 0$ *has an asymptotic expansion in the Poincaré sense on this disk and the expansion is the Taylor series about* 0*:*
Let f *an analytic function on* $D_r(0)$*; there exists an entire series* $\sum_{m\geq 0} b_m\ \varepsilon^m$ *convergent in* $D_r(0)$ *such that,* $\forall \varepsilon \in D_r(0)$, $f(\varepsilon) = \sum_{m\geq 0} b_m\ \varepsilon^m$.
As $\sum_{m\geq 0} b_m\ \varepsilon^m$ *is the Taylor series of* f *about* $\varepsilon = 0$, *we have:*

$$\forall \varepsilon \in D_r(0), \quad f(\varepsilon) = \sum_{m=0}^{N-1} \frac{\varepsilon^m}{m!} f^{(m)}(0) + \frac{\varepsilon^N}{(N-1)!} \int_0^1 (1-t)^{N-1} f^{(N)}(\varepsilon t) dt$$

$$\text{and} \quad \mid f(\varepsilon) - \sum_{m=0}^{N-1} \frac{\varepsilon^m}{m!} f^{(m)}(0) \mid \leq \frac{\mid \varepsilon \mid^N}{(N-1)!}\ Sup_{|\eta| \leq |\varepsilon|} \mid f^{(N)}(\eta) \mid$$

□

We denote by $\mathcal{A}(S)$ the space of all functions analytic on S having an asymptotic expansion in the Poincaré sense as $\varepsilon \to 0$, ε in the open sector S.

Properties of $\mathcal{A}(S)$

Proposition 1.7 *Let* S *be an open sector whose vertex is at the origin. Then* $(\mathcal{A}(S),\ +,\ .,\ \varepsilon^2 \frac{d}{d\varepsilon})$ *is a* $\mathbb{C}$*-differential algebra.*

[5] Let r a positive real and $x_0 \in \mathbb{C}$. We denote by $D_r(x_0)$ the open disk of radius r and center x_0.

Remark 1.13 *Some functions, analytic on a sector S have no asymptotic expansion in the Poincaré sense as $\varepsilon \to 0$: let $S = \{\varepsilon \in \mathbb{C} \ / \ \mathrm{Re}\,(\varepsilon) > 0\}$ the open half-plane and let $f(\varepsilon) = \frac{\varepsilon Log(\varepsilon)}{1+\varepsilon^2}$; this function is analytic and bounded on $\mathrm{Re}\,(\varepsilon) > 0$, but it has no asymptotic expansion as $\varepsilon \to 0$ in the Poincaré sense.*

Proposition 1.8 *Let S be an open sector whose vertex is at the origin. The following conditions are equivalent:*
(i) $f \in \mathcal{A}(S)$
(ii) f is analytic on S and there exists a sequence $(b_m)_{m\in\mathbb{N}}$ such that

$$\forall\ S' \prec S, \quad lim_{\varepsilon\to 0,\ \varepsilon\in S'} f^{(m)}(\varepsilon) = m!\ b_m$$

Proof: Let $f \in \mathcal{A}(S)$ and let $S' \prec S$. We have

$$f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m = \varepsilon^{N-1}\varphi(\varepsilon) \ \ with\ \varphi(\varepsilon) \to 0 \ as\ \varepsilon \to 0,\ \varepsilon \in S'$$

It is clear that $\forall\ S' \prec S,\ lim_{\varepsilon\to 0,\ \varepsilon\in S'} f(\varepsilon) = b_0$.
For the derivatives:

$$f'(\varepsilon) - \sum_{m=1}^{N-1} m\ b_m \varepsilon^{m-1} = \varepsilon^{N-2}(\varepsilon\varphi'(\varepsilon) + (N-1)\varphi(\varepsilon))$$

We have $\varepsilon\varphi'(\varepsilon) \to 0\ as\ \varepsilon \to 0,\ \varepsilon \in S'$:
Let $S'' \prec S'$ and $\lambda > 0$ such that $\forall\ \varepsilon \in S'',\ D_{\lambda|\varepsilon|}(\varepsilon) \subset S'$.

$$\varphi(\varepsilon) = \frac{1}{2i\pi}\int_\gamma \frac{\varphi(s)}{s-\varepsilon}ds \ \ where\ \gamma = \partial D_{\lambda|\varepsilon|}(\varepsilon)$$

$$\varepsilon\varphi'(\varepsilon) = \frac{\varepsilon}{2i\pi}\int_\gamma \frac{\varphi(s)}{(s-\varepsilon)^2}ds$$

$$|\ \varepsilon\varphi'(\varepsilon)\ | \le \frac{|\ \varepsilon\ |}{2\pi}\int_\gamma \frac{|\ \varphi(s)\ |}{\lambda^2\ |\ \varepsilon\ |^2}ds \le \frac{1}{\lambda} Sup_{D_{\lambda|\varepsilon|}(\varepsilon)}\ |\ \varphi(s)\ |$$

and this supremum tends to 0 as $\varepsilon \to 0,\ \varepsilon \in S''$ because $D_{\lambda|\varepsilon|}(\varepsilon) \subset S'$.
For the sufficient condition, we consider the Taylor series of f between ε_0 and ε in $S' \prec S$, then $\varepsilon_0 \to 0$. So

$$|\ f(\varepsilon) - \sum_{m=0}^{N-1} b_m\varepsilon^m\ | \le \frac{|\ \varepsilon\ |^N}{N!}\ Sup_{\ t\in S'\cap D_\varepsilon(0)}\ |\ f^{(N)}(t)\ |$$

and the supremum is bounded in the neighbourhood of 0 because $\frac{f^{(N)}(t)}{N!} \to b_N,\ as\ t \to 0,\ t \in S'$. □

Remark 1.14 *So, if $f \in \mathcal{A}(S)$ then there exists a series $\widehat{f}(\varepsilon) = \sum_{m\geq 0} b_m\varepsilon^m$ which is the asymptotic expansion of f at $\varepsilon = 0$. We can consider the map J*

$$\begin{array}{rcl} J : \mathcal{A}(S) & \to & \mathbb{C}[[\varepsilon]] \\ f & \mapsto & \widehat{f}(\varepsilon) \end{array}$$

J is a homomorphism of commutative differential algebras over $\mathbb{C}$ [CL55, Was65].

Theorem 1.9 Borel-Ritt Theorem.
Let S be an open sector whose vertex is at the origin. Then J: $\mathcal{A}(S) \to \mathbb{C}[[\varepsilon]]$ is surjective.

Proof: (see [CL55, Mal95, Was65]).

Remark 1.15 *However, J is not injective even if the opening of S is greater than 2π. Effectively, on $S_{r,-2\pi,2\pi}$, $r > 0$, the functions 0 and $e^{-(1/\varepsilon)^{1/4}}$ have $0 + 0\varepsilon + 0\varepsilon^2 + \ldots$ as asymptotic expansion at $\varepsilon = 0$.*

Definition 1.11 *We call the set of functions* infinitely flat at the origin, *the set of the functions analytic on S that have $0 + 0\varepsilon + 0\varepsilon^2 + \ldots$ as asymptotic expansion at 0. We denote by $\mathcal{A}^{<0}(S)$ this set; it is the kernel of the homomorphism J.*

Gevrey Asymptotic expansions We consider a subset of $\mathcal{A}(S)$ which precises the constant $C_{S',N}$ in the upperbound:

$$| f(\varepsilon) - \sum_{m=0}^{N-1} a_m \varepsilon^m | \leq C_{S',N} \ | \varepsilon |^N$$

As for Gevrey series, we introduce the *asymptotic expansions with Gevrey estimates* where:

$$C_{S',N} \leq C_{S'} \ A^{N/k} \ \Gamma(\alpha + N/k)$$

where $A,\ \alpha,\ k > 0$. So we will consider a new map J, with $J(f) \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k}}$.

Definition 1.12 *Let S be an open sector whose vertex is at the origin. Let f an analytic function on S, let $\widehat{f}(\varepsilon) = \sum_{m \geq 0} b_m \varepsilon^m \in \mathbb{C}[[\varepsilon]]$ a formal series and let $A,\ k$ two positive reals. We will say that f* admits $\widehat{f}(\varepsilon)$ as asymptotic expansion of Gevrey order $1/k$ and type A as $\varepsilon \to 0$ on S *if there is a positive constant $\alpha > 0$ and for every closed subsector $S' \prec S$, there is a constant $C_{S'} > 0$ such that*

$$(1.4) \qquad \forall\ \varepsilon \in S',\ \forall N \in \mathbb{N}^*, \quad | f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m | \leq C_{S'} \ A^{N/k} \ \Gamma(\alpha + N/k) \ \ | \varepsilon |^N$$

We say also that $f(\varepsilon)$ is *Gevrey*-$1/k$ *asymptotic of type A to $\widehat{f}(\varepsilon)$ on S.*

Definition 1.13 *We denote by $\mathcal{A}_{\frac{1}{k},A}(S)$ the set of all functions admitting asymptotic expansions of Gevrey order $1/k$ and type A as $\varepsilon \to 0$, $\varepsilon \in S$ and $\mathcal{A}_{\frac{1}{k}}(S) = \cup_{A>0} \mathcal{A}_{\frac{1}{k},A}(S)$.*

Definition 1.14 *Let S be an open sector whose vertex is at the origin and let $D_r(0)$ a disk. Let $f(x, \varepsilon)$ an analytic function of x and ε, for x in $D_r(0)$ and for ε in S. Let $\widehat{f}(x, \varepsilon) = \sum_{m \geq 0} b_m(x) \varepsilon^m \in \mathbb{C}\{x\}[[\varepsilon]]$ a formal power series and let A and k two positive reals. We say that f* admits $\widehat{f}(x, \varepsilon)$ as asymptotic expansion of Gevrey order $1/k$ and type A as $\varepsilon \to 0$ on S, *uniformly for x in $D_r(0)$ if there exists $\alpha > 0$ such that for all closed subsector S' of S, there exists a constant $C_{S'} > 0$ such that $\forall\ \varepsilon \in S',\ \forall N \in \mathbb{N}^*$*

$$| f(x, \varepsilon) - \sum_{m=0}^{N-1} b_m(x) \varepsilon^m | \leq C_{S'} \ A^{N/k} \ \Gamma(\alpha + N/k) \ \ | \varepsilon |^N \ , \quad \forall\ x \in D_r(0)$$

1.2.3 Properties of $\mathcal{A}_{\frac{1}{k},A}(S)$; Examples

Properties of $\mathcal{A}_{\frac{1}{k},A}(S)$

Remark 1.16 *Let $k_1 > 0$ and let S be an open sector whose vertex is at the origin.*
i) If $f \in \mathcal{A}_{\frac{1}{k_1}}(S)$ then $f \in \mathcal{A}(S)$.
ii) Let k_2 such that $0 < k_1 < k_2$ and let $A > 0$. If $f \in \mathcal{A}_{\frac{1}{k_2},A}(S)$ then $f \in \mathcal{A}_{\frac{1}{k_1},A}(S)$ with the same asymptotic expansion.
iii) Let A and B such that $0 < A < B$. If $f \in \mathcal{A}_{\frac{1}{k_1},A}(S)$ then $f \in \mathcal{A}_{\frac{1}{k_1},B}(S)$ with the same asymptotic expansion.

Proposition 1.10 *Let $k > 0$ and let S be an open sector whose vertex is at the origin. Then $f \in \mathcal{A}_{\frac{1}{k}}(S)$ if and only if $f \in \mathcal{A}(S)$ and $\forall\ S' \prec S,\ \exists\ C'_S > 0,\ A,\ \alpha > 0$ such that*

$$\forall\ N \in \mathbb{N},\ \forall\ \varepsilon \in S',\quad \frac{\mid f^{(N)}(\varepsilon) \mid}{N!} \leq C_{S'}\ A^{N/k}\ \Gamma(\alpha + N/k)$$

Proof: If $f \in \mathcal{A}_{\frac{1}{k}}(S)$ then $f \in \mathcal{A}(S)$, so in particular

$$\forall\ S' \prec S,\ \forall\ N \in \mathbb{N},\ lim_{\varepsilon\to 0, \varepsilon \in S'} \frac{f^{(N)}(\varepsilon)}{N!} = b_N$$

Let $S'' \prec S' \prec S$ and let $\lambda > 0$ such that $\forall \varepsilon \in S'',\ D_{\lambda|\varepsilon|}(\varepsilon) \subset S'$.
Let $\phi(\varepsilon) := f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m$, then $\phi^{(N)}(\varepsilon) = f^{(N)}(\varepsilon)$.
The Cauchy's formula gives:

$$\frac{\phi^{(N)}(\varepsilon)}{N!} = \frac{1}{2i\pi} \int_\gamma \frac{\phi(t)}{(t-\varepsilon)^{N+1}} dt$$

where γ is the boundary of $D_{\lambda|\varepsilon|}(\varepsilon)$. As $f \in \mathcal{A}_{\frac{1}{k}}(S)$, there exist $C'_S > 0,\ A,\ \alpha > 0$ such that

$$\forall\ t \in \gamma,\quad \mid \phi(t) \mid \leq C_{S'}\ A^{N/k}\ \Gamma(\alpha + N/k) \mid t \mid^N$$

$$So\quad \mid \frac{f^{(N)}(\varepsilon)}{N!} \mid = \mid \frac{\phi^{(N)}(\varepsilon)}{N!} \mid \leq C_{S'}\ B^{N/k}\ \Gamma(\alpha + N/k)\ where\ B > A.$$

Conversely, if $f \in \mathcal{A}(S)$, f admits $\sum_{m\geq 0} a_m \varepsilon^m$ as asymptotic expansion on S, we have: $\forall S' \prec S$, $\exists\ C'_S > 0,\ A,\ \alpha > 0$ such that

$$\forall\ N \in \mathbb{N},\ lim_{\varepsilon\to 0, \varepsilon\in S'} \mid \frac{f^{(N)}(\varepsilon)}{N!} \mid = \mid b_N \mid \leq C_{S'} A^{N/k} \Gamma(\alpha + N/k)$$

Let ε_0 and ε in S'; we consider a Taylor expansion of f between ε_0 and ε on $S' \prec S$ then $\varepsilon_0 \to 0$. So

$$f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m = \int_0^\varepsilon \frac{(\varepsilon - t)^{N-1}}{(N-1)!}\ f^{(N)}(t) dt$$

Let $t = u\ \varepsilon = u \mid \varepsilon \mid e^{i\phi}$

$$f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m = \int_0^1 \mid \varepsilon \mid^N \ e^{iN\phi} \frac{(1-u)^{N-1}}{(N-1)!}\ f^{(N)}(u \mid \varepsilon \mid e^{i\phi}) du$$

$$\mid f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m \mid \leq \frac{\mid \varepsilon \mid^N}{N!}\ Sup_{\ \xi \in S' \cap D_\varepsilon(0)} \mid f^{(N)}(\xi) \mid \leq \mid \varepsilon \mid^N C_{S'} A^{N/k} \Gamma(\alpha + N/k)$$

□

Proposition 1.11 *Let $k, A > 0$ and let S be an open sector whose vertex is at the origin. If $f \in \mathcal{A}_{\frac{1}{k},A}(S)$ then its asymptotic expansion $\sum_{m\geq 0} b_m \varepsilon^m$ is Gevrey of order $1/k$ and same type A.*

Proof: We majorize $\mid b_N \mid$ when $N \geq 1$. We have

$$\mid f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m \mid \leq C_{S'} A^{N/k}\ \Gamma(\alpha + N/k)\ \ \mid \varepsilon \mid^N$$

$$and\ \ \mid f(\varepsilon) - \sum_{m=0}^{N} b_m \varepsilon^m \mid \leq C_{S'} A^{\frac{N+1}{k}}\ \Gamma(\alpha + \frac{N+1}{k})\ \ \mid \varepsilon \mid^{N+1}$$

we deduce

$$\mid b_N \mid \leq C_{S'} A^{N/k}\ \Gamma(\alpha + N/k)\ + C_{S'} A^{N/k}\ \Gamma(\alpha + N/k)\ \ \mid \varepsilon \mid$$

then we make ε tend to 0 in the inequality above and we obtain $\mid b_N \mid \leq C_{S'} A^{N/k}\ \Gamma(\alpha + N/k)$. □

From proposition (1.11), we conclude that $\mathcal{A}_{\frac{1}{k}}(S)$ is a sub-algebra of $\mathcal{A}(S)$ as a commutative differential algebra over $\mathbb{C}$. Moreover, with the proposition above, we can consider a restriction of J, called *canonical homomorphism* ([Tou89]) that we still denote by J

(1.5)
$$\begin{array}{rcl} J : \mathcal{A}_{\frac{1}{k},A}(S) & \longrightarrow & \mathbb{C}[[\epsilon]]_{\frac{1}{k},A} \\ f & \longmapsto & \widehat{f}(\varepsilon) = \sum_{m\geq 0} b_m \varepsilon^m \end{array}$$

This new map is still a homomorphism of commutative differential algebras on $\mathbb{C}$.

Examples

Example 1.3 *The Euler's series $\sum_{m\geq 0} (-1)^m\ m!\ \varepsilon^{m+1}$ is Gevrey of order 1 and type 1 and divergent. Moreover, the Euler series is the asymptotic expansion of the Gevrey order 1 of the holomorphic function $f(\varepsilon) = \int_0^\infty e^{-\frac{\lambda}{\varepsilon}}\ \frac{d\lambda}{1+\lambda}$ as $\varepsilon \to 0$, for ε in $S_{r,-\pi/2,\pi/2}$, where $r > 0$. Indeed*

$$\frac{1}{1+\lambda} = \sum_{m=0}^{N-1} (-1)^m \lambda^m + (-1)^N \frac{\lambda^N}{1+\lambda}$$

$$\int_0^\infty e^{-\frac{\lambda}{\varepsilon}}\ \frac{d\lambda}{1+\lambda} = \sum_{m=0}^{N-1} (-1)^m \int_0^\infty e^{-\frac{\lambda}{\varepsilon}} \lambda^m d\lambda + (-1)^N \int_0^\infty e^{-\frac{\lambda}{\varepsilon}} \frac{\lambda^N}{1+\lambda} d\lambda$$

$$or \int_0^{\infty} e^{-\frac{\lambda}{\varepsilon}} \lambda^m d\lambda = m! \ \varepsilon^{m+1} \ for \ \varepsilon \in S_{r,-\pi/2,\pi/2}$$

$$So \int_0^{\infty} e^{-\frac{\lambda}{\varepsilon}} \ \frac{d\lambda}{1+\lambda} - \sum_{m=0}^{N-1} (-1)^m \ m! \ \varepsilon^{m+1} = (-1)^N \int_0^{\infty} e^{-\frac{\lambda}{\varepsilon}} \frac{\lambda^N}{1+\lambda} d\lambda$$

$$and \ \left| \int_0^{\infty} e^{-\frac{\lambda}{\varepsilon}} \frac{\lambda^N}{1+\lambda} d\lambda \right| \leq N! \ \mid \varepsilon \mid^{N+1}$$

□

Example 1.4 *Let us consider* $f(\varepsilon) = \exp(-1/\varepsilon)$. *This function is not analytic in the neighbourhood of* 0*; However,* f *is analytic on the domain* $\{\varepsilon, \ \varepsilon \neq 0, \ \mathrm{Re}\,(\varepsilon) \geq 0\}$. *So in particular,* f *is analytic on every open sector whose vertex is at the origin, bissected by* $\mathbb{R}^+$ *and of opening* $< \pi$. *When* $\arg \varepsilon$ *increases,* $\mathrm{Re}\,(\varepsilon)$ *decreases and* $\mid f(\varepsilon) \mid$ *increases*[6] *and expand at the paramount if* $\mathrm{Re}\,(\varepsilon) < 0$.
Besides, this function admits the series $0 + 0\varepsilon + 0\varepsilon^2 + ...$ *as asymptotic expansion with Gevrey estimates of order* 1 *as* $\varepsilon \to 0$, $\mathrm{Re}\,(\varepsilon) > 0$.

1.3 Truncated Laplace transform

1.3.1 Borel transform.

Let S a sector whose opening is stricly greater than π, bissected by a direction $d_\phi = \{r \ e^{i\phi}, \ r > 0\}$, $\phi \in [0, 2\pi[$. We denote by S the open sector $S_{R,\phi-\theta/2,\phi+\theta/2}$ where $R > 0$, $\theta > \pi$.
We consider $a(\varepsilon)$ an holomorphic function on S.

Definition 1.15 *We call* Borel transform *of level* 1 *of* a*, the function*

$$\mathcal{B}_1(a)(\lambda) = \mathbf{a}(\lambda) = \frac{1}{2i\pi} \int_\gamma a(\varepsilon) \ \frac{e^{\lambda/\varepsilon}}{\varepsilon^2} d\varepsilon$$

where γ *is a LACET of* S *qui aboutit en* 0 *where* $\frac{e^{\lambda/\varepsilon}}{\varepsilon^2}$ *decreases rapidly.*

Proposition 1.12 *If* $a(\varepsilon) = \varepsilon^\alpha$, $\alpha > 0$*, then* $\mathcal{B}_1(a)(\lambda) = \frac{\lambda^{\alpha-1}}{\Gamma(\alpha)}$ *where*

$$\frac{1}{\Gamma(\alpha)} = \frac{\lambda^{1-\alpha}}{2i\pi} \int_\gamma \varepsilon^\alpha \ \frac{e^{\lambda/\varepsilon}}{\varepsilon^2} d\varepsilon \quad (Hankel's formula)$$

Proposition 1.13 *If* $\mid a(\varepsilon) \mid \leq C \mid \varepsilon \mid^\alpha$, $\alpha > 0$ *for* $\varepsilon \in S_{R,\phi-\theta/2,\phi+\theta/2}$*, where* $R > 0, \ \theta > \pi$*, then* $\mathbf{a}(\lambda) \leq K \ C \ \frac{|\lambda|^{\alpha-1}}{\Gamma(\alpha)} e^{|\lambda|/R}$ *on the sector* $V = S_{\phi-\theta'/2-\pi/2,\phi+\theta'/2+\pi/2}$ *where* $\theta > \theta' > \pi$.

Remark 1.17 *This inequality controls the behaviour of* $\mathbf{a}$ *near* 0 *and near to infinity on a sector bissected by* d_ϕ.

[6]Effectively, $\mid \exp(-\frac{1}{\varepsilon}) \mid = \exp(-\frac{\mathrm{Re}\,(\varepsilon)}{|\varepsilon|})$.

1.3.2 Laplace transform

Definition 1.16 *Let* $\mathbf{a}(\lambda)$ *an analytic function near* $d_\phi = \{r\ e^{i\phi},\ r > 0\}$, $\phi \in [0, 2\pi[$. *We call* Laplace transform *of level* k *of* $\mathbf{a}$, *the function*

$$\mathcal{L}_{\phi,k}(\mathbf{a})(\varepsilon) = k \int_{d_\phi} e^{-\lambda^k/\varepsilon^k}\ \mathbf{a}(\lambda)\ \frac{\lambda^{k-1}}{\varepsilon^{k-1}} d\lambda$$

Remark 1.18 *When* $\mathbf{a}$ *has an exponentially increasing of level at most* k *on* d_ϕ*: i.e. when there exists* $K, \gamma > 0$ *such that*

$$|\ \mathbf{a}(\lambda)\ | \leq K\ \exp(\gamma\ |\ \lambda\ |^k)\ \ \forall\ \lambda \in d_\phi$$

and if $\mathbf{a}$ *is integrable in the neighbourhood of* 0*, then this Laplace transform defines an analytic function on the bounded domain* $\{\varepsilon\ /\gamma - \cos(k\ arg\ \varepsilon - k\ \phi)|\ \varepsilon\ |^{-k} < 0\}$ *(see subsection 2.6).*

In the particular case where $\mathbf{a}(\lambda) = \frac{\lambda^m}{\Gamma(1+m/k)}$, the Laplace transform is defined on $S = \{\varepsilon\ /\phi - \pi/2k \leq arg\ \varepsilon \leq \phi + \pi/2k\}$. Effectively, $e^{-\lambda^k/\varepsilon^k}$ must not expand at the paramount in the neighbourhood of $\varepsilon = 0$, so we must have $\mathcal{R}e(\lambda^k/\varepsilon^k) \geq 0$ i.e. $\varepsilon \in S$.

Lemma 1.14 *Let* $S = \{\varepsilon\ /\ \ |\ \phi - arg\ \varepsilon\ | \leq \pi/2k\}$*, then* $\forall \varepsilon \in S$*, we have*

$$\forall\ m \geq 0, \quad \mathcal{L}_{\phi,k}(\frac{\lambda^m}{\Gamma(1+m/k)}) = \varepsilon^{m+1}$$

This identity, easy to prove, will be useful. It shows that $\mathcal{L}_{\phi,k}$ is the "inverse" transform of the formal Borel transform $\widehat{\mathcal{B}}_k$: let d_ϕ be a direction, $\forall\ k > 0,\ \forall\ \varepsilon \in S$

$$\mathcal{L}_{\phi,k}(\widehat{\mathcal{B}}_k(\varepsilon^{m+1})) = \varepsilon^{m+1} \quad \forall m \geq 0$$

Proposition 1.15 *Let* $k = 1$ *and let* d_ϕ *be a direction. The transforms* $\mathcal{B}_1$ *and* $\mathcal{L}_1$ *in the direction* d_ϕ *are inverse one together:*

$$\mathcal{B}_1 \circ \mathcal{L}_1(\mathbf{f}) = \mathbf{f} \ \ \textit{and} \ \ \mathcal{L}_1 \circ \mathcal{B}_1(f) = f$$

The main properties of the Laplace transform for $k = 1$ are gathered in the table 1.3 where we denote $\mathcal{L}_{\phi,1}$ by $\mathcal{L}_\phi$.

$$\mathcal{L}_\phi(\mathbf{a})(\varepsilon) = \int_{d_\phi} e^{-\lambda/\varepsilon}\ \mathbf{a}(\lambda)\ d\lambda$$

$\mathbf{a}(\lambda)$	$\mathcal{L}_\phi(\mathbf{a})(\varepsilon)$
$\mathbf{a} * \mathbf{b}(\lambda)$	$\mathcal{L}_\phi(\mathbf{a})\ .\ \mathcal{L}_\phi(\mathbf{b})(\varepsilon)$
$\int_0^\lambda \mathbf{a}(u)du$	$\varepsilon\ \mathcal{L}_\phi(\mathbf{a})(\varepsilon)$
$(\frac{d}{d\lambda})\ \mathbf{a}(\lambda)$	$-\mathbf{a}(0) + \frac{1}{\varepsilon}\ \mathcal{L}_\phi(\mathbf{a})(\varepsilon)$
$\lambda\ \mathbf{a}(\lambda)$	$\varepsilon^2\ (\frac{d}{d\varepsilon})\mathcal{L}_\phi(\mathbf{a})(\varepsilon)$

table 1.3: Some properties of $\mathcal{L}_\phi$.

1.3.3 Truncated Laplace transform

If $\widehat{a}(\varepsilon)$ is a Gevrey series of order $1/k$, the series $\widehat{\mathcal{B}}_k(\widehat{a})(\lambda)$ is a convergent series and there exists $r > 0$ such that $\widehat{\mathcal{B}}_k(\widehat{a})$ defines a holomorphic function $\mathbf{a}(\lambda)$ on the neighbourhood[7] of $\bar{D}_r(0)$.
If we don't know the singularities of $\mathbf{a}$ in the λ-plane (called Borel plane)[8], we introduce the *truncated Laplace transform* of level k, denoted by $\mathcal{L}_{r,\phi,k}$:

$$\mathcal{L}_{r,\phi,k}(\mathbf{a})(\varepsilon) = k \int_{d_{\phi,r}} e^{-\lambda^k/\varepsilon^k} \ \mathbf{a}(\lambda) \ \frac{\lambda^{k-1}}{\varepsilon^{k-1}} d\lambda$$

where $d_{\phi,r}$ is the line-segment $[0, r] \subset d_\phi$.
We want that $\mathcal{L}_{r,\phi,k}(\mathbf{a})(\varepsilon)$ will be defined in the neighbourhood of $\varepsilon = 0$ so we must have $\phi - \frac{\pi}{2k} \leq arg\ \varepsilon \leq \phi + \frac{\pi}{2k}$ (see figure 1.1).
Now we consider $\mathcal{L}_{r,0,1}(\mathbf{a})(\varepsilon) = \int_0^r e^{-\lambda/\varepsilon} \ \mathbf{a}(\lambda) \ d\lambda$. Here $\phi = 0$ and $k = 1$; we denote by $\mathcal{L}_r$ this incomplete transform and $M_{\mathbf{a}}$ the supremum of $\mid \mathbf{a}(\lambda) \mid$ for $\lambda \in \bar{D}_r(0)$.

The classic properties of the Laplace transform remain for $\mathcal{L}_r$ with exponentially small corrections ([Ca91]).

Proposition 1.16 *Let $\varepsilon > 0, \ r > 0$.*

$$\mathcal{L}_r(1) = \varepsilon - \varepsilon e^{-r/\varepsilon}$$

Proof: We have

$$\mathcal{L}_r(1) = \int_0^r e^{-\lambda/\varepsilon} \ \ d\lambda = [-\varepsilon e^{-\lambda/\varepsilon}]_0^r$$

□

Proposition 1.17 *Let $\mathbf{a}(\lambda)$ an holomorphic function on a neighbourhood of $\bar{D}_r(0)$ and let $\varepsilon > 0$.*

$$\mathcal{L}_r(\int_0^\lambda \mathbf{a}(u)du)(\varepsilon) = \varepsilon \ \mathcal{L}_r(\mathbf{a})(\varepsilon) - \varepsilon \ e^{-r/\varepsilon} \int_0^r \mathbf{a}(u)du, \ \forall \ \lambda \in \bar{D}_r(0)$$

Proof: (see Exercises II).

Proposition 1.18 *Let $\mathbf{a}(\lambda)$ an holomorphic function on a neighbourhood of $\bar{D}_r(0)$.*

$$\mathcal{L}_r\Big((\frac{d}{d\lambda}) \ \mathbf{a}(\lambda)\Big)(\varepsilon) = -\mathbf{a}(0) + \mathbf{a}(r) \ e^{-r/\varepsilon} + \frac{1}{\varepsilon} \ \mathcal{L}_r(\mathbf{a})(\varepsilon)$$

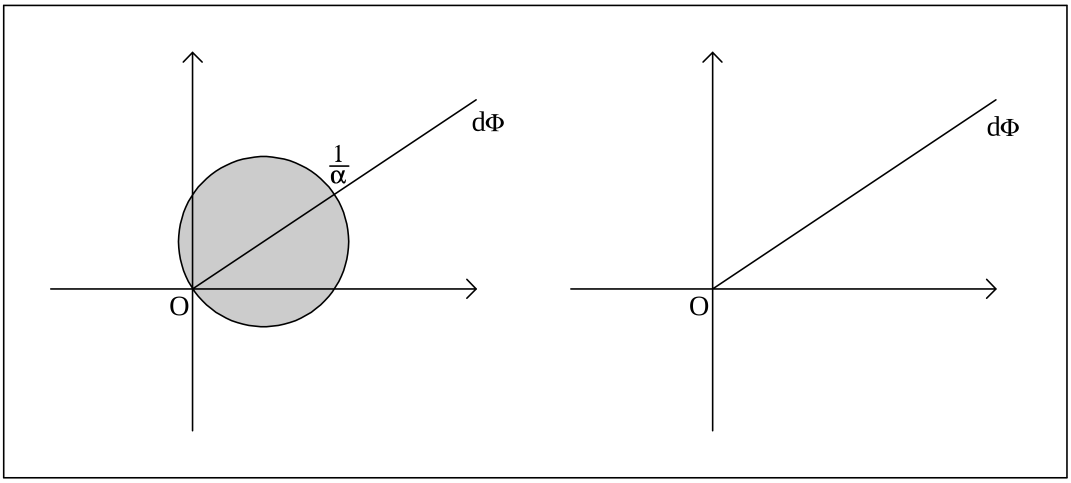
Proof: (see Exercises II).

Proposition 1.19 *Let $\mathbf{a}(\lambda)$ and $\mathbf{b}(\lambda)$ two holomorphic functions on a neighbourhood of $\bar{D}_r(0)$.*

$$\mathcal{L}_r(\mathbf{a} * \mathbf{b})(\varepsilon) = \mathcal{L}_r(\mathbf{a})(\varepsilon) \ . \ \mathcal{L}_r(\mathbf{b})(\varepsilon) - E(\varepsilon)$$
$$with \quad \mid E(\varepsilon) \mid \leq r^2 \ M_{\mathbf{a}} M_{\mathbf{b}} \ e^{-r/\varepsilon} \quad \forall \varepsilon, \ \varepsilon > 0$$

where $M_{\mathbf{a}}$ is the supremum of $\mid \mathbf{a}(\lambda) \mid$ on $\bar{D}_r(0)$.

[7] We denote by $D_r(0)$ the open disk of center 0 and radius r and $\bar{D}_r(0)$ the closed disk.
[8] This is the case, for example, if the series coefficients of $\widehat{a}(\varepsilon)$ are only majorized.



Sector of analyticity of $\mathcal{L}_{r,\phi,k}(\mathbf{a})(\varepsilon)$ The line-segment $d_{\phi,r}$ and the singularities of $\mathbf{a}(\lambda)$

ε-plane λ-plane

figure 1.1

Proof: $\mathcal{L}_r(\mathbf{a}*\mathbf{b}(\lambda))(\varepsilon) = \int_0^r e^{-\lambda/\varepsilon}(\int_0^\lambda \mathbf{a}(u)\mathbf{b}(\lambda-u)du)d\lambda$
Let $\eta = \lambda - u$

$$\begin{aligned}\mathcal{L}_r(\mathbf{a}*\mathbf{b}(\lambda))(\varepsilon) &= \int_0^r e^{-u/\varepsilon}\mathbf{a}(u)(\int_0^{r-u} e^{-\eta/\varepsilon}\ \mathbf{b}(\eta)d\eta)du \\ &= \int_0^r e^{-u/\varepsilon}\mathbf{a}(u)\Big(\int_0^r e^{-\eta/\varepsilon}\ \mathbf{b}(\eta)d\eta - \int_{r-u}^r e^{-\eta/\varepsilon}\ \mathbf{b}(\eta)d\eta\Big)du\end{aligned}$$

$$\mathcal{L}_r(\mathbf{a}*\mathbf{b})(\varepsilon) = \mathcal{L}_r(\mathbf{a})(\varepsilon)\ .\ \mathcal{L}_r(\mathbf{b})(\varepsilon) - E(\varepsilon)$$

with $E(\varepsilon) = \int_0^r e^{-u/\varepsilon}\ \mathbf{a}(u)(\int_{r-u}^r e^{-\eta/\varepsilon}\ \mathbf{b}(\eta)d\eta)du$

$$\begin{aligned}\mid E(\varepsilon)\mid &\leq \int_0^r e^{-u/\varepsilon} \mid \mathbf{a}(u)\mid (\int_{r-u}^r e^{(-r+u)/\varepsilon} \mid \mathbf{b}(\eta)\mid d\eta)du \quad car\ \ -\eta \leq -r+u \\ &\leq e^{-r/\varepsilon} \int_0^r\int_0^r \mid \mathbf{a}(u)\mid\mid \mathbf{b}(\eta)\mid du\ d\eta \\ &\leq e^{-r/\varepsilon}\ r^2 M_{\mathbf{a}}\ M_{\mathbf{b}}\end{aligned}$$

□

These different properties are gathered in the table 2.4.

$$\mathcal{L}_r(\mathbf{a})(\varepsilon) = \int_0^r e^{-\lambda/\varepsilon}\ \mathbf{a}(\lambda)\ d\lambda$$

$\mathbf{a}(\lambda)$	$\mathcal{L}_r(\mathbf{a})(\varepsilon),\ \varepsilon > 0$
1	$\varepsilon - \varepsilon\ e^{-r/\varepsilon}$
$\mathbf{a} * \mathbf{b}(\lambda)$	$\mathcal{L}_r(\mathbf{a})\ .\ \mathcal{L}_r(\mathbf{b})(\varepsilon) - E(\varepsilon)$ $\mid E(\varepsilon) \mid \leq r^2\ M_{\mathbf{a}} M_{\mathbf{b}}\ e^{-r/\varepsilon}\ \forall\varepsilon,\ \varepsilon > 0$
$\int_0^\lambda \mathbf{a}(u)du,\ \lambda \in \bar{D}_r(0)$	$\varepsilon\ \mathcal{L}_r(\mathbf{a})(\varepsilon) - \varepsilon\ e^{-r/\varepsilon} \int_0^r \mathbf{a}(u)du$
$(\frac{d}{d\lambda})\ \mathbf{a}(\lambda)$	$-\mathbf{a}(0) + \mathbf{a}(r)\ e^{-r/\varepsilon} + \frac{1}{\varepsilon}\ \mathcal{L}_r(\mathbf{a})(\varepsilon)$

table 2.4: some properties of $\mathcal{L}_r$.

1.4 Properties of J

1.4.1 Flat functions

Definition 1.17 *Let $A, k > 0$, let S an open sector whose vertex is the origin and let f an analytic function on S. The function f is said to be* flat in the Gevrey sense of order $1/k$ and type A, as $S \ni \varepsilon \to 0$, *if f admits the nil formal power series as asymptotic expansion with Gevrey estimates of order $1/k$ and type A, i.e. if there exists $\alpha > 0$ such that for all closed subsector S' of S, there exists a positive constant $C_{S'} > 0$ such that* $\forall\ \varepsilon \in S'$, $\forall N \in \mathbb{N}^*$

$$\mid f(\varepsilon) \mid \leq C_{S'}\ A^{N/k}\ \Gamma(\alpha + N/k)\ \ \mid \varepsilon \mid^N$$

Proposition 1.20 *Let $A, k > 0$ and let S be an open sector whose vertex is at the origin. A function f is flat in the Gevrey sense of order $1/k$ and type A if and only if it has an exponential decay of level k and type A, uniformly on every closed subsector S' of S i.e. there exists $\rho \leq 0$ such that*

$$\forall\ S' \prec S,\ \exists\ C_{S'} > 0,\ \mid f(\varepsilon) \mid \leq C_{S'}\ \ \mid \varepsilon \mid^\rho\ \ e^{-\frac{1}{A|\varepsilon|^k}},\quad \forall\varepsilon \in S'$$

Proof ([Wat12, Ram78, Sib90-1]):
Sufficient condition: Let $S' \prec S$ and suppose there exists $C_{S'} > 0$ such that

$$\forall\varepsilon \in S',\quad \mid f(\varepsilon) \mid \leq C_{S'}\ \ \mid \varepsilon \mid^\rho\ \ e^{-\frac{1}{A|\varepsilon|^k}}$$

We show that f is flat in the Gevrey sense of order $1/k$ and type A, i.e. $\widehat{f}(\varepsilon) = 0 + 0\varepsilon + \cdots$ and there exist $\beta > 0,\ \tilde{C} > 0$ such that

$$\forall\ N \in \mathbb{N}^*,\quad \mid f(\varepsilon) - 0 \mid \leq \tilde{C}\ A^{N/k}\ \Gamma(\beta + N/k) \mid \varepsilon \mid^N$$

By hypothesis, $\mid f(\varepsilon) \mid \leq C_{S'}\ \ \mid \varepsilon \mid^\rho\ \ e^{-\frac{1}{A|\varepsilon|^k}}$ so $\frac{|f(\varepsilon)|}{|\varepsilon|^N} \leq C_{S'}\ \ \mid \varepsilon \mid^{\rho - N}\ \ e^{-\frac{1}{A|\varepsilon|^k}}$.
We apply the next lemma to the function $\phi(\varepsilon) = \frac{f(\varepsilon)}{|\varepsilon|^N}$ and we conclude.

Lemma 1.21 *Let S' be an open sector whose vertex is at the origin and let ρ a real. If there exists $\tau > 0$ such that, $\forall\ N \geq 1,\ \forall\ \varepsilon \in S'$,*

$$\begin{array}{rcl} \mid \phi(\varepsilon) \mid & \leq & C \ \mid \varepsilon \mid^{\rho-N} e^{-\frac{1}{\tau\mid\varepsilon\mid^k}} \\ then \quad \mid \phi(\varepsilon) \mid & \leq & \bar{C}\ \tau^{N/k}\ \Gamma(1/2-\rho/k+N/k) \end{array}$$

Proof of the lemma: We observe that the function $t \mapsto \frac{e^{-\frac{1}{\tau\ t^k}}}{t^{N-\rho}}$ reaches a maximum at $t_0 = (\frac{k}{(N-\rho)\tau})^{\frac{1}{k}}$.So

$$\mid \phi(\varepsilon) \mid \leq \tilde{C}\ \tau^{N/k}\ e^{-(N-\rho)/k}(\frac{N-\rho}{k})^{(N-\rho)/k} \leq \bar{C}\ \tau^{N/k}\ \Gamma(1/2-\rho/k+N/k)$$

with the Stirling's formula. □

Necessary condition: Let $f \in \mathcal{A}_{\frac{1}{k},A}(S)$ such that $J(f) = 0$. By hypothesis, there exists $\alpha > 0$ such that $\forall\ S' \prec S,\ \exists C_{S'} > 0$

$$\mid f(\varepsilon) - 0 \mid \leq C_{S'}\ A^{N/k}\ \Gamma(\alpha + N/k) \mid \varepsilon \mid^N,\ \forall \varepsilon \in S',\ \forall\ N \geq 1$$

We apply the next lemma to the function f:

Lemma 1.22 *Let A and α two positive constants. If $\forall \varepsilon \in S',\ \forall\ N \geq 1$*

$$\mid f(\varepsilon) \mid \leq C_{S'}\ A^{N/k}\ \Gamma(\alpha + N/k) \mid \varepsilon \mid^N$$

$$then \quad \mid f(\varepsilon) \mid \leq \bar{C}_{S'}\ \mid \varepsilon \mid^{-k\alpha+k/2}\ e^{-\frac{1}{A\mid\varepsilon\mid^k}} \leq \tilde{C}_{S'}\ e^{-\frac{1}{B\mid\varepsilon\mid^k}} \quad where\ B > A.$$

Proof: (see Exercises II).

We denote by $\mathcal{A}_{\overline{A}}^{\leq -k}(S)$ the set of all functions in $\mathcal{A}(S)$ with an exponential decay of level k and type A. Thus we remark

$$\mathcal{A}_{\overline{A}}^{\leq -k}(S) = \mathcal{A}_{\frac{1}{k},A}(S) \cap \mathcal{A}^{<0}(S) = Ker\ J$$

Remark 1.19 *1) Let k_1 and k_2 two reals such that $0 < k_1 < k_2$, let A a positive real and S an open sector. Then*

$$\mathcal{A}_{\overline{A}}^{\leq -k_2}(S) \subset \mathcal{A}_{\overline{A}}^{\leq -k_1}(S)$$

2) let $k > 0$ a fixed level, if $0 < A < B$ and if S is an open sector then

$$\mathcal{A}_{\overline{A}}^{\leq -k}(S) \subset \mathcal{A}_{\overline{B}}^{\leq -k}(S)$$

Example 1.5 *The flatness* in the Gevrey sense *assigns Gevrey conditions on the asymptotic expansion: the function $f(\varepsilon) = e^{-1/\sqrt{\varepsilon}}$, that has an exponential decay of level $1/2$ and type 1, is flat in the Gevrey sense of order 2 and type 1 for $\varepsilon \in \mathbb{C}\backslash\mathbb{R}_-$. But it is* not flat in the Gevrey sense of order 1, *although it is* flat *in 0 in the half-plane* $\mathrm{Re}\,\varepsilon > 0$.

We have the same definitions and the same results for an analytic function $f(x,\varepsilon)$ of x, for x in the disk $D_r(0)$:

Definition 1.18 *Let $k > 0$, let $D_r(0)$ be an open disk and let S be an open sector whose vertex is at the origin. Let $f(x, \varepsilon)$ an analytic function of x and ε, for x in $D_r(0)$ and for ε in S. The function $f(x, \varepsilon)$ is said to be* flat in the Gevrey sense of order $1/k$ and type A, uniformly for x in $D_r(0)$, *as $\varepsilon \to 0$, $\varepsilon \in S$, if f admits the formal series nil as asymptotic expansion with Gevrey estimates of order $1/k$ and type A, uniformly for x in $D_r(0)$.*

Proposition 1.23 *Let $k > 0$, let $D_r(0)$ be a disk and let S be an open sector whose vertex is at the origin. A function $f(x, \varepsilon)$ The function $f(x, \varepsilon)$ is flat in the Gevrey sense of order $1/k$ and type A, uniformly for x in $D_r(0)$ if and only if it has an exponential decay of level k and type A, uniformly on every closed subsector S' of S and uniformly for x in $D_r(0)$, i.e. there exists $\rho \leq 0$ such that*

$$\forall\ S' \prec S,\ \exists\ C_{S'} > 0,\ \mid f(x, \varepsilon) \mid \leq C_{S'} \ \mid \varepsilon \mid^{\rho}\ e^{-\frac{1}{A|\varepsilon|^k}},\ \ \forall \varepsilon \in S',\ \forall\ x \in D_r(0)$$

Conclusion: The difference between two functions in $\mathcal{A}_{\frac{1}{k},A}(S)$, having the same asymptotic expansion, is exponentially small of level k and type A. So we have a sequence of differential algebras:

$$\mathcal{A}_A^{\leq -k}(S) \ \longrightarrow\ \mathcal{A}_{\frac{1}{k},A}(S)\ \longrightarrow\ \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$$

Now we ask two questions:
1) What are the conditions for J to be injective ? (Watson theorem [Wat12] will give the answer).
2) What are the conditions for J to be surjective ? (Borel-Ritt Gevrey theorem [Wat12] will give the answer).

1.4.2 Injectivity of J: Watson theorem.

Let $f \in \mathcal{A}_{\frac{1}{k}}(S)$. Watson theorem [Wat12] gives condition on the opening of the sector S so that the asymptotic expansion of f determines uniquely the function f.

Theorem 1.24 Watson theorem.
Let $k > 0$, let S be a sector whose opening is $> \pi/k$ and let $f \in \mathcal{A}(S)$. If f has an exponential decay of level k in S, then $f \equiv 0$.

Proof: The proof of the Watson theorem is similar to the *Phragmén-Lindelöf lemma*'s one, and this last proof is a variant of the Maximum Principal.
Let S a sector bissected by the positive real axes, whose opening is $> \pi/k$. Then there exists S' a closed subsector of S, $S' = S_{r,-\eta-\pi/2k,\eta+\pi/2k}$ whose opening $\mid S' \mid= 2(\eta + \pi/2k)$ where $\eta > 0$ and k' such that $k' < k$, $\ k'(\pi/2k + \eta) > \pi/2$. We majorize the function

$$g(\varepsilon) = f(\varepsilon)\ \exp(\lambda\varepsilon^{-k'} - \lambda r^{-k'})$$

for $\varepsilon \in S'$, $\forall\ \lambda > 0$.

On the boundary of S', if $arg\ \varepsilon = \theta = \pm(\pi/2k + \eta)$ then

$$\mid f(\varepsilon)\ \exp(\lambda\varepsilon^{-k'} - \lambda r^{-k'}) \mid \leq \mid f(\varepsilon) \mid\mid e^{\lambda\varepsilon^{-k'}}\ \mid e^{-\lambda r^{-k'}}\ \leq \mid f(\varepsilon) \mid \leq C$$

because $| \ e^{\lambda\varepsilon^{-k'}} \ |= e^{\lambda cos(k'\theta)|\varepsilon|^{-k'}} \leq 1 \ (cos(k'\theta) < 0)$ and $| \ f(\varepsilon) \ |\leq Ce^{-\frac{M}{|\varepsilon|^k}} \leq C$ for all $\varepsilon \in S$, S open, therefore for all $\varepsilon \in S'$.
If $| \ \varepsilon \ |= r$, $arg \ \varepsilon \in] -\eta - \pi/2k, \eta + \pi/2k[$,

$$| \ f(\varepsilon) \ \exp(\lambda\varepsilon^{-k'} - \lambda r^{-k'}) \ |\leq| \ f(\varepsilon) \ |\leq C$$

Into the sector S' :

$$| \ f(\varepsilon) \ \exp(\lambda\varepsilon^{-k'} - \lambda r^{-k'}) \ |\leq Ce^{-M|\varepsilon|^{-k}} e^{\lambda cos(k'\theta)|\varepsilon|^{-k'}} e^{-\lambda r^{-k'}} \leq Ce^{-M|\varepsilon|^{-k}(1-\frac{\lambda cos(k'\theta)}{M}|\varepsilon|^{k-k'})}$$

$$and \ \ 1 - \frac{\lambda cos(k'\theta)}{M} \ | \ \varepsilon \ |^{k-k'} \rightarrow 1 \ as \ \varepsilon \rightarrow 0 \ \ (k > k')$$

so $1 - \frac{\lambda cos(k'\theta)}{M} \ | \ \varepsilon \ |^{k-k'} > 0$ for $\varepsilon \rightarrow 0$ and the majorant $Ce^{-M|\varepsilon|^{-k}(1-\frac{\lambda cos(k'\theta)}{M}|\varepsilon|^{k-k'})}$ tends to 0 as $\varepsilon \rightarrow 0$. So

$$| \ f(\varepsilon) \ \exp(\lambda\varepsilon^{-k'} - \lambda r^{-k'}) \ |\rightarrow 0 \ as \ \varepsilon \rightarrow 0, \ \varepsilon \in S'$$

Since the function $g(\varepsilon)$ is majorized by C on the boundary of S' and $g(\varepsilon) \rightarrow 0$ as $| \ \varepsilon \ |\rightarrow 0$, we apply the Maximum Principle: g is majorized by C in the interior of S'

$$| \ f(\varepsilon) \ \exp(\lambda\varepsilon^{-k'} - \lambda r^{-k'}) \ |\leq C, \ \forall \ \varepsilon \in S', \ \forall \ \lambda > 0$$

Let r' such that $0 < r' < r$. On the line-segment $[0, r']$,

$$| \ f(\varepsilon) \ |\leq C \ e^{-\lambda \ |\varepsilon|^{-k'}} e^{\lambda r^{-k'}} \leq C \ e^{-\lambda(r'^{-k'} - r^{-k'})}$$

(here, $\alpha = 0$) and $r'^{-k'} - r^{-k'} > 0$. So, $e^{-\lambda(r'^{-k'} - r^{-k'})} \rightarrow 0$ as $\lambda \rightarrow +\infty$. Therefore $f \equiv 0$ on the line-segment $[0, r']$ and $f \equiv 0$ dans S because f is analytic on S. $\square$

Proposition 1.25 *Let S be a sector. If the opening of S is $> \pi/k$, then the homomorphism J defined by*

$$\begin{array}{rcl} J : \mathcal{A}_{\frac{1}{k},A}(S) & \longrightarrow & \mathbb{C}[[\epsilon]]_{\frac{1}{k},A} \\ f & \longmapsto & \widehat{f}(\varepsilon) \end{array}$$

is injective.

Proof: When $f \in \mathcal{A}_{\frac{1}{k},A}(S)$ and $J(f) = 0$, f has an exponential decay of order k. The Watson theorem leads us to conclude. $\square$

Example 1.6 *([Bal94]) Let S be an open sector whose vertex is at the origin and whose opening is $\leq \pi/k$, $k > 0$ and let $f(\varepsilon) = e^{-c/\varepsilon^k}$, $c > 0$. The function $f \in \mathcal{A}_{\frac{1}{k}}(S)$ and $J(f)(\varepsilon) = 0 + 0\varepsilon + 0\varepsilon^2 + ...$ The map J is not injective because the nil function has too $0 + 0\varepsilon + 0\varepsilon^2 + ...$ as asymptotic expansion.*
Moreover, if $g \in Ker(J)$, if h is analytic on S and if there exists α such that $\varepsilon^\alpha \ h(\varepsilon)$ is bounded at the origin, then the function $g \cdot h \in \mathcal{A}_{\frac{1}{k}}(S)$ and $g \cdot h \in Ker(J)$ (J is non injective !).

1.4.3 Surjectivity of J: Borel-Ritt Gevrey theorem

We consider the map

$$\begin{array}{rcl} J : \mathcal{A}_{\frac{1}{k}}(S) & \longrightarrow & \mathbb{C}[[\epsilon]]_{\frac{1}{k}} \\ f & \longmapsto & \widehat{f} = \sum_{m \geq 0} b_m \varepsilon^m \end{array}$$

If the opening of the sector S is $2\theta < \pi/k$, we show that J is surjective. We can prove precisely that if a series $\widehat{f}$ is Gevrey of order $1/k$ and type A, the function $f \in \mathcal{A}_{\frac{1}{k}}(S)$ such that $J(f) = \widehat{f}$ is *at most* of type $\frac{A}{(\cos k\theta)^k}$ (see appendix B). In order to do that, we define a new formal Borel transform and so a new truncated Laplace transform.
let $\widehat{f} = \sum_{m \geq 1} b_m \varepsilon^m \in \mathbb{C}[[\epsilon]]_{\frac{1}{k}, A}$. So

$$\exists\ \alpha > 0,\ \exists\ K > 0,\ \forall m \geq 1\ \mid b_m \mid \leq K\ A^{m/k}\ \Gamma(\alpha + m/k)$$

Definition 1.19 *Let $\widehat{f} = \sum_{m \geq 1} b_m \varepsilon^m$; we define the formal Borel transform still denoted by $\widehat{\mathcal{B}}_k$:*

$$\widehat{\mathcal{B}}_k(\widehat{f})(t) = \sum_{m \geq 1} \frac{b_m}{\Gamma(\alpha + 1 + 1/k + m/k)} t^m$$

Lemma 1.26 *The series $\widehat{\mathcal{B}}_k(\widehat{f})(t)$ is absolutely convergent for $\mid t \mid \leq (1/A)^{1/k}$.*

Proof: (see Exercises II).

Remark 1.20 *In the Borel-plane the radius of convergence of the series is $\geq (1/A)^{1/k}$.*

Lemma 1.27 *With the properties above, we have*

$$\sum_{n \geq 1} \frac{b_n}{\Gamma(\alpha + 1 + 1/k + n/k)} t^n = \sum_{n=1}^{N-1} \frac{b_n}{\Gamma(\alpha + 1 + 1/k + n/k)} t^n + t^N\ \phi_N(t)$$

$$\textit{and} \quad \mid t^N\ \phi_N(t) \mid \leq C \times \mid t \mid^N A^{N/k} \quad \textit{for} \quad \mid t \mid \leq (1/A)^{1/k}$$

Proof: (see Exercises II). We define the new formal Laplace transform:

Definition 1.20 *Let $\mathbf{f}(t)$ an analytic function on the neighbourhood of $\bar{D}_{(1/A)^{1/k}}(0)$. We call* truncated Laplace transform of $\mathbf{f}$, *the function*

$$\mathcal{L}_A(\mathbf{f})(\varepsilon) = k\ \varepsilon^{-k\alpha - k - 1} \int_0^{(1/A)^{1/k}} e^{-t^k/\varepsilon^k}\ \mathbf{f}(t)\ t^{k\alpha + k}\ dt$$

This function is analytic on $S = S_{-\theta,\theta}$, $\theta < \pi/(2k)$.

Remark 1.21 *The new transforms are inverse one another. Let $\alpha > 0$ and $\theta < \pi/(2k)$. Then $\forall\ \varepsilon \in S_{-\theta,\theta}$, $\forall n \geq 0$*

$$\varepsilon^n = k\ \varepsilon^{-k\alpha - k - 1} \int_0^{+\infty} e^{-t^k/\varepsilon^k}\ \frac{t^n}{\Gamma(\alpha + 1 + 1/k + n/k)}\ t^{k\alpha + k}\ dt$$

Theorem 1.28 *:* **Borel-Ritt Gevrey theorem***.*
Let $k > 0$ and $A > 0$. Let $\widehat{f} = \sum_{n\geq 1} b_n\varepsilon^n$ a Gevrey series of order $1/k$ and type A and let $\widehat{\mathcal{B}}_k(\widehat{f})(t) = \sum_{n\geq 1} \frac{b_n}{\Gamma(\alpha+1+1/k+n/k)}t^n$ its Borel-series convergent for $\mid t \mid\leq (1/A)^{1/k}$. It defines a function $\mathbf{f}(t)$ analytic on the neighbourhood of $\bar{D}_{(1/A)^{1/k}}(0)$ and continue on this closed disk. Thus $f(\varepsilon) = \mathcal{L}_A(\mathbf{f})(\varepsilon) = k\ \varepsilon^{-k\alpha-k-1}\ \int_0^{(1/A)^{1/k}}\ e^{-t^k/\varepsilon^k}\ \mathbf{f}(t)\ t^{k\alpha+k}\ dt$ is an analytic function on the open sector $S_{-\pi/2k,\pi/2k}$ and f has $\widehat{f}$ as asymptotic expansion Gevrey of order $1/k$: $\forall\ \theta < \pi/(2k)$, there exists $\bar{K} > 0$ such that, $\forall\varepsilon \in S_{-\theta,\theta}, \forall N \geq 2$

$$\mid f(\varepsilon) - \sum_{n=1}^{N-1} b_n\varepsilon^n \mid\leq \bar{K}\ \Gamma(\alpha+1+1/k+N/k)\ (\frac{A^{1/k}}{\cos k\theta})^N\mid \varepsilon\mid^N$$

i.e. $f \in \mathcal{A}_{\frac{1}{k}}(S_{-\theta,\theta})$ and its type is $\frac{A}{(\cos k\theta)^k}$. Moreover the type $\frac{A}{(\cos k\theta)^k}$ is optimum.

Proof: The proof is given in the case $k = 1$ where $\widehat{\mathcal{B}}_1(\widehat{f})(t) = \sum_{n\geq 1} \frac{b_n}{\Gamma(\alpha+2+n)}t^n$ and $\mathcal{L}_A(\mathbf{f})(\varepsilon) = \varepsilon^{-\alpha-2}\ \int_0^{1/A}\ e^{-t/\varepsilon}\ \mathbf{f}(t)\ t^{\alpha+1}\ dt$.
The next identity is straightforward:

Lemma 1.29 *: Let $\alpha > 0$ and let $\varepsilon \in S_{-\theta,\theta}$ where $\theta < \pi/2$. Then*

$$\forall\varepsilon \in S_{-\theta,\theta},\ \forall n \geq 0\ \ \varepsilon^n = \varepsilon^{-\alpha-2}\ \int_0^{\infty}\ e^{-t/\varepsilon}\ \frac{t^{n+\alpha+1}}{\Gamma(\alpha+2+n)}\ dt$$

For $N \geq 2$,

$$\begin{aligned}
\mid f(\varepsilon) - \sum_{n=1}^{N-1} b_n\varepsilon^n \mid\ &=\ \mid \varepsilon^{-\alpha-2}\int_0^{1/A}\ e^{-t/\varepsilon}\ (\sum_{n=1}^{\infty}\frac{b_n}{\Gamma(\alpha+2+n)}t^n)\ t^{\alpha+1}\ dt - \sum_{n=1}^{N-1} b_n\varepsilon^n \mid\\
&=\mid \int_0^{1/A}\ \varepsilon^{-\alpha-2}\ e^{-t/\varepsilon}\ t^{\alpha+1}\ (\sum_{n=1}^{N-1}\frac{b_n}{\Gamma(\alpha+2+n)}t^n + t^N\phi_N(t))\ dt\\
&- \sum_{n=1}^{n=N-1} b_n \times \int_0^{\infty}\ \varepsilon^{-\alpha-2}\ e^{-t/\varepsilon}\ \frac{t^{n+\alpha+1}}{\Gamma(\alpha+2+n)}\ dt \mid\\
&=\mid f_1(\varepsilon) - f_2(\varepsilon) \mid
\end{aligned}$$

$$\textit{where}\ \ f_1(\varepsilon) = \int_0^{1/A}\ t^{N+\alpha+1}\ \varepsilon^{-\alpha-2}\ e^{-t/\varepsilon}\ \phi_N(t)\ dt$$

$$\textit{and}\ \ f_2(\varepsilon) = \int_{1/A}^{\infty}\ \varepsilon^{-\alpha-2}\ e^{-t/\varepsilon}\ t^{\alpha+1}\ (\sum_{n=1}^{N-1}\frac{b_n}{\Gamma(n+\alpha+2)}t^n)\ dt$$

Upperbounds of $f_1(\varepsilon)$: Let $S_{-\tilde{\theta},\tilde{\theta}}$ be a subsector of $S_{-\theta,\theta}$ with $0 < \tilde{\theta} < \theta < \pi/2$ and let $\varepsilon \in S_{-\tilde{\theta},\tilde{\theta}}$. As $\varepsilon =\mid \varepsilon \mid e^{i\theta_0}$ with $-\tilde{\theta} \leq\ \theta_0 \leq\ \tilde{\theta} < \theta$, $\mid e^{-t/\varepsilon} \mid= e^{-\frac{t\cos\theta_0}{\mid\varepsilon\mid}} \leq e^{-\frac{t\cos\ \tilde{\theta}}{\mid\varepsilon\mid}}$ and there exists $C > 0$ such that

$$\mid f_1(\varepsilon) \mid\leq C\mid \varepsilon \mid^{-\alpha-2}\int_0^{1/A}\ e^{-\frac{t\cos\tilde{\theta}}{\mid\varepsilon\mid}}\ t^{N+\alpha+1}\ A^N dt$$

(cf. lemma (1.27)). Let $u = \frac{t\cos\tilde\theta}{|\varepsilon|}$, therefore we have

$$\begin{aligned}
| f_1(\varepsilon) | &\leq C| \varepsilon |^{-\alpha-2} \int_0^\infty e^{-u} | \varepsilon |^{N+\alpha+2} u^{N+\alpha+1} A^N \frac{1}{\cos\tilde\theta^{N+\alpha+2}} du \\
&\leq C| \varepsilon |^N \frac{1}{\cos\tilde\theta^{N+\alpha+2}} A^N \int_0^\infty e^{-u} u^{N+\alpha+1} du
\end{aligned}$$

So

$$| f_1(\varepsilon) | \leq \bar C \, (\frac{A}{\cos\tilde\theta})^N \, \Gamma(\alpha+2+N) | \varepsilon |^N \tag{1.6}$$

Upperbounds of $f_2(\varepsilon)$ *:*

$$\begin{aligned}
| f_2(\varepsilon) | = & | \int_{1/A}^\infty \varepsilon^{-\alpha-2} \, e^{-t/\varepsilon} \, t^{\alpha+1} \, (\sum_{n=1}^{N-1} \frac{b_n}{\Gamma(\alpha+2+n)} t^n) \, dt \, | \\
\leq & \int_{1/A}^\infty | \varepsilon |^{-\alpha-2} \, | e^{-t/\varepsilon} | \, t^{\alpha+1} \, K \, (tA)^N \, dt
\end{aligned}$$

with the lemma

Lemma 1.30 *If* $t \geq 1/A > 0$ *then there exists* $K > 0$ *such that*

$$\forall N \geq 2, \quad | \sum_{n=1}^{N-1} \frac{b_n}{\Gamma(\alpha+2+n)} t^n | \leq K \times (tA)^N$$

The lemma is proved in Exercises II.

$$\text{So} \quad | f_2(\varepsilon) | \leq K A^N \, | \varepsilon |^{-\alpha-2} \int_{1/A}^\infty e^{-\frac{t\cos\tilde\theta}{|\varepsilon|}} \, t^{N+\alpha+1} \, dt$$

Let $u = \frac{t\cos\tilde\theta}{|\varepsilon|}$, we have

$$| f_2(\varepsilon) | \leq K (\frac{A}{\cos\tilde\theta})^N \, \Gamma(\alpha+2+N) | \varepsilon |^N \tag{1.7}$$

Using the inequalities (1.6) and (1.7), $\forall \varepsilon \in S_{-\tilde\theta,\tilde\theta}$, $\tilde\theta < \theta$, $\forall \; N \geq 2$, we have

$$| f(\varepsilon) - \sum_{n=1}^{N-1} b_n \varepsilon^n | = | f_1(\varepsilon) - f_2(\varepsilon) | \leq \bar K (\frac{A}{\cos\tilde\theta})^N \, \Gamma(\alpha+2+N) | \varepsilon |^N \tag{1.8}$$

Remark 1.22 *The constant* $\bar K$ *depends on* $\tilde\theta$*: it has* $1/\cos\tilde\theta^{\alpha+2}$ *as factor.*

Remark 1.23 *As* $\tilde\theta < \theta$*, we have* $1/\cos\tilde\theta < 1/\cos\theta$ *and*

$$| f(\varepsilon) - \sum_{n=1}^{N-1} b_n \varepsilon^n | \leq \bar K \times \Gamma(\alpha+2+N) \times \frac{A^N}{(\cos\theta)^N} | \varepsilon |^N$$

We have built a function f *analytic on* $S_{-\theta,\theta}$ *that has* $\widehat{f}$ *as asymptotic expansion with Gevrey estimates of order* 1 *and type*[9] $A/\cos\theta$*. In Appendix A, we show that* $A/\cos\theta$ *is optimum.*

[9] The type must depend on the sector $S_{-\theta,\theta}$ and not depend on $S_{-\tilde\theta,\tilde\theta}$.

With the next proposition, we can recognize the functions in $\mathcal{A}_{\frac{1}{k}}(S)$.

Proposition 1.31 *([RS96]) Let $k>0$ and let S be an open sector whose opening $\leq \pi/k$ bissected by d_ϕ. Let f be an analytic function on S such that $f\in\mathcal{A}(S)$. Let $\widehat{f}$ be the asymptotic expansion of f (in the Poincaré sense). We suppose that $\widehat{f}\in\mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$ and we denote by $R>0$ the radius of convergence of $\widehat{\mathcal{B}}_k(\widehat{f}-f(0))$. We choose a positive number r such that $0<r<R$, then the following conditions are equivalent:*
(i) $f\in\mathcal{A}_{\frac{1}{k},\frac{A}{(\cos k\theta)^k}}(S)$
(ii) $f-\mathcal{L}_{(\frac{1}{A})^{1/k},\phi,k}(\widehat{\mathcal{B}}_k(\widehat{f}))\in\mathcal{A}^{\leq -k}_{\frac{A}{(\cos k\theta)^k}}(S)$

Proof: As the map J is surjective if $|\ S\ |\leq \pi/k$, let $r=(1/A)^{1/k}$ so the function

$$\mathcal{L}_{r,\phi,k}(\widehat{\mathcal{B}}_k(\widehat{f}))\in\mathcal{A}_{\frac{1}{k},\frac{A}{(\cos k\theta)^k}}(S)$$

and it has $\widehat{f}$ as asymptotic expansion with Gevrey estimates of order $1/k$ and type $\frac{A}{(\cos k\theta)^k}$.
Thus if $f\in\mathcal{A}_{\frac{1}{k},\frac{A}{(\cos k\theta)^k}}(S)$ then
$f-\mathcal{L}_{r,\phi,k}(\widehat{\mathcal{B}}_k(\widehat{f}))\in Ker(J)$ and $Ker(J)=\mathcal{A}^{\leq -k}_{\frac{A}{(\cos k\theta)^k}}(S)$.
Conversely, let $\widehat{f}=\sum_{m=0}^{+\infty}b_m\varepsilon^m$ and let $S'\prec S$; we have to prove that there exist $C,\alpha>0$ such that

$$\forall\ N\geq 1,\ \forall\ \varepsilon\in S',\ \ |\ f(\varepsilon)-\sum_{m=0}^{N-1}b_m\varepsilon^m\ |\leq C\ (\frac{A}{(\cos k\theta)^k})^{N/k}\Gamma(\alpha+N/k)\ |\ \varepsilon\ |^N$$

With the inequality (1.8) where $k>0$ and the hypothesis (ii), we obtain the result (because if $\gamma>0$ then $\mathcal{O}(\exp(-\gamma\ |\ \varepsilon\ |^{-k}))\leq\bar{C}\ |\ \varepsilon\ |^N,\ \forall\ N\geq 1$). □

Remark 1.24 *Let k_1 and k_2 two reals such that $0<k_1<k_2$. If a series $\widehat{f}$ is Gevrey of order $1/k_2$ and type $A>0$ then it is Gevrey of order $1/k_1$ and type A. This series $\widehat{f}$ can be "realised" (in a non unique way) by an analytic function $\mathcal{L}_{r,\phi,k_2}(\widehat{\mathcal{B}}_{k_2}(\widehat{f}))$, $r>0$, on a sector S_2 whose opening is $<\pi/k_2$. It can also be realised by an analytic function $\mathcal{L}_{r,\phi,k_1}(\widehat{\mathcal{B}}_{k_1}(\widehat{f}))$ on a sector S_1 with a* greater *opening ($|\ S_1\ |<\pi/k_1$) but the estimates of the error is* worse*: the error term is like* $\exp(-\frac{1}{A|\varepsilon|^{k_1}})$ *instead of* $\exp(-\frac{1}{A|\varepsilon|^{k_2}})$.

1.5 Cut-off asymptotic; Summation to the least term

In this subsection, we introduce the notion of *cut-off asymptotic* [RS96]) and we explain the precise equivalence between the existence of a Gevrey asymptotic expansion and the exponential precision of a "lest term' cut-off procedure.

Definition 1.21 *Let $k>0, A>0$ and let S be an open sector whose vertex is at the origin. We will say that a function f analytic on S has $\widehat{f}=\sum_{m=0}^{+\infty}b_m\varepsilon^m$ as* a cut-off asymptotic of order $1/k$ and type A *if and only if there is a constant* $\rho \leq 0$ *and for each closed subsector* S' *of* S*, there is a positive constant* $C_{S'}$ *such that*

$$\forall\ \varepsilon \in S', \quad |\ f(\varepsilon) - \sum_{m=0}^{[\frac{k}{A|\varepsilon|^k}]} b_m \varepsilon^m \ |\leq C_{S'} \ \ |\ \varepsilon\ |^\rho \ \ e^{-\frac{1}{A|\varepsilon|^k}}$$

The set of all $f \in \mathcal{O}(S)$ *having such an asymptotic will be denoted by* $\mathcal{C}_{\frac{1}{k},A}(S)$ *and* $N(\varepsilon) = [\frac{k}{A|\varepsilon|^k}]$ *will be the index of the cut-off asymptotic.*

Remark 1.25 *We take* $\rho \leq 0$ *in order to keep the same constant* A *in the next properties.*

Remark 1.26 *The* truncated *series* $\sum_{m=0}^{N(\varepsilon)} b_m \varepsilon^m$ *is exponentially closed to the function* $f(\varepsilon)$.

We had already the property:
If $f \in \mathcal{A}_{\frac{1}{k},A}(S)$ then $J(f) \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$. We have also the proposition

Proposition 1.32 *Let* $k, A > 0$. *If* $f \in \mathcal{C}_{\frac{1}{k},A}(S)$ *then* $J(f) \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$.

Proof: See exercises II.

Theorem 1.33 *([RS96]) A function* f *analytic on an open sector* S *has a Gevrey-*$1/k$ *asymptotic expansion of type* A *if and only if it has a cut-off asymptotic with Gevrey estimates of order* $1/k$ *and type* A *on that sector.*

Proof: (see [RS96], theorem 6.9, page 365 and exercises II). The authors show that if $f \in \mathcal{C}_{\frac{1}{k},A}(S)$, i.e. if there exists $\rho \leq 0$ such that $\forall\ S' \prec S,\ \exists\ C_{S'} > 0$ with

$$\forall\ \varepsilon \in S', \quad |\ f(\varepsilon) - \sum_{m=0}^{\frac{k}{A|\varepsilon|^k}} b_m \varepsilon^m \ |\leq C_{S'} \ \ |\ \varepsilon\ |^\rho \ \ e^{-\frac{1}{A|\varepsilon|^k}}$$

then $\forall\ N \geq 1,\ \forall\ \varepsilon \in S'$

$$|\ f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m \ |\leq \bar{C}_{S'} \ A^{N/k}\ \Gamma((N+\beta)/k + 1/2) \ \ |\ \varepsilon\ |^N$$

where $\beta = k/2 - \rho$. So

$$|\ f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m \ |\leq \bar{C}_{S'} \ A^{N/k}\ \Gamma(1 - \rho/k + N/k) \ \ |\ \varepsilon\ |^N \tag{1.9}$$

Conversely, if $\forall\ S' \prec S,\ \exists\ C_{S'},\ \alpha > 0$ such that $\forall\ N \geq 1,\ \forall\ \varepsilon \in S'$

$$|\ f(\varepsilon) - \sum_{m=0}^{N-1} b_m \varepsilon^m \ |\leq C_{S'} \ A^{N/k}\ \Gamma(\alpha + N/k) \ \ |\ \varepsilon\ |^N$$

then ,

$$|\ f(\varepsilon) - \sum_{m=0}^{N^*-1} b_m \varepsilon^m \ |\leq \tilde{C}_{S'} \ \ |\ \varepsilon\ |^{-k\alpha + k/2} \ \ e^{-\frac{1}{A|\varepsilon|^k}} \tag{1.10}$$

where $N^* = [\frac{k}{A|\varepsilon|^k}] - 1$ □

Remark 1.27 *Thus, let* $\mu := 1 - \rho/k$ *in (1.9). Then the power of* $|\ \varepsilon\ |$ *in (1.10) is equal to* $-k\mu + k/2 = \rho - k/2$*: we have only lost* $k/2$ *in regard of the exponant* ρ *of* $|\ \varepsilon\ |^\rho$ *in the definition of the cut-off asymptotic.*

This theorem gives us a computational process to approach a function with a known asymptotic expansion with Gevrey estimates of order $1/k$ and type A.

Definition 1.22 *Let* $A, k > 0$*. Let* $\sum_{m\geq 0} b_m \varepsilon^m$ *be a Gevrey series of order* $1/k$ *and type* A*. We denote by* $N(\varepsilon)$*,* $[\frac{k}{A|\varepsilon|^k}]$*. We call* quasi-summation to the least term *of this series, the sum* $\sum_{m=0}^{N(\varepsilon)} b_m \varepsilon^m$*.*

In fact, this quasi-sum is the summation to the least term of the majorant series whose general term is $C\ A^{m/k}\ \Gamma(\alpha + m/k)$: the index $N(\varepsilon)$ of the last term considered in the truncated series corresponds to the index of the least term of the sequence $\{CA^{m/k}\Gamma(\alpha + m/k) \mid \varepsilon \mid^m\}_m$.
So we justify the H. Poincaré's techniques of "summation to the least term", where we cut the series at N^*such that $|\ b_{N^*}\ |\ |\ \varepsilon\ |^{N^*}$ is minimum ([Poi90, Poi92]).

2 Introduction to k-summability

2.1 Borel-Laplace summation

We remind that if $\widehat{a}(\varepsilon)$ is a Gevrey series of order $1/k$ then the series $\widehat{\mathcal{B}}_k(\widehat{a})(\lambda)$ is a convergent series and there exists a disk $D_r(0)$ in the λ-plane such that $\widehat{\mathcal{B}}_k(\widehat{a})$ defines an holomorphic function $\mathbf{a}(\lambda)$ on the neighbourhood of $\bar{D}$.
If $\mathbf{a}(\lambda)$ has an analytic continuation in some directions $d_\phi = \{r\ e^{i\phi},\ r > 0\}$, $\phi \in [0, 2\pi[$, we still denote by $\mathbf{a}(\lambda)$ this new function and we consider its Laplace transform:

$$\mathcal{L}_{\phi,k}(\mathbf{a})(\varepsilon) = k \int_{d_\phi} e^{-\lambda^k/\varepsilon^k}\ \mathbf{a}(\lambda)\ \frac{\lambda^{k-1}}{\varepsilon^{k-1}} d\lambda \tag{2.1}$$

Now, $k = 1$. When $\mathbf{a}$ has an exponential increasing of level ≤ 1 on the half-line d_ϕ, the function $\mathcal{L}_{\phi,1}(\mathbf{a})(\varepsilon)$ is at least defined on the disk centered on d_ϕ and having 0 on its boundary (see figure 2.1). More precisely

Definition 2.1 *With the previous notations, if* $|\ \mathbf{a}(\lambda)\ | \leq K\ \exp(\gamma\ |\ \lambda\ |),\ \forall\ \lambda \in d_\phi$ *where* $K,\ \gamma > 0$*, then the function* $\mathcal{L}_{\phi,1}(\mathbf{a})(\varepsilon)$ *is defined on the disk*[10] $\{\varepsilon\ /\ \mathrm{Re}\,(\frac{e^{i\phi}}{\varepsilon}) > \gamma\}$ *containing* 0 *on its boundary and having a diameter on* d_ϕ*, whose length is* $1/\gamma$*. We call it the* Borel-disk in the direction d_ϕ.

Remark 2.1 *We suppose that the function* $\mathbf{a}$ *has an exponential increasing of level at most* k *on* d_ϕ*, i.e. there exist* $\gamma, K > 0$ *such that*

$$|\ \mathbf{a}(\lambda)\ | \leq K\ \exp(\gamma|\ \lambda\ |^k),\ \forall\ \lambda \in d_\phi$$

[10]Effectively, if $\lambda = |\ \lambda\ |\ e^{i\phi}$ then $|\ e^{-\frac{\lambda}{\varepsilon}}\ \mathbf{a}(\lambda)\ | = e^{-|\lambda|\ \mathrm{Re}\,(\frac{e^{i\phi}}{\varepsilon})}\ \ |\ \mathbf{a}(\lambda)\ |$.

If $k > 1$, the Borel-disk becomes a "tear" T_k defined by $T_k = \{\varepsilon \ / \ \gamma - cos(k \ arg \ \varepsilon - k \ \phi) | \ \varepsilon \ |^{-k} < 0\}$ whose angle's measure at the origin is equal to π/k and whose length is equal to $(1/\gamma)^{1/k}$. If $0 < k < 1$, $\mathcal{L}_{\phi,k}(\mathbf{a})(\varepsilon)$ is analytic on a domain whose angle's measure at the origin is $> \pi$. We remark that for all $k \neq 0$, this domain is bounded by an half-lemniscate of Bernouilli (see figure 2.1).

Remark 2.2 Analytic continuation.
If the series $\widehat{a}(\varepsilon) = \sum_{m\geq 1} a_m \varepsilon^m$ is convergent, it defines an analytic function $a(\varepsilon)$ on $D_R(0)$. Moreover, $\forall \ r > 0, \ r < R, \ \exists \ A_r > 0$ such that $| \ a_m \ | \leq A_r r^{-m-1}, \ \forall m \geq 1$. The radius of convergence of the series $\widehat{\mathcal{B}}_1(\widehat{a})(\lambda) = \sum_{m\geq 0} a_{m+1} \frac{\lambda^m}{m!}$ is infinite and its sum is an entire function, denoted by $\mathbf{a}(\lambda)$, such that $\forall \ \lambda \in \mathbb{C}$, $| \ \mathbf{a}(\lambda) \ | \leq A_r \ \exp(\frac{|\lambda|}{r})$. Let d_ϕ be a direction, the integral $S(\varepsilon) = \int_{d_\phi} e^{-\lambda/\varepsilon} \mathbf{a}(\lambda) \ d\lambda$ is convergent if ε belongs to the Borel-disk centered on d_ϕ, the length of its diameters is equal to R and defined by $\{\varepsilon \ / \ \mathrm{Re}\,(\frac{e^{i\phi}}{\varepsilon}) > 1/R\}$. We recognize the Laplace transform of $\mathbf{a}(\lambda)$: $S(\varepsilon) = \mathcal{L}_{\phi,1}(\mathbf{a})(\varepsilon)$ and $S(\varepsilon) = \sum_{m\geq 0} a_{m+1} \varepsilon^{m+1} = a(\varepsilon)$. Besides, $S(\varepsilon) = \int_{d_\phi} e^{-\lambda/\varepsilon} (\sum_{m\geq 0} a_{m+1} \frac{\lambda^m}{m!}) \ d\lambda$ is the sum of the series $\widehat{a}(\varepsilon)$ by the Borel's exponential method *[Can89, Dum87].*
Thus, the convergent series $\widehat{a}(\varepsilon)$ has an analytic continuation on its Borel star *constitued by the open Borel-disks having 0 on their boundary and if $| \ \mathbf{a}(\lambda) \ | \leq C \ \exp(\gamma | \ \lambda \ |)$ where $1/\gamma > R$ then this Borel-method sums the series $\widehat{a}(\varepsilon)$ for ε in a domain where the series was divergent.*
With the Borel and Laplace transforms of level k, we can do more: let us consider the Mittag-Leffler's star[11] *of $\widehat{a}(\varepsilon)$, we can compute the sum of $\widehat{a}(\varepsilon)$ in its Mittag-Leffler's star with the operators $\mathcal{L}_{\phi,k}$ and $\widehat{\mathcal{B}}_k$. Moreover, we have:*

Proposition 2.1 *[Bor28] Let $\widehat{a}(\varepsilon)$ be a convergent series. If ε_0, $\varepsilon_0 \neq 0$ is a fixed point of the Mittag-Leffler's star of $\widehat{a}(\varepsilon)$ then there exists a real k^* such that for all $k > k^*$, the Borel-domain T_k contains ε_0 and is included in its star. Then we can compute $a(\varepsilon_0)$ with the Borel -Laplace method of all levels $k > k^*$. Let ϕ be the argument of ε_0, $\mathcal{L}_{\phi,k}(\widehat{\mathcal{B}}_k)(\widehat{a})(\varepsilon)$ converges and is equal to $a(\varepsilon_0)$.*

Let $k^* < k_1 < k_2$. If $\widehat{\mathcal{B}}_{k_2}(\widehat{f})(\lambda)$ is majorized by $\exp(\gamma | \ \lambda \ |^{k_2})$ in the direction d_ϕ (resp. $\widehat{\mathcal{B}}_{k_1}(\widehat{f})(\lambda)$ is majorized by $\exp(\gamma | \ \lambda \ |^{k_1})$ on d_ϕ), its Laplace transform defines a function f_2 on the "tear" T_{k_2} with opening π/k_2, and length $(1/\gamma)^{1/k_2}$ (resp. it defines a function f_1 on the "tear" T_{k_1} with opening π/k_1, and length $(1/\gamma)^{1/k_1}$). If $\gamma > 1$ then the domain T_{k_2} is more EFFILE than the domain T_{k_1} ($1/k_2 < 1/k_1$ and $(1/\gamma)^{1/k_2} > (1/\gamma)^{1/k_1}$): XXXX"Plus on veut voir f loin de 0 in the direction d_ϕ, plus on restreint la vision angulaire" [MR88]XXXX. Nevertheless, we can observe numerically that if k is too large (k is chosen such that $\varepsilon_0 \in T_k$), the domain T_k is XXXX encore plus effilé,XXXX and the numerical convergence of the Borel-Laplace process of level k is less good: we must be careful for the choice of k.

Definition 2.2 *Let k be a positive real and let d_ϕ be a direction. A series $\widehat{f} \in \mathbb{C}[[\varepsilon]]$ is* Borel-Laplace-summable *of level k in the direction d_ϕ if the series is Gevrey of order $1/k$ and if the sum of the convergent series $\widehat{\mathcal{B}}_k(\widehat{f})(\lambda)$ has an analytic continuation $\mathbf{f}(\lambda)$ that is*

[11] The Mittag-Leffler's star of $\widehat{a}(\varepsilon)$ is an open maximal star-domain (with respect to the origin) including the disk of convergence of the series. For example, the star- domain of $\sum_{m\geq 0} x^m$ is equal to $\mathbb{C}\backslash[1, \infty[$.

holomorphic and has an exponential increasing of order at most k at the infinite on an open sector V in the neighbourhood of d_ϕ.

Analyticity domain of $\mathcal{L}_{\phi,k}(\mathbf{a})(\varepsilon)$	$\mid \mathbf{a}(\lambda) \mid \leq K \ \exp(\gamma \mid \lambda \mid^k), \ \forall \ \lambda \in d_\phi$
ε-plane	*λ-plane*

figure 2.1

In these conditions, we say that

$$f(\varepsilon) = \mathcal{L}_{\phi,k}(\mathbf{f})(\varepsilon) = k\int_{d_\phi} e^{-\lambda^k/\varepsilon^k}\ \mathbf{f}(\lambda)\ \frac{\lambda^{k-1}}{\varepsilon^{k-1}} d\lambda \tag{2.2}$$

is the sum of $\widehat{f}$ in the direction d_ϕ *in the Borel-Laplace sense.*

Remark 2.3 *We can also define* $f(\varepsilon)$ *with* $\widehat{\mathcal{B}}_1$*, with the Laplace transform of level* 1 *and with the* ramification operators ρ_k *and* ρ_k^{-1} *(see the example 2.1).*

XXXNous allons voir que cette definition est équivalente à la definition d'une series k-*summable*.XXXX

2.2 k-summability.

In the late 1970s, J.-P. Ramis introduce the notion of k-summability [Ram78, Ram80] of a formal series, with a geometric approach of the ideas of E. Borel [Bor99] , E. Leroy [Ler00], G.N. Watson [Wat12] and F. Nevanlinna [Nev19].

Definition 2.3 *Let* k *be a positive real and let* d_ϕ *be a direction. A formal series* $\widehat{f} \in \mathbb{C}[[\varepsilon]]$ *is said to be* k-summable in the direction d_ϕ *if there exists an holomorphic function* f *on a sector* S*, bissected by* d_ϕ*, with opening* $> \pi/k$ *and* f *has* $\widehat{f}$ *as asymptotic expansion with Gevrey estimates of order* $1/k$ *on* S*.*

In these conditions, $\widehat{f}$ is a Gevrey series of order $1/k$ and the *sum* f is unique (see Watson's theorem) [Wat12]. We will say that f is the *sum of* $\widehat{f}$ *in the direction* d_ϕ in the sense of the k-summability.
This notion of k-summability is easily *verified* with the Ramis-Sibuya's theorem [RSi89] as we will see it in the subsection 2.3 On the opposite, this notion is not a *computational* method but is equivalent to the definition 2.2.

Proposition 2.2 *Let* $k > 0$ *and let* d_ϕ *be a direction. Let* $\widehat{f}(\varepsilon) \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k}}$*. Then the series* $\widehat{f}$ *is summable by the Borel-Laplace method of level* k *in the direction* d_ϕ *if and only if* $\widehat{f}$ *is* k*-summable in the direction* d_ϕ*.*

Proof: Let $k = 1$.
Necessary condition (cf. [Can95]): We suppose that $\widehat{f}$ is summableby the Borel-Laplace method of level 1 in the direction d_ϕ. Then the function f defined by 2.2 has an analytic continuation on a sector S with opening $> \pi$:
Let $V = \{\lambda \ / \ \phi_1 < arg\ \lambda < \phi_2\}$. We consider the directions $d_\psi,\ \phi_1 < \psi < \phi_2$. Then $f(\varepsilon)$ has an analytic continuation on the union of the Borel-disks $D_\psi,\ \phi_1 < \psi < \phi_2$. Then there exists a sector S included in the union of the disks $D_\psi,\ \phi_1 < \psi < \phi_2$ with radius $\rho > 0$ such that $\mid S \mid > \pi$ (see figure 2.2).
The function $f(\varepsilon)$ is unique (by Watson's theorem ([Wat12]) and has $\widehat{f}$ as asymptotic expansion with Gevrey estimates of order 1 on S (f has $\widehat{f}$ as asymptotic expansion with Gevrey

estimates of order 1 on sectors with opening $< \pi$ and we recover S with a finite number of these sectors).
Sufficient condition: we consider the sector $S = S_{R,\phi-\theta/2,\phi+\theta/2}$ where $\theta > \pi$. We apply the Borel transform to the function f to prove the sufficient condition:

$$\mathcal{B}(f)(\lambda) = \mathbf{f}(\lambda) = \frac{1}{2i\pi} \int_{\gamma} f(\varepsilon)\, \frac{e^{\lambda/\varepsilon}}{\varepsilon^2} d\varepsilon$$

We use the result of the proposition 1.13 and we remark that $\widehat{\mathcal{B}}_1(\widehat{f})$ defines an holomorphic function that COINCIDE with $\mathcal{B}(f)$ on a disk. □

Definition 2.4 *Let k be a positive real. A formal series $\widehat{f} \in \mathbb{C}[[\varepsilon]]$ is said to be* k-summable *if the series is k-summable in every direction d except a finite number of directions. These singular directions are called* anti-Stokes lines[12].

ε-plane
sector S of analyticity of f
(the opening of S is $> \pi$)

λ-plane
$| \mathbf{f}(\lambda) | \leq M \exp(B \mid \lambda \mid),\ \forall\ \lambda \in V$

figure 2.2

Example 2.1 *([LR90]) The Leroy's series $\widehat{f}_2(x) = \sum_{m\geq 0}(-1)^m m!\ x^{2m+2}$ is Gevrey of order 1/2. With the change of variables $x = t^{1/2}$, we obtain the Euler's series $\widehat{f}_1(t) = \sum_{m\geq 0}(-1)^m m!\ t^{m+1}$. Then we consider the two* operators of ramification*:*

$$\begin{aligned} \rho_2 : f(x) &\longmapsto \rho_2(f)(t) = f(t^{1/2}) \\ \textit{and}\ \ \rho_{1/2} : f(t) &\longmapsto \rho_{1/2}(f)(x) = f(x^2) \end{aligned}$$

[12] These directions are called *Stokes lines* by some authors: "Half the discontinuity in form occurs on reaching the Stokes ray, and half on leaving it the other side ([Din73]).

Thus, $\rho_2(\widehat{f_2})(x) = \widehat{f_1}(t)$. We compute the Borel transform of the series $\widehat{f_2}(x)$ with the transformation:

$$(\rho_{1/2} \circ \widehat{\mathcal{B}}_1 \circ \rho_2)(\widehat{f_2})(\lambda)$$

As $\widehat{\mathcal{B}}_1(\widehat{f_1})(\xi) = \widehat{\mathcal{B}}_1(\sum_{m\geq 0}(-1)^m m!\ t^{m+1}) = \sum_{m\geq 0}(-1)^m\ \xi^m = \frac{1}{1+\xi}$, we obtain the next result: the Borel transform of the series $\widehat{f_2}(x)$ is a convergent series on $D_1(0)$ and defines the analytic function $\mathbf{f_2}(\lambda) = \rho_{1/2}(\frac{1}{1+\xi}) = \frac{1}{1+\lambda^2}$ in the neighbourhood of $D_1(0)$. It has two poles $-i$, $+i$ and has an analytic continuation, with an exponential increasing of level at most 2 on the sector $\{\lambda\ /\ -\pi/2 < arg\ \lambda < +\pi/2\}$ bissected by $\mathbb{R}_+$: the series $\widehat{f_2}(x)$ is 2-summable in the direction $\mathbb{R}_+$ and in every direction d_θ except for $\theta = -\pi/2$, $\theta = +\pi/2$ ($i\mathbb{R}_+$ and $i\mathbb{R}_-$ are the anti-Stokes lines).
For computing the sum of the series $\widehat{f_2}(x)$, we give a direction d_θ. With the operator ρ_2, this direction becomes the direction $d_{2\theta}$. So we can compute the sum $\widehat{f_1}(t)$ in the open half-plane bissected by $d_{2\theta}$ and we obtain the function: $\mathcal{L}_{2\theta,1} \circ \widehat{\mathcal{B}}_1(\widehat{f_1})(t)$ analytic in the open half-plane bissected by $d_{2\theta}$: $\{t\ /\ 2\theta - \pi/2 < arg\ t < 2\theta + \pi/2\}$. We define the sum of $\widehat{f_2}(x)$ by

$$f_2(x) := \rho_{1/2}[\mathcal{L}_{2\theta,1} \circ \widehat{\mathcal{B}}_1(\widehat{f_1})](x)$$

for x in the quadrant bissected by d_θ whose opening is $\pi/2$. Then when 2θ varies between $-\pi$ and $+\pi$ (as for the Euler's series [Can89]), we obtain a sum of $\widehat{f_2}(x)$ for $x \in S = \{x / -3\pi/4 < arg\ x < 3\pi/4\}$.

Thus, J.-P. Ramis showed the k-summability of formal solutions of generic differential equations [Ram80] and asked for the problem of *multisummability* of formal solutions of all the analytic differential equations. Effectively, all formal solutions of meromorphic linear differential systems are not k-summable (cf. [RSi89], page 90).

Example 2.2 *The series $\widehat{y} = \sum_{m\geq 0}(-1)^m m!\ x^{m+1} + \sum_{m\geq 0}(-1)^m m!\ x^{2m+2}$ is Gevrey of order 1 and is a formal solution of the equation*

$$(\frac{d}{dx})^5\{x^5(2-x)\frac{d^2y}{dx^2} - x^2(-2x^3+5x^2+4)\frac{dy}{dx} - 2(x^2-x+2)y\} = 0$$

but $\widehat{y}$ is not k-summable, $\forall\ k > 0$.

In fact, there are several definitions of the multisummability: those of J. Martinet and J.-P. Ramis [MR91] using the Ecalle approach; in a general context, those of J. Ecalle [Eca93] iterating integral formulae with accelerating nuclears[13] (*accelerating-summability*) and those of B. Malgrange and J.-P. Ramis [MalR92] using the *quasi-functions* and a relative version of the Watson lemma due to B. Malgrange [Mal91].

The $(k_1, ..., k_r)$-*sum* of the formal series $\widehat{f} \in \mathbb{C}[[x]]_{\frac{1}{k_1}}$ is then defined in a unique way by one of the summation process deduced of the three definitions described above except in a finite number of singular directions.
We give an example for one of the previous definitions:

[13]The nuclears generalize the Laplace and Laplace inverse nuclear of the classic Borel-summation.

Example 2.3 *([LR95]) Let $0 < k_1 < k_2$. A series $\widehat{f} \in \mathbb{C}[[\varepsilon]]$ is (k_1, k_2)-summable if:*
(1) the series $(\widehat{\mathcal{B}}_{k_1}\widehat{f})(\lambda_1)$ is convergent (i.e. $\widehat{f} \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k_1}}$),
(2) its sum has an analytic continuation $\mathbf{f}_1$ on an open sector $S_{k_1\theta_1,k_1\theta_2}$ including the direction $d_{k_1\theta}$. This continuation $\mathbf{f}_1$ has an exponential increasing of level $\frac{k_2}{k_2-k_1}$.
(3) We consider the operator $\rho_{(k_1,k_2);k_1\theta}$ called accelerating operator of power k_2/k_1 in the direction $k_1\theta$*:*

$$\rho_{(k_1,k_2);k_1\theta}\ (\phi)(\tau) := \frac{2i\pi}{\tau}\int_{d_{k_1\theta}} \mathcal{C}_{\frac{k_2}{k_1}}\ (\frac{\xi}{\tau^{k_1/k_2}})\phi(\xi)\ d\xi$$

where $\mathcal{C}_a(\xi) := \frac{1}{2i\pi}\int_{\mathcal{H}} e^{u-u^{1/a}\xi}\ du$ with $\mathcal{H}$ Hankel curve around $\mathbb{R}_-$ (as a is a rationnel number, the nuclear $\mathcal{C}_a$ is a Meijer function G).
Let $\mathbf{f}_2 := \rho_{(k_1,k_2);k_1\theta}\ (\mathbf{f}_1)$; this function is defined and has an an exponential increasing of level 1 on a sector $S_{k_2\theta_1,k_2\theta_2}$.
(4) Its Laplace transform, written as a fonction of ε, is the sum.
For example, the series $\sum_{m\geq 0}(-1)^m m!\ x^{m+1} + \sum_{m\geq 0}(-1)^m m!\ x^{2m+2}$ is $(1,2)$-summable.

2.3 Ramis-Sibuya's theorem

In order to improve various results of the Poincaré asymptotics by means of the Gevrey asymptotics, the following theorem (*Ramis-Sibuya's theorem*) is fundamental. This theorem claims that the characterization of flatness also characterizes the Gevrey asymptotics itself. As we will see in two examples, it can be employed in problems concerning singularly perturbed differential equations.

Results We have a first result:

Lemma 2.3 *Let $k > 0$, S an open sector whose vertex is at the origin and let $f : S \to \mathbb{C}$, analytic in ε, for $\varepsilon \in S$. We suppose that f admits $\widehat{f}$ as asymptotic expansion of Gevrey order $1/k$ and type A. Let $\tilde{A} > A$, then there exists a good covering $\{S_1, ..., S_m\}$ of $D^* = \{\varepsilon \in \mathbb{C},\ 0 <\mid \varepsilon \mid< r\}$ and there exist m functions $f_1, ..., f_m$ such that*
i) $f_j,\ j = 1...m$, are analytic and bounded on S_j,
ii) $S = S_1,\ f = f_1$,
iii) the functions f_j have $\widehat{f}$ as asymptotic expansion of Gevrey order $1/k$ and type B, $A < B < \tilde{A}$,
iv) we have

$$\forall\ \varepsilon \in S_j \cap S_{j+1},\quad \mid f_j(\varepsilon) - f_{j+1}(\varepsilon) \mid \leq C\ \exp(\frac{-1}{\tilde{A}\mid \varepsilon \mid^k}),\ \forall\ j = 1, ..., m \text{ where } C > 0$$

The functions $\{f_j\}_{j=1,...,m}$ make an *holomorphic* 0-*cochain* and the differences $\{f_{i,j} := f_i - f_j\}_{i,j=1,...,m}$ make a *cocycle.*

Proof: We have $f \in \mathcal{A}_{\frac{1}{k},A}(S)$, so $J(f) = \widehat{f}$ is a Gevrey series of order $1/k$ and type A. Therefore, $\widehat{\mathcal{B}}_k(\widehat{f})$ is a convergent series on $\bar{D}_R(0)$, $R > 0$: this convergent series defines an analytic function $\mathbf{f}(\lambda)$ in the neighbourhood of $\bar{D}_R(0)$. We assume that the sector S is bissected by the direction d. Let B a positive real such that $A < B < \tilde{A}$. We construct a good covering $\{S_1, ..., S_m\}$ of the punctured disk in the ε-plane $D^* = \{\varepsilon \in \mathbb{C},\ 0 <\mid \varepsilon \mid< \rho\}$ with $S_1 = S$ and $\forall\ j,\ j = 2, ..., m,\ \mid S_j \mid= 2\theta < \pi/k$ where $\frac{A}{(cos\ k\theta)^k} = B$. Let d_j the half-line bissecting the sector S_j for $j = 2, ..., m$ $(d_1 = d)$. Let $0 < r < R$; we define, for $\varepsilon \in S_j$

$$f_j(\varepsilon) := k \int_{d_{j,r}} e^{-\lambda^k/\varepsilon^k}\ \mathbf{f}(\lambda)\ \frac{\lambda^{k-1}}{\varepsilon^{k-1}}\ d\lambda$$

where $d_{j,r}$ is the line-segment $[0, r] \subset d_j$ in the λ-plane. The functions f_j are analytic in ε for $\varepsilon \in S_j$. Utilizing the Gevrey Borel-Ritt's theorem, ($\forall\ j,\ j = 2, ..., m,\ \mid S_j \mid= 2\theta < \pi/k$ and the map J is surjective), these functions have $\widehat{f}$ as asymptotic expansion of Gevrey order $1/k$ and type $\frac{A}{(cos\ k\theta)^k} = B$. Thus, on $S_j \cap S_{j+1}$

$$f_j - f_{j+1} \in Ker(J) = \mathcal{A}^{\leq -k}_{\frac{A}{(cos\ k\theta)^k}}(S_j \cap S_{j+1})$$

i.e. there exist a real β, $\beta \leq 0$ and a positive real C such that

$$\forall\ \varepsilon\ \in S_j \cap S_{j+1},\quad \mid f_j(\varepsilon) - f_{j+1}(\varepsilon) \mid\leq C\ \ \mid \varepsilon \mid^\beta\ \exp(\frac{-1}{B \mid \varepsilon \mid^k}) \leq \bar{C}\ \exp(\frac{-1}{\tilde{A} \mid \varepsilon \mid^k})$$

thus, the lemma is proved. □

Remark 2.4 *We can* directly *majorize the differences* $\mid f_{j+1} - f_j \mid$ *on* $S_{j+1} \cap S_j$ *for* $j = 2, ..., m-1$.

We have a similar result with only a Gevrey series of order $1/k$ and type A:

Lemma 2.4 *Let* $k > 0$ *and let* $\widehat{f}(\varepsilon)$ *a Gevrey series of order* $1/k$ *and type* A. *Let* $\tilde{A} > A$, *then there exists a good covering* $\{S_1, ..., S_m\}$ *of* $D^* = \{\varepsilon \in \mathbb{C},\ 0 <\mid \varepsilon \mid< r\}$ *and there exist* m *functions* $f_1, ..., f_m$ *such that*
i) $\mid S_j \mid= 2\theta < \pi/k$ *for* $j = 1, ..., m$ *with* $\frac{A}{(cos\ k\theta)^k} < \tilde{A}$,
ii) $f_j,\ j = 1...m$, *are analytic and bounded on* S_j,
iii) the functions f_j *have* $\widehat{f}$ *as asymptotic expansion of Gevrey order* $1/k$ *and type* B, $A < B < \tilde{A}$,
iv) we have

$$\forall\ \varepsilon \in S_j \cap S_{j+1},\quad \mid f_j(\varepsilon) - f_{j+1}(\varepsilon) \mid\leq C\ \exp(\frac{-1}{\tilde{A}\mid \varepsilon \mid^k}),\ \forall\ j = 1, ..., m \text{ where } C > 0$$

Proof: The proof is the same as below. □

Example 2.4 *The Leroy series,* $\widehat{f_2}(x) = \sum_{m \geq 0}(-1)^m m!\ x^{2m+2}$, *is a Gevrey series of order* $1/2$ *and type* 1. *Let* d_θ *be a direction. When* 2θ *is varying between* $-\pi$ *and* $+\pi$, *we obtain a sum* $f_2(x)$ *of the series* $\widehat{f_2}(x)$ *for* $x \in S = \{x\ /\ -3\pi/4 < arg\ x < 3\pi/4\}$. *We obtain a new sum* $\tilde{f_2}(x)$ *of this series, for* $x \in \tilde{S} = \{x\ /\ \frac{\pi}{4} < arg\ x < \frac{7\pi}{4}\}$, *when* 2θ *is varying between* π *and* 3π. *These two sums are asymptotic to* $\widehat{f_2}(x)$ *with Gevrey estimates of order* $1/2$ *and* *type* 1 *on their one sector of definition; however, on the intersection of these two sectors, there is not unicity:*

$S \cap \tilde{S} = S_1 \cup S_2$ *avec* $S_1 = \{x \ / \ \pi/4 < arg\ x < 3\pi/4\}$ *and* $S_2 = \{x \ / \ -3\pi/4 < arg\ x < 7\pi/4\}$

On S_1 *(and on* S_2*),*

$$f_2(x) - \tilde{f}_2(x) = 2i\pi \ e^{1/x^2}$$

and we remark that this difference is exponentially flat[14] *of order* 2.

Definition 2.5 *[Ram93] Let* S *be an open sector whose vertex is at the origin and let* $k > 0$. *We call* quasi-function k-precise *on the sector* S*, an holomorphic* 0*-cochain* $\{f_i\}_{i=1,...,n}$ *associated to a covering* $\{S_i\}_{i=1,...,n}$ *of* S*, the functions* $f_{i+1} - f_i$ *having an exponential decay of level* k *on* $S_{i+1} \cap S_i$.

We identify $(\{f_i\}_i, \ \{S_i\}_i)$ and $(\{g_j\}_j, \ \{T_j\}_j)$ if $\forall \ i, \ j$ such that $S_i \cap T_j \neq \emptyset$, then $f_i - g_j$ has an exponential decay of level k on $S_i \cap T_j$.

Remark 2.5 *So we have associated a Gevrey series of order* $1/k$ *and a unique quasi-function* k*-precise* $(\{f_i\}_i, \ \{S_i\}_i)$*, modulus the identification made below (*D^* *must be covered by sectors whose opening is* $< \pi/k$*). We call it* its quasi-sum.

XXXXThe previous lemma has a converse: This converse, qui s'énonce de façon équivalente avec des faisceaux quotients, est un résultat "clé" dans les problèmes d'équations différentielles, in the mesure where effectuer une différence linéarise ou simplifie le problème.XXXX

Theorem 2.5 Ramis-Sibuya's theorem *([Ram78, Sib81]).*
Assume that $\{S_1, ..., S_m\}$ *is a good covering of the punctured disk* $D^* = \{\varepsilon \in \mathbb{C}, \ 0 <\mid \varepsilon \mid< r\}$ *and that functions* $f_1(\varepsilon), f_2(\varepsilon), ..., f_m(\varepsilon)$*, satisfy the following conditions:*
i) $f_j : S_j \rightarrow \mathbb{C}$ *is holomorphic and bounded on* S_j*,*
ii) We have

$$\forall \ \varepsilon \in S_j \cap S_{j+1}, \quad \mid f_j(\varepsilon) - f_{j+1}(\varepsilon) \mid = \mathcal{O}(\exp(\frac{-1}{A \mid \varepsilon \mid^k}))$$

where $A > 0$, $k > 0$ *independent of* j *and* $S_{m+1} = S_1$, $f_{m+1} = f_1$.
Then there exists a formal power series $\widehat{f}(\varepsilon) \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k},A}$ *and the functions* f_j *have* $\widehat{f}(\varepsilon)$ *as the same asymptotic expansion with Gevrey estimates of order* $1/k$ *and type* A *on* S_j.

Remark 2.6 *We don't assume that the functions* f_j *have an asymptotic expansion on* S_j*. The Cauchy-Heine's formula gives the existence of the asymptotic expansion*[15]*).*

Sketch of proof: ([Sib90-1])
We use the *Cauchy-Heine's formula*:
For $k = 1, ..., m$, we consider $\varepsilon_k \in S_k \cap S_{k+1}$, $\mid \varepsilon_k \mid = r' < r$. We denote by $\int_{\varepsilon_{k-1}}^{\varepsilon_k}$ the integral

[14](because $\mid e^{1/x^2} \mid = e^{\frac{cos(2arg\ x)}{|x|^2}}$ and, on S_1 and S_2, $cos(2arg\ x) < 0$.)
[15]See the Principe des singularités inexistantes de Riemann: an holomorphic function, bounded on D^*, is the sum of a convergent series.

on the circular arc joining ε_{k-1} to ε_k and $\int_0^{\varepsilon_k}$ the integral on the radius $[0,\varepsilon_k]$.
For $j \in \{1,...,m\}$, let $\varepsilon \in S_j$ such that $arg\ \varepsilon_{j-1} < arg\ \varepsilon < arg\ \varepsilon_j$, then

$$f_j(\varepsilon) = \frac{1}{2i\pi}\sum_{k=1}^{m}\int_{\varepsilon_{k-1}}^{\varepsilon_k}\frac{f_k(\xi)}{\xi-\varepsilon}\,d\xi + \frac{1}{2i\pi}\sum_{k=1}^{m}\int_0^{\varepsilon_k}\frac{f_{k+1}(\xi)-f_k(\xi)}{\xi-\varepsilon}\,d\xi$$

We remark that

$$\frac{1}{\xi-\varepsilon} = \sum_{n=1}^{N-1}\frac{\varepsilon^n}{\xi^{n+1}} + \frac{\varepsilon^N}{\xi^{N+1}} \times \frac{1}{1-\varepsilon/\xi}\ ,$$

therefore, the coefficients b_n of the asymptotic expansion $\widehat{f}(\varepsilon) = \sum_{n\geq 0} b_n\varepsilon^n$ of the functions f_j are given by

$$b_n = \frac{1}{2i\pi}\sum_{k=1}^{m}\int_{\varepsilon_{k-1}}^{\varepsilon_k}\frac{f_k(\xi)}{\xi^{n+1}}\,d\xi + \frac{1}{2i\pi}\sum_{k=1}^{m}\int_0^{\varepsilon_k}\frac{f_{k+1}(\xi)-f_k(\xi)}{\xi^{n+1}}\,d\xi \quad \forall\ n \geq 0$$

and we can estimate, in a Gevrey sense of order $1/k$ and type A, the difference $\mid f_j(\varepsilon) - \sum_{n=0}^{N-1} b_n\varepsilon^n \mid$ for $\varepsilon \in S_j$. □

Remark 2.7 *We can also prove this theorem with the next lemma ([Sib92], p. 210–212, [Mal95], p. 176–177):*

Lemma 2.6 *Let $k > 0$ and let $\{S_1,...,S_m\}$ a good covering of $D^* = \{\varepsilon \in \mathbb{C},\ 0 <\mid \varepsilon \mid< r\}$. We consider m functions $\delta_j : S_j \cap S_{j+1} \to \mathbb{C}$ ($S_{m+1} = S_1$ and $f_{m+1} = f_1$) such that*
1) the functions $\delta_j(\varepsilon)$ are holomorphic on $S_j \cap S_{j+1}$,
2) $\forall\ j,\ j = 1,...,m,\ \forall\ \varepsilon \in S_j \cap S_{j+1}$,

$$\mid \delta_j(\varepsilon) - \delta_{j+1}(\varepsilon) \mid = \mathcal{O}(\exp(\frac{-1}{A\mid \varepsilon \mid^k}))\ where\ A > 0$$

Then there exist m functions $\psi_1,...,\psi_m$ such that $\forall\ j,\ j = 1,...,m$, ψ_j holomorphic and bounded on S_j, $\psi_{m+1} = \psi_1$, $\delta_j = \psi_j - \psi_{j+1}$ on $S_j \cap S_{j+1}$ and $\psi_j \in \mathcal{A}_{\frac{1}{k},A}(S_j)$.

Corollary 2.7 *([Sib90-1]) Let $k > 0$ and let $\{S_1,...,S_m\}$ a good covering of $D^* = \{\varepsilon \in \mathbb{C},\ 0 <\mid \varepsilon \mid< r\}$. We consider m analytic and flat functions $f_j : S_j \to \mathbb{C}$ such that, $\forall\ j,\ j = 1,...,m$:*

$$\forall\ \varepsilon \in S_j \cap S_{j+1},\quad \mid f_j(\varepsilon) - f_{j+1}(\varepsilon) \mid = \mathcal{O}(\exp(\frac{-1}{A\mid \varepsilon \mid^k}))\ where\ A > 0$$

($S_{m+1} = S_1$ and $f_{m+1} = f_1$).
Then, $\forall\ j,\ j = 1,...,m$, the functions f_j are flat in the Gevrey sense of order $1/k$ and type A:

$$\forall\ \varepsilon \in S_j,\quad \mid f_j(\varepsilon) \mid = \mathcal{O}(\exp(\frac{-1}{A\mid \varepsilon \mid^k}))$$

Proof: Utilizing the Ramis-Sibuya's theorem, we have $f_j \in \mathcal{A}_{\frac{1}{k}}(S_j)$ with some type $\leq A$ and $\widehat{f_j} = 0 + 0\varepsilon + ...$
Thus $f_j \in \mathcal{A}^{\leq -k}(S_j)$ with a type $\leq A$ too (and if $\mid S_j \mid> \pi/k$ then $f_j \equiv 0$). □

Remark 2.8 *With the Ramis-Sibuya's theorem, we can obtain the k-summability of the series $\hat{f}(\varepsilon)$ if we add conditions on the opening of the sectors. Effectively, if we suppose that the functions $f_j - f_{j+1}$ have an exponential decay of level k on a sector with opening $\geq \pi/k$ then the series $\hat{f}(\varepsilon)$ is k-summable. In the example XXXX about the Leroy's series, this last one has two sums f_2 and $\tilde{f}_2$ that have an exponential decay of level 2 on $S_1 \cup S_2$ where $\mid S_1 \mid = \mid S_2 \mid = \pi/2$: $\widehat{f_2}$ is 2-summable.*
Therefore, the k-summability defined by definition XXXX is easily verified: we consider the cocycle $\{f_{j,j+1}\}_{j=1,...,m}$ and we REGARDE if the functions $f_{j,j+1}$ have an exponential decay decay of level k on maximal *sectors (i.e. sectors with opening $\geq \pi/k$).*

Applications Here are two applications of the Ramis-Sibuya's theorem[16].

1) A new proof of proposition 1.4:

Proposition 2.8 *([Mal95]) Let $\Phi(\varepsilon, x_1, ..., x_p)$ an holomorphic function in a neighbourhood $D = D_{\varepsilon_0}(0) \times D_{r_1}(0) \times ... \times D_{r_p}(0)$ of $0 \in \mathbb{C}^{p+1}$ and let $\widehat{u}_1(\varepsilon), ..., \widehat{u}_p(\varepsilon)$, p formal series, Gevrey of order $1/k$ and type A such*[17] *that $\widehat{u}_1(0) = ... = \widehat{u}_p(0) = 0$. Let $\tilde{A}, A$ two positive reals such that $\tilde{A} > A$. Then $\widehat{f}(\varepsilon) = \Phi(\varepsilon, \widehat{u}_1(\varepsilon), ..., \widehat{u}_p(\varepsilon)) \in \mathbb{C}[[\varepsilon]]$ is a Gevrey series of order $1/k$ and type $\tilde{A}$.*

Proof: In the case $p = 2$, let $\widehat{f}(\varepsilon) = \Phi(\varepsilon, \widehat{u}(\varepsilon), \widehat{u}(\varepsilon))$. Let $\tilde{A} > A$ and let $\{U_i\}_{i=1,...,m}$ a good covering of $D_{\varepsilon_0}(0)$ such that, for $i = 1, ..., m$, $\mid U_i \mid = 2\theta < \pi/k$ where θ verifies $A < \frac{A}{(\cos\ k\theta)^k} < \tilde{A}$. As the formal series $\widehat{u}(\varepsilon)$ (resp. $\widehat{v}(\varepsilon)$) is Gevrey of order $1/k$ and type A and the opening of $U_i < \pi/k$, we can represent this series on each sector by a function $u_i \in \mathcal{A}_{\frac{1}{k}}(U_i)$ with type $\frac{A}{(\cos\ k\theta)^k}$ (resp. $v_i \in \mathcal{A}_{\frac{1}{k}}(U_i)$ with type $\frac{A}{(\cos\ k\theta)^k}$). The lemma 2.3 implies that the 0-cochains $\{u_i\}_i$ and $\{v_i\}_i$ satisfy:

$$\forall\, \varepsilon \in U_i \cap U_{i+1}, \quad \mid u_i(\varepsilon) - u_{i+1}(\varepsilon) \mid \leq C\ \exp(\frac{-1}{\tilde{A} \mid \varepsilon \mid^k}),\ \textit{where } C > 0$$

$$\forall\, \varepsilon \in U_i \cap U_{i+1}, \quad \mid v_i(\varepsilon) - v_{i+1}(\varepsilon) \mid \leq \bar{C}\ \exp(\frac{-1}{\tilde{A} \mid \varepsilon \mid^k}),\ \textit{where } \bar{C} > 0$$

In the same way, $\Phi(., \widehat{u}, \widehat{v})$ is represented by $\Phi(., u_i, v_i)$ on each sector U_i. This last function belongs to $\mathcal{A}(U_i)$ and has $\Phi(\varepsilon, \widehat{u}(\varepsilon), \widehat{v}(\varepsilon))$ as asymptotic expansion (the series is obtained by substitution). We have to show that $\Phi(\varepsilon, u_i, v_i) - \Phi(\varepsilon, u_{i+1}, v_{i+1})$ has an exponential decay of order k and type $\tilde{A}$ on $U_i \cap U_{i+1}$.
The functions $\delta_{i,i+1} = u_i - u_{i+1}$ and $\mu_{i,i+1} = v_i - v_{i+1}$ have an exponential decay of order k and type $\tilde{A}$ on this set. Besides, we have the result (see [Mal95]):

Lemma 2.9 *Let Φ an holomorphic function in a neighbourhood of $0 \in \mathbb{C}^3$, then*

$$\Phi(\varepsilon, y_1 + z_1, y_2 + z_2) - \Phi(\varepsilon, y_1, y_2) = z_1\ \nu_1(\varepsilon, y_1, y_2, z_1, z_2) + z_2\ \nu_2(\varepsilon, y_1, y_2, z_1, z_2)$$

The functions ν_1 and ν_2 are holomorphic in the neighbourhood of 0.

[16]See [Mal95] for additional results about summability.
[17]We can replace this condition by the condition $(0, \widehat{u}_1(0), ..., \widehat{u}_p(0)) \in D$.

So,

$$\begin{aligned} \Phi(\varepsilon, u_i, v_i) - \Phi(\varepsilon, u_{i+1}, v_{i+1}) &= \\ \Phi(\varepsilon, u_{i+1} + \delta_{i,i+1}, v_{i+1} + \mu_{i,i+1}) - \Phi(\varepsilon, u_{i+1}, v_{i+1}) &= \\ \delta_{i,i+1}\ \nu_1(\varepsilon, u_{i+1}, v_{i+1}, \delta_{i,i+1}, \mu_{i,i+1}) &+ \mu_{i,i+1}\ \nu_2(\varepsilon, u_{i+1}, v_{i+1}, \delta_{i,i+1}, \mu_{i,i+1}) \end{aligned}$$

where ν_1 and ν_2 are holomorphic.
Thus, the left hand-side has an exponential decay of order k and type $\tilde{A}$. We apply the Ramis-Sibuya's theorem and we conclude

$$\forall\ i, i = 1, ..., m, \quad \Phi(\varepsilon, u_i, vi) \in \mathcal{A}_{\frac{1}{k}}(U_i) \textit{ and type } \tilde{A}$$

and $\hat{f}(\varepsilon) = \Phi(\varepsilon, \widehat{u}(\varepsilon), \widehat{v}(\varepsilon)) \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k}, \tilde{A}}$. □

Remark 2.9 *In the case when $\Phi(\varepsilon, x_1, ..., x_p)$ is an holomorphic function on $V \times D_{r_1}(0) \times ... \times D_{r_p}(0)$, having an asymptotic expansion $\hat{\Phi}(\varepsilon)$ in $\mathbb{C}\{x_1, ..., x_p\}[[\varepsilon]]$ with Gevrey estimates of order $1/k$ and type N on an open sector V whose vertex is at the origin, uniformly in $(x_1, ..., x_p)$, the proposition below is still true:*

Proposition 2.10 *Let $\Phi(\varepsilon, x_1, ..., x_p)$ be an holomorphic function on $V \times D_{r_1}(0) \times ... \times D_{r_p}(0)$, having an asymptotic expansion $\hat{\Phi}(\varepsilon)$ in $\mathbb{C}\{x_1, ..., x_p\}[[\varepsilon]]$ with Gevrey estimates of order $1/k$ and type N on an open sector V whose vertex is at the origin, uniformly in $(x_1, ..., x_p)$. If $u_1(\varepsilon), ..., u_p(\varepsilon)$ are analytic functions on V having $\widehat{u}_1(\varepsilon), ..., \widehat{u}_p(\varepsilon)$ as asymptotic expansion on V with Gevrey estimates of order $1/k$ and type A with $\widehat{u}_1(0) = ... = \widehat{u}_p(0) = 0$ then the function $f(\varepsilon) = \Phi(\varepsilon, u_1(\varepsilon), ..., u_p(\varepsilon))$ is an analytic function on V having $\hat{\Phi}(\varepsilon, \widehat{u}_1(\varepsilon), ..., \widehat{u}_p(\varepsilon))$ as asymptotic expansion on V with Gevrey estimates of order $1/k$ and type T, where $T > Max\{A, N\}$.*

Remark 2.10 *We can also prove the next result about Gevrey functions.*

Proposition 2.11 *Let $\Phi(z)$ be an analytic function on a sector $V \in \mathbb{C}$ having an asymptotic expansion $\widehat{\Phi}(z)$ on V with Gevrey estimates of order $1/k$ and let $u(\varepsilon)$ an analytic function on a sector U having $\widehat{u}(\varepsilon)$ as asymptotic expansion on U with Gevrey estimates of the same order $1/k$. Moreover, we suppose $\widehat{u} = \varepsilon + ...$ and $\widehat{\Phi}(0) = 0$. Then the function $\Phi \circ u(\varepsilon)$ is an analytic function on a subsector of U and $\Phi \circ u(\varepsilon)$ has $\widehat{\Phi}(\widehat{u}) \in \mathbb{C}[[\varepsilon]]_{\frac{1}{k}}$ as asymptotic expansion with Gevrey estimates of order $1/k$.*

This result was proved in the first time by E. Gevrey [Gev18] in the $\mathcal{C}^\infty$ case.

2) Gevrey solution of a singularly perturbed differential equation: the invertible case

We consider a system of singularly perturbed differential equations

$$\varepsilon^\sigma \frac{dy}{dx} = f(x, \varepsilon) + A(x, \varepsilon)y + \sum_{|p| \geq 2} f_p(x, \varepsilon) y^{\mathbf{p}} \tag{2.3}$$

where $x \in \mathbb{C},\ y \in \mathbb{C}^n,\ \varepsilon$ is a complex parameter and σ is a positive integer.
The functions f and $f_p:\ \mathbb{C}^2 \to \mathbb{C}^n$ are holomorphic on $D_{r_0}(0) \times D_{r_1}(0)$.

The coefficients of the matrix $A(x,\varepsilon)$, $n\times n$ are holomorphic on $D_{r_0}(0)\times D_{r_1}(0)$.
We denote by $\mid p\mid = p_1+...+p_n$ and $y^{\mathbf{p}} = y_1^{p_1}...y_n^{p_n}$. The series $\sum_{|p|\geq 2} f_p(x,\varepsilon)y^{\mathbf{p}}$ is uniformly convergent on each compact subset of $D_{r_0}(0)\times D_{r_1}(0)$ and $\parallel y\parallel = \ Max_{j=1}^{j=n}\mid y_j\mid < r_2$.
Moreover, we suppose $f(0,0)=0$ and $A(0,0)$ invertible.
Then we have the next result:

Theorem 2.12 *([Sib90-2]) Under the hypothesis below, the system (2.3) admits a unique formal solution* $\widehat{y}(x,\varepsilon) = \sum_{m\geq 1} y_m(x)\varepsilon^m$ *with coefficients* $y_m(x)$ *holomorphic on* $D_{r_0}(0)$*. Moreover, there exist four positive numbers* $\delta,\ r,\ \alpha,\ T$ *such that (2.3) admits a solution* $y(x,\varepsilon)$ *holomorphic on* $D_\delta(0)$, $\varepsilon\in S_{r,-\alpha,\alpha}$ *and such that* $y(x,\varepsilon)$ *has* $\widehat{y}(x,\varepsilon)$ *as asymptotic expansion with Gevrey estimates of order* $1/\sigma$ *and type* T *as* $\varepsilon\longrightarrow 0$, $\varepsilon\in S_{r,-\alpha,\alpha}$*, uniformly in* $x\in D_\delta(0)$*.*

The series $\widehat{y}(x,\varepsilon)$ is then Gevrey of order $1/\sigma$ and type T, uniformly in x (i.e. $\widehat{y}(x,\varepsilon)\in\mathbb{C}\{x\}[[\varepsilon]]_{\frac{1}{\sigma},T}$).
Here are the main steps of the proof of theorem 2.12 (see [Sib90-2]):
- The system (2.3) admits a unique formal solution $\widehat{y}(x,\varepsilon)$.
- For each direction θ in the ε-plane, there exists a sector, denoted by S_θ, $S_\theta = \{\varepsilon\ /\ \mid arg\ \varepsilon-\theta\mid<\alpha(\theta),\ 0<\mid\varepsilon\mid<\omega(\theta)\}$ and there exists a disk $\Delta_\theta = \{x/\mid x\mid< r(\theta)\}$ such that (2.3) has a solution $\phi(x,\varepsilon;\theta)$, holomorphic on $\Delta_\theta\times S_\theta$ having $\widehat{y}(x,\varepsilon)$ as asymptotic expansion in the Poincaré sense [Sib58].
- We choose $\theta_1,...,\theta_N$ such that $S_1,...,S_N$ is a good covering of $\{\varepsilon\ /\ 0<\mid\varepsilon\mid<\rho_0\}$ with $0<\rho_0\leq Min_{j=1}^{j=N}\omega(\theta_j)$.
We denote $\phi_j(x,\varepsilon) := \phi(x,\varepsilon;\theta_j)$ and $D_0 = \{x\in\mathbb{R}/\mid x\mid< r\leq Min_{j=1}^{j=N} r(\theta_j)\}$,
- We have:

$$\mid(\phi_{j+1}-\phi_j)(x,\varepsilon)\mid\leq\gamma\ \exp(-\frac{1}{T\mid\varepsilon\mid^\sigma})\ \ sur D_0\times(S_j\cap S_{j+1})$$

where $\gamma\geq 0,\ T>0$.
Effectively, if we denote $F(x,\varepsilon,y) := \sum_{|p|\geq 2} f_p(x,\varepsilon)y^p$ *and* $\delta_j(x,\varepsilon) := \phi_{j+1}(x,\varepsilon)-\phi_j(x,\varepsilon)$, the function $\delta_j(x,\varepsilon)$ satisfies:

$$\varepsilon^\sigma\frac{d\delta_j(x,\varepsilon)}{dx} = A(x,\varepsilon)\delta_j(x,\varepsilon)+B(x,\varepsilon)\delta_j(x,\varepsilon)$$

where $B(x,\varepsilon)=\int_0^1\frac{\partial F}{\partial y}(x,\varepsilon,\phi_{j+1}(x,\varepsilon)+t\delta_j(x,\varepsilon))dt$.
Thus, $\delta_j(x,\varepsilon)$ is a solution of the linear system

$$\varepsilon^\sigma\frac{dv}{dx} = [A(x,\varepsilon)+B(x,\varepsilon)]\ v$$

where $A(0,0)$ is invertible and $B(0,0)=0$. Moreover, $\widehat{\delta}_j(x,\varepsilon)=0$. We conclude with the bloc-diagonalization theorem of Y. Sibuya [Sib58]. There exist $\gamma\geq 0,\ T>0$ such that

$$\delta_j(x,\varepsilon)\leq\gamma\ \exp(-\frac{1}{T\mid\varepsilon\mid^\sigma})\ \ on D_0\times U_j$$

- We apply the Ramis-Sibuya's theorem to the functions ϕ_j. Thus, the solutions $\phi_j(x,\varepsilon)$ have $\widehat{y}(x,\varepsilon)$ as asymptotic expansion with Gevrey estimates of order $1/\sigma$ and type T and we deduce $\widehat{y}(x,\varepsilon)\in\mathbb{C}\{x\}[[\varepsilon]]_{\frac{1}{\sigma},T}$. □

Remark 2.11 *In the case when $A(0,0)$ is non invertible we use the same method to show the existence of "quasi-solutions" of the equation (see exercises IV about section 3).*

3 Application to Van der Pol equation

We can find a very good dictionnary between solutions and formal solutions of singular differential equations ([RSi89]). This one is based on Asymptotic Expansion Theory in the Poincaré sense ([Poi81]) improved with Gevrey estimates by G.N. Watson ([Wat12]) and F. Nevanlinna ([Nev19]). This last theory, rediscovered by J.-P. Ramis in the seventies and systematically developped since, is named *Gevrey Asymptotic Theory.*

This chapter is more specially concerned with *singularly perturbed differential equations.* These, are differential equations where a small parameter (named ε) multiplies the highest derivative. The link between formal solutions and solutions is partially established for singularly perturbed differential equations but we think that most of the problems implying singular perturbations may be solved with Gevrey Theory.
Let us consider a singularly perturbed differential equation with *turning point* I mean the linear part of the right-hand side of the equation is not invertible[18]. Gevrey Theory consists in studying the existence of formal series solution (the formal series is power series into the parameter ε and the series coefficients are *holomorphic* functions in the variable of derivation).
First we establish the *Gevrey character* of some order[19] of this series.
Then the formal Borel and truncated Laplace transforms ([Bor99]) of the formal series provide " quasi-solutions" i.e. functions that satisfy the differential equation except for an exponentially small error.
On final, we prove the existence of solutions of the singularly perturbed differential equation.

In this chapter, we apply the above described method to the famous Van der Pol equation ([VdP26]).

3.1 The equations

Consider the forced Van der Pol equation

$$\varepsilon\ \ddot{x} + (x^2 - 1)\ \dot{x} + x = a \tag{3.4}$$

where a is a real auxiliary parameter and ε is a fixed infinitely small real number.
On the Liénard plane [Lie28] (x, y) where $y = \varepsilon\ \dot{x} + x^3/3 - x$, (3.4) is equivalent to the system:

$$\left\{ \begin{array}{rcl} \varepsilon\dot{x} & = & y - (x^3/3 - x) \\ \dot{y} & = & a - x \end{array} \right. \tag{3.5}$$

This system is called a *slow-fast system*: the variable y is slow because $\dot{y}$ takes finite values at all finite points of the plane. The variable x is rapid because $\dot{x}$ takes infinite large values at some finite points of the plane.

[18] The case *invertible* has been treated by asymptotic methods ([Sib58, Was65, Sib90-1]).
[19] We often remark the existence of a natural Gevrey frontier in singularly perturbed problems.

We define the *slow-curve* $\mathcal{L}$ as the set of points at which the time derivative of the rapid variable vanishes. For the system (3.5)

$$\mathcal{L} = \{(x, y),\ /y = f(x) = x^3/3 - x\}$$

Here the slow curve is the graph of the cubic function f.
When we study the vector field, as ε is a small parameter, the slow curve presents an attractive part and a repelling one: far from the slow-curve, $\dot{x} = \frac{-x^3/3+x+y}{\varepsilon}$ is infinitely large, so the vector field corresponding to the system (3.5) is almost horizontal; above $\mathcal{L}$ it is directed to the right and below $\mathcal{L}$ to the left (see figure 3.1). It is obvious that the two parts of the graph where f increases are attracting (that is, locally in the neighbourhood of these parts the rapid movement is directed towards them) and the part of the graph where f is decreasing is repelling.
We are specially interested in trajectories that move from the attractive part of the slow curve to the repelling part[20]. We call *canard solution*[21] a solution of (3.5) which has such a behaviour. More precisely, we have to consider the couple: the solution $(x(t), y(t))$ of (3.5) and the value of the parameter a such that this behaviour occurs.

In [BCDD81] the existence of Van der Pol equation's canard solutions is proved and if we consider two canard solutions $((x_1(t), y_1(t)), a_1)$ and $((x_2(t), y_2(t)), a_2)$, then the difference $a_1 - a_2$ is exponentially small with respect to ε. These results lead J.-P. Ramis, in the eighties, to conjecture the *Gevrey character of order* 1 [Ram78, Ram80] of the asymptotic expansion in powers of ε of the parameter a.

figure 3.1: Van der Pol equation canard solution.

In order to apply Gevrey theory, we suppose that now, all variables are complex and we consider the Van der Pol equation as a complex differential equation.
We make the change of variables $(X = x - 1, z = \dot{X})$ in (3.5) and we obtain the following system:

$$\begin{cases} \dot{X} & = & z \\ \varepsilon\, \dot{z} & = & -X((2+X)z+1) + a - 1 \end{cases} \tag{3.6}$$

or

$$\varepsilon\, z\, \frac{dz}{dX} = -X((2+X)z+1) + a - 1 \tag{3.7}$$

where $z = \phi_0(X) = -\frac{1}{2+X}$ is the equation of the new slow curve when $a = 1$.

We want to study the existence of well behaved so called *overstable solutions* of (3.7); these are functions of X and ε that solve (3.7) and tend, as $\varepsilon \to 0$, to the slow curve $\phi_0(X)$ (they remain bounded, as $\varepsilon \to 0$ uniformly in X in a full neighbourhood of $X = 0$).

Definition 3.1 *An* overstable solution *of (3.7) is a couple* $(z^*(X, \varepsilon), a^*(\varepsilon))$ *of functions such that:*

[20] For example in the neighbourhooh of the point $(x_0 = -1, y_0 = 2/3)$ that separates the slow curve in an attractive part and a repelling one.
[21] More generally, we consider a complex differential equation and we are interested in *overstable solutions* ([Wall90],[CRSS00].)

- *$z^*(X,\varepsilon)$ is holomorphic in X in $D_r(0)$, $r>0$,*
- *$z^*(X,\varepsilon)$ and $a^*(\varepsilon)$ are holomorphic in ε on a sector V and satisfy*

$$\varepsilon\, z^* \,\frac{dz^*}{dX} = -X(2+X)(z^* + \frac{1}{2+X}) + a^* - 1$$

  for $X \in D_{r'}(0),\ r' < r,$ and $\varepsilon \in V$,
- *$a^*(\varepsilon) \to 1$ as $V \ni \varepsilon \to 0$,*
- *$z^*(X,\varepsilon) \to \phi_0(X)$ as $V \ni \varepsilon \to 0$, uniformly with respect to $X \in D_{r'}(0),\ r' < r$.*

3.2 Gevrey formal solution

We are interested in a *couple*[22] $(\widehat{z}(X,\varepsilon) = \sum_{j\geq 0} b_j(X)\varepsilon^j, \widehat{a}(\varepsilon) = \sum_{j\geq 0} c_j\varepsilon^j)$ satisfying the formal equation associated to (3.7):

$$\varepsilon\, \widehat{z}\, \frac{d\widehat{z}}{dX} = -X((2+X)\widehat{z} + 1) + \widehat{a} - 1 \tag{3.8}$$

In order to obtain overstable solutions we must impose that the functions $b_j(X)$ are analytic in $D_{r'_1}(0)$, $b_0(X) = \phi_0(X)$ and $c_0 = 1$.

There exist recurrence formulae due to M.A. Shubin and A.K. Zvonkin [ShZv84], which determine the coefficients c_j and the functions $b_j(X)$, and give a practical way to compute them. In ([CDG89]), the 50 first terms of the expansions were computed with the formal computation software *Macsyma*. According to these results, the expansion seemed divergent and Gevrey of order 1. More precisely, the sequence $(c_j\varepsilon^j)_j$, $\varepsilon \in \mathbb{R}^+$, decreases then increases. If we notice $N_0(\varepsilon)$ the index of the minimum of this sequence (the smallest term), the finite sum $\sum_{j=0}^{N_0(\varepsilon)} c_j\varepsilon^j$ is a numerical value of the parameter a that provides a numerical solution[23] $z(X,\varepsilon)$ *regular* in the neighbourhood of $X=0$ (see figure 3.2).

figure 3.2 : A Van der Pol equation's canard solution, for $\varepsilon = 0.05$ and $\bar{a} = \sum_{j=0}^{N_0(\varepsilon)} c_j\varepsilon^j = -0.006509067491...$

[22] The parameter a is a kind of *control parameter*.

[23] (For a given initial condition and with a numerical integration method.)

3.2.1 Results

We show the Gevrey character of the formal series solutions. Moreover these series are divergent.

Theorem 3.1 *Consider the transformed Van der Pol equation*

$$\varepsilon\ z\ \frac{dz}{dX} = -X(X+2)(z+\frac{1}{X+2})+a-1$$

1. It has a unique formal solution $\widehat{z}(X,\varepsilon) = \sum_{j\geq 0} b_j(X)\varepsilon^j$, $\widehat{a}(\varepsilon) = \sum_{j\geq 0} c_j\varepsilon^j$ *and the functions* $b_j(X)$ *are analytic in* $D_r(0)$, $r<2$, $b_0(X) = \frac{-1}{X+2}$, $c_j \in \mathbb{C}$ *and* $c_0 = 1$.
2. $\widehat{z}$ *is Gevrey of order* 1 *on every subdisk* $D_{\tilde{r}}(0)$, $\tilde{r}<2$ *and* $\widehat{a}$ *is Gevrey of order* 1 *too, i.e. for every* $\tilde{r}<2$, *there are numbers* $M,\ N>0$ *such that* $\forall\ j\geq 1$:

$$Sup_{|X|\leq\tilde{r}} \mid b_j(X)\mid \leq M\ N^j\ \Gamma(1+j/\sigma)\ ,\quad \mid c_j\mid \leq M\ N^j\ \Gamma(1+j/\sigma)$$

3. Moreover, let $\bar{b}_j(X)\in\mathbb{C}[[X]]$ *the expansion associated to* $-b_j(-X)$ *so* $\bar{b}_0(X) = 1/2\sum_{n\geq 0}(X/2)^n$. *One has the following minoration*[24]*:*

$$\bar{b}_j(X)\ >>\frac{j!}{2^j}\ \bar{b}_0(X)^{3j+1}\quad ,\quad \mid c_j\mid \geq (\frac{1}{2})^{4j-1}\ j!$$

3.2.2 Proof

1. M.A. Shubin and A.K. Zvonkin gave recurrence formulae which determine the numbers c_j and the functions $b_j(X)$. We can rewrite these formulae as:

(3.9)
$$\begin{aligned} c_0 &= 1,\ \ b_0(X) = -1/(2+X) \\ c_{j+1} &= \sum_{k=0}^{j} b'_k(0)\ b_{j-k}(0),\ \forall\ j,\ j\geq 0 \\ b_{j+1} &= \mathcal{S}(\sum_{k=0}^{j} b'_k\ b_{j-k})(b_0),\ \forall\ j,\ j\geq 0 \end{aligned}$$

where $\mathcal{S}$ is the so-called *Shift operator* defined by

$$\begin{aligned} \mathcal{S}f(x) &:= \frac{f(x)-f(0)}{x}\quad if\ x\neq\ 0 \\ \mathcal{S}f(0) &:= f'(0) \end{aligned}$$

for f holomorphic in a neighbourhood of 0. The shift operator is closely related to division by x, but it avoids making 0 a pole.
Consider the b_j as $\mathbb{C}$-valued functions defined on $\mathbb{C}$, (3.9) proves that b_0 is holomorphic in $D_r(0)$ when $r<2$, and the same is true for the b_j, according to (3.9). □

[24]We say that a formal power series $A(X) = \sum_{j\geq 0}\alpha_j X^j$ overestimate the formal power series $B(X) = \sum_{j\geq 0}\beta_j X^j$ and we denote $>>$ if and only if $\mid\alpha_j\mid\geq\mid\beta_j\mid$, $\forall\ j\geq 0$.

2. We use modified Nagumo norms to prove the Gevrey character of formal solutions.
Nagumo norms Let numbers $0 < \rho < r$ be given. Consider the following function d on the open circle $D_r = D_r(0)$.

$$d(x) = \begin{cases} r - |x| & \text{if } |x| \geq \rho \\ r - \rho & \text{if } |x| < \rho \end{cases}$$

This is a modification of the function denoting the distance from x to the boundary of D_r and also depends upon ρ. We have the following property

Proposition 3.2 *If $x, y \in D_r$ then $|d(x) - d(y)| \leq |x - y|$.*

Now we introduce (modified) Nagumo norms (cf. [CRSS00]) on $\mathcal{H}(D_r)$. For nonnegative integers p and for f holomorphic in D_r we put

$$||f||_p := \sup_{|x|<r} |f(x)|\, d(x)^p \quad .$$

Note that the norms depend upon ρ but we do not indicate this. Of course $||f||_p$ is infinite for certain $f \in \mathcal{H}(D_r)$. If f is also continuous on the closure of D_r, then

$$||f||_p \leq (r-\rho)^p \sup_{|x|<r} |f(x)| \quad .$$

In any case

$$\begin{aligned} |f(x)| &\leq ||f||_p\, d(x)^{-p} \quad \text{for all} \quad |x| < r \ , \\ |f(x)| &\leq ||f||_p \delta^{-p} \quad \text{if} \quad |x| \leq r - \delta,\ 0 < \delta < r - \rho \ . \end{aligned} \tag{3.10}$$

The larger p is, the larger is the set of functions f having finite norm $||f||_p$. Of course we have

$$\begin{aligned} ||f+g||_p &\leq ||f||_p + ||g||_p \\ ||\alpha f||_p &= |\alpha|\, ||f||_p \end{aligned}$$

for $f, g \in \mathcal{H}(D_r)$ (except if $\alpha = 0, ||f||_p = \infty$).
The norms are also compatible with multiplication

$$||fg||_{p+q} \leq ||f||_p ||g||_q \tag{3.11}$$

for f, $g \in \mathcal{H}(D_r)$ and nonngative integers p, q (except for $f = 0, ||g||_q = \infty$ or $g = 0, ||f||_p = \infty$). The most important property is

Lemma 3.3 *For $f \in \mathcal{H}(D_r)$ one has $||f'||_{p+1} \leq e\,(p+1)\, ||f||_p$.*

Here the norm of the derivative of a holomorphic function is estimated; it is important that it is estimated not only on a subset of D_r.
Proof: See Appendix B.

Lemma 3.4 $||Sf||_p \ \leq \ \frac{2}{\rho}\ ||f||_p$ *for $f \in \mathcal{H}(D_r),\ p \in \mathbb{N}$.*
where S designs the shift operator.

It is important for us that the operator S is 'better' than differentiation in the sense that it does not need an increased index of the norm. This is the motivation for using modified Nagumo norms (the original norms are those where $\rho = 0$).
Proof: See Appendix B.

Let us consider the equation (3.7). As $\phi_0(0) = -\frac{1}{2} \neq 0$ we introduce the following change of variables $u = (2+X)z + 1$ and we consider:

$$\begin{aligned} \varepsilon \frac{du}{dX} &= -X(2+X)^2 u - (2+X)^2(a-1) + \varepsilon \frac{u-1}{2+X} \\ &\quad - X(2+X)^2 \frac{u^2}{1-u} - (2+X)^2(a-1)\frac{u}{1-u} \end{aligned} \tag{3.12}$$

The equation of the slow curve is now $u = 0$ for $a_0 = 1$.
The right hand side of (3.12) can be rewritten

$$\begin{aligned} A_0(X)\ u + B_0(X)\ (a-1) \ &+\ A(X,\varepsilon)\ u + \phi(X,\varepsilon) \\ &+ \sum_{p+q\geq 2} f_{pq}(X,\varepsilon)\ u^p (a-1)^q \end{aligned} \tag{3.13}$$

where $A_0(X) = -X(2+X)^2,\ B_0(X) = -(2+X)^2$.

Lemma 3.5 *The mapping $\mathcal{H}(D_r) \times \mathbb{C} \to \mathcal{H}(D_r)$ defined by $(u,a) \mapsto A_0 \cdot u + B_0 \cdot (a-1)$ is bijective for sufficiently small r, $0 < r < 2$. Here $\mathcal{H}(D_r)$ denotes the set of functions that are holomorphic on $D_r(0)$.*

Proof: Hence we can rewrite (3.12) as

$$A_0(X)\ u + B_0(X)\ (a-1) = \varepsilon \frac{du}{dX} - \mathcal{R}(X,u,a,\varepsilon) \ .$$

To solve the formal problem, we are lead to consider equations of the form
$A_0(X)\ u + B_0(X)\ (a-1) = g(X)$ for arbitrary functions g. Then

$$u(X) = \frac{\Big[g(X)\ -\ B_0(x)\ (a-1)\Big]_l}{-X(2+X)^2}$$

We remark that $A_0(0) = 0$, so, in order to obtain a function u holomorphic in a neighbourhood of 0, we must have

$$g(0) = B_0(0)\ (a-1)$$

As $B_0(0) = -4 \neq 0$, this last equation has a unique solution a and then

$$u(X) = \frac{-1}{(2+X)^2} \mathcal{S}[g - B_0\ (a-1)](X)$$

where $\mathcal{S}$ is the so-called *shift operator*. Moreover, u is analytic for $X \in D_r(0),\ r < 2$. □

For the proof of 2., we rewrite (3.12) as an equation for formal series

$$
\begin{aligned}
(3.14) \qquad A_0(X)\ u + B_0(X)\ (a-1) \ &= \ \varepsilon \frac{du}{dX} - A(X,\varepsilon)\ u - \phi(X,\varepsilon) \\
&\quad - \sum_{p+q\geq 2} f_{pq}(X,\varepsilon)\ u^p (a-1)^q
\end{aligned}
$$

In order to show the Gevrey property of u and a, we must study the inverse of the mapping introduced above. We denote by

$$
(3.15) \qquad \begin{aligned} P_1 : \mathcal{H}(D_r) \ &\longrightarrow \ \mathcal{H}(D_r) \\ P_2 : \mathcal{H}(D_r) \ &\longrightarrow \ \mathbb{C} \end{aligned}
$$

the uniquely determined linear mappings satisfying

$$
(3.16) \quad A_0(X)(P_1 f)(X) + B_0(X)(P_2 f)(X) = f(X) \quad \text{for } X \in D_r(0),\ f \in \mathcal{H}(D_r) \ .
$$

Then we have

Lemma 3.6 *There is a constant $K > 0$ such that*

$$
||P_1 f||_p \leq K\, ||f||_p \ \textit{ and } \ |P_2 f| \leq K\, d(0)^{-p}\, ||f||_p
$$

for nonnegative integers p and $f \in \mathcal{H}(D_r)$.

Proof: See Appendix B.

We consider the series $v := A_0(X)u + B_0(X)(a-1)$. It gives back u and a using the projections P_1 and P_2 of (3.15). We have $u = P_1 v,\ a = P_2 v$. Here we use the convention that (like multiplication by $A_0(X), B_0(X)$), P_1 is applied to the coefficient of each power ε^j of $V = \sum_{j=1}^{\infty} v_j(X)\varepsilon^j$.

Now (3.14) reads

$$
(3.17) \ \ v = \varepsilon \frac{d}{dX}(P_1 v) - \phi(X,\varepsilon) - A(X,\varepsilon)(P_1 v) - \sum_{p+q\geq 2} f_{pq}(X,\varepsilon)\ (P_1 v)^p (P_2 v - 1)^q \ \ .
$$

Definition 3.2 We say that a series $g = \sum_{k=0}^{\infty} g_k(x)\varepsilon^k$ is majorized ('$\ll$') by a series $h(z) = \sum_{l=0}^{\infty} h_l z^l$ if and only if

$$
||g_j||_j \leq h_j\, j! \ \text{ for } j = 0, 1, 2...
$$

We have the following relations for $\ll$.

Lemma 3.7 *Assume that $g \ll h(z)$ and $\tilde{g} \ll \tilde{h}(z)$. Then*

$$
\begin{aligned}
g\,\tilde{g} \ &\ll \ h(z)\,\tilde{h}(z) \\
\varepsilon \frac{d}{dx} g \ &\ll \ e\, z\, h(z)
\end{aligned}
$$

Proof: See Appendix B.

We extend the notion of majorisation to series of vectors by using the maximum norm and to series of matrices by using a compatible matrix norm.

By applying the $\frac{|| \, ||_j}{j!}$ to the coefficient of ε^j of the functions in equation (3.17), we find majorant (scalar) series $\hat{\phi}(z), \hat{A}(z), \hat{f}_{pq}(z)$ of $\phi(x,\varepsilon), A(x,\varepsilon)$ and $f_{pq}(x,\varepsilon)$. By Cauchy's estimate for the coefficients of a convergent power series, all these series have a common positive radius of convergence, and furthermore, the series $\sum_{p+q\geq 2} \hat{f}_{pq}(z) g^{p+q}$ is convergent if $|z|$ and $|g|$ are sufficiently small. We can now consider the following so-called *majorant equation* related to (3.17)

$$
\begin{aligned}
w(z) &= \hat{\phi}(z) + eKzw(z) + \hat{A}(z)Kw(z) \\
&\quad + \sum_{\nu=2}^{\infty} \left(\sum_{p+q=\nu} \hat{f}_{pq}(z) \right) K^{\nu} w(z)^{\nu} \quad .
\end{aligned}
\tag{3.18}
$$

Here $K > 0$ denotes the constant of lemma 3.6. As $\hat{\phi}(0) = \hat{A}(0) = 0$, it is easy to see that (3.18) has a unique formal solution $w(z) = \sum_{j=1}^{\infty} w_j z^j$ and that all w_j are nonnegative.

Furthermore, (3.18) is an implicit equation for $w(z)$ and therefore its solution series $w(z)$ converges, i.e. there are constants $M, N > 0$ such that $w_j \leq MN^j, \ j = 1, 2, ...$

Now, we claim that $w(z)$ majorizes our solution v of (3.17). To make this clear in a formal way, denote the right hand side of (3.17) by Rv, the right hand side of (3.18) by $\hat{R}w(z)$. Then our lemmas 3.7, 3.6 show that

$$
y \ll w(z) \Rightarrow Ry \ll \hat{R}w(z) \ . \tag{3.19}
$$

Here we also need that $\hat{\phi}, \hat{A}, \hat{f}_{pq}$ have been chosen as majorants of the corresponding terms ϕ, A, f_{pq}.

Let us start with $v_0 = \sum_j 0\varepsilon^j$ and $w_0(z) = \sum_j 0 z^j$. We clearly have $v_0 \ll w_0(z)$. We define recursively $v_k = Rv_{k-1}$ and $w_k(z) = \hat{R}w_{k-1}(z)$ for $k = 1, 2, ...$, and have by (3.19) that $v_k \ll w_k(z)$ for all k.

Now, remark that the coefficients of $\varepsilon, ..., \varepsilon^l$ in v (or $\hat{v}$) determine those of $\varepsilon, ..., \varepsilon^{l+1}$ in Rv (or $\hat{R}$). This implies that the coefficients of $\varepsilon^1, ..., \varepsilon^k$ in each v_k (or $w_k(z)$) agree with those of the formal solution v of (3.17) (or w of (3.18) respectively). Therefore, the fact that $v_k \ll w_k(z)$ for all k implies that $v \ll w(z)$ for the formal solutions v of (3.17) and w of (3.18).

The relation (3.10) now implies for every $0 < \delta < r - \rho$ that

$$
|v_k(X)| \leq ||v_k||_k \, \delta^{-k} \leq k! \, w_k \, \delta^{-k}
$$

for X with $|X| \leq r - \delta$ and hence v is Gevrey of order 1 on each subdisk $D_{\tilde{r}}, \ \tilde{r} < r$ of D_r. Using P_1 and P_2 (and lemma 3.6) again we obtain that $u = P_1 v, \ a = P_2 v$ are Gevrey of order 1. This proves our theorem. □

Remark 3.1 *We can prove the result 2. by direct upperbounds ([Ca91]). See exercises IV.*

3. See ([Ca91]), p. 10 - 12.

Remark 3.2 *These results have been improved by A. Fruchard et R. Schäfke [FS96-1]: they use the* relief method *of J.-L. Callot [Cal93] to obtain the theorem:*

Theorem 3.8 *The coefficients c_j of the series $\widehat{a}(\varepsilon)$ verify:*

$$\forall\ j \geq 0,\ \mid c_j \mid = l\ j!\ (3/4 + o(1))^j$$

where $l = \mathcal{O}(1)$ as $j \to +\infty$.

Moreover, the formal solution $\widehat{a}(\varepsilon)$ is 1-summable in every direction except $\mathbb{R}_+$ ([FS96-1]).

Remark 3.3 *Recently, E. Matzinger gives an equivalent for the c_j as $j \to +\infty$.*

Remark 3.4 *We are here in a "pure" situation, as it can be frequently observed. In general, the coefficients of a series Gevrey of order $1/k$ and type A, $\sum_{j\geq 0} c_j\varepsilon^j$ verify:*

$$\mid c_j \mid \sim C A^{j/k}\Gamma(\alpha + j/k)$$

So we can consider the notion of cut-off ([RS96]) and the index of the sum at the smallest term, $N(\varepsilon)$ is equal to $[\frac{k}{A|\varepsilon|^k}]$.
Here, we can compare $N(\varepsilon) = [\frac{1}{A|\varepsilon|}]$ with $A = 3/4$ and the observed index $N_0(\varepsilon)$ of the minimum of the sequence $(c_j\varepsilon^j)_j$ and we notice that these index are similar (see table 4.1).

ε	$N_0(\varepsilon)$	$N(\varepsilon) = [\frac{1}{A\|\varepsilon\|}]$
0.05	27	26
0.1	13	13
0.033	40	40

table 4.1: Index $N_0(\varepsilon)$ and $N(\varepsilon)$
for formal solutions of Van der Pol equation.

3.3 Existence of solutions

This subsection is concerned with the existence of couples $(z(X,\varepsilon), a(\varepsilon))$ of functions called *overstable solution.* These functions must verify

$$\varepsilon\ z\ \frac{dz}{dX} = -X((2+X)z+1) + a - 1$$

for all ε in a sector V and for all $X \in D_r(0)$.
We remind that z and a must be holomorphic in ε in V and z holomorphic in X for X in $D_r(0)$. Moreover (z, a) will have $(\widehat{z}, \widehat{a})$ as asymptotic expansion of Gevrey order 1 as $V \ni \varepsilon \to 0$, uniformly for X in $D'_r(0)$, $r' < r$.

3.3.1 Quasi-solutions

With the unique formal solution $(\widehat{z}, \widehat{a})$ of (3.8) we construct a couple $(\tilde{z}(X,\varepsilon), \tilde{a}(\varepsilon))$ of quasi-solution of (3.7), that is to say the couple satisfy the differential equation except for an exponentially small error.

Let $\widehat{U} = \widehat{z} + \frac{1}{2+X}$, $\widehat{\alpha} = \widehat{a} - 1$. These series are the unique formal solution of

$$\varepsilon\ (\widehat{U} - \frac{1}{2+X})\ \frac{d\widehat{U}}{dX} = -X(2+X)\widehat{U} + \widehat{\alpha} \tag{3.20}$$

As $(\widehat{U}, \widehat{\alpha})$ is a couple of series Gevrey of order 1 in ε, we define powers series called formal Borel transforms of $(\widehat{U}, \widehat{\alpha})$.

$$\widehat{\mathcal{B}}_1(\widehat{U})(\lambda) = \sum_{j\geq 0} \frac{b_{j+1}(X)}{j!}\lambda^j$$

$$\widehat{\mathcal{B}}_1(\widehat{\alpha})(\lambda) = \sum_{j\geq 0} \frac{c_{j+1}}{j!}\lambda^j$$

The series defined above are convergent and there exists $T > 0$ such that $\widehat{\mathcal{B}}_1(\widehat{U})$ and $\widehat{\mathcal{B}}_1(\widehat{\alpha})$ define two holomorphic functions $\mathbf{U}(X,\lambda)$ and $\alpha(\lambda)$ in the λ-plane, at the neighbourhood of $\bar{D}_T(0)$.
One can thus compute truncated Laplace transforms of these functions[25]

$$\tilde{U}(X,\varepsilon) := \mathcal{L}_T(\mathbf{U})(\varepsilon) = \int_0^T e^{-\lambda/\varepsilon}\ \mathbf{U}(X,\lambda)\ d\lambda$$

$$\tilde{\alpha}(\varepsilon) := \mathcal{L}_T(\alpha)(\varepsilon) = \int_0^T e^{-\lambda/\varepsilon}\ \alpha(\lambda)\ d\lambda$$

They are analytic for $\varepsilon \in S_{-\pi/2,\pi/2}$ and $X \in D_r(0),\ r < 2$ and they admit $(\widehat{U}, \widehat{\alpha})$ as their asymptotic expansion of the Gevrey order 1, uniformly in $X \in \bar{D}_{r'}(0),\ r' < r$.

Theorem 3.9 *Let $\tilde{\alpha}(\varepsilon) = \int_0^T e^{-\lambda/\varepsilon}\ \alpha(\lambda)\ d\lambda$. Then, $\tilde{U}(X,\varepsilon)$ satisfy the equation*

$$\varepsilon\ (\tilde{U} - \frac{1}{2+X})\ \frac{d\tilde{U}}{dX} = -X(X+2)\tilde{U} + \tilde{\alpha}(\varepsilon) + \exp(-T/\varepsilon)P(X,\varepsilon) \tag{3.21}$$

where $P(X,\varepsilon)$ is an holomorphic function in the neighbourhood of $\bar{D}_{r'}(0) \times \mathbb{R}+$ and we have:

$$\| P(X,\varepsilon) \|_{r_1} \leq T^2 M(\mathbf{U}'_X)\ M(\int \mathbf{U}) + 1, \quad \forall\ \varepsilon > 0,\ \varepsilon \to 0$$

where $M(\mathbf{f})$ is the supremum of $\| \mathbf{f} \|$ when $X \in \bar{D}_{r'}(0)$ and $\lambda \in \bar{D}_T(0)$.

[25] We suppose ε positive real.

The equation (3.21) is similar to the Van der Pol equation except an exponentially small term $\exp(-T/\varepsilon)P(X,\varepsilon)$ with $P(X,\varepsilon)$ limited. The couple

$$(\tilde{z}(X,\varepsilon) = \tilde{U}(X,\varepsilon) - \frac{1}{X+2}, \tilde{a}(\varepsilon) = \tilde{\alpha}(\varepsilon) + 1)$$

is a quasi-solution of the Van der Pol equation.
Proof: We apply the formal Borel transform to the equation (3.20). We obtain:

$$\int_0^\lambda (\mathbf{U} - (\frac{1}{2+X})\ \delta)(u)\ du\ \star \frac{d\mathbf{U}}{dX} = -X(2+X)\mathbf{U} + \alpha \tag{3.22}$$

Then we apply the truncated Laplace transform:

$$\mathcal{L}_T(\int_0^\lambda (\mathbf{U} - (\frac{1}{2+X})\ \delta)(u)\ du)\ .\ \mathcal{L}_T(\frac{d\mathbf{U}}{dX}) - E(\varepsilon) = -X(2+X)\tilde{U} + \tilde{\alpha}$$

with $|\ E(\varepsilon)\ | \leq T^2 M(\mathbf{U}'_X)\ M(\int \mathbf{U})$. Or

$$\Big(\varepsilon\ (\tilde{U} - \frac{1}{2+X}) - \varepsilon e^{-T/\varepsilon} \int_0^T (\mathbf{U} - (\frac{1}{2+X})\ \delta)\quad (u)\quad du\Big).\frac{d\tilde{U}}{dX} = E(\varepsilon) - X(X+2)\tilde{U} + \tilde{\alpha} \tag{3.23}$$

So

$$\varepsilon\ (\tilde{U} - \frac{1}{2+X})\ \frac{d\tilde{U}}{dX} = -X(X+2)\tilde{U} + \tilde{\alpha}(\varepsilon) + \exp(-T/\varepsilon)P(X,\varepsilon)$$

with $\|\ P(X,\varepsilon)\ \|_{r'} \leq T^2 M(\mathbf{U}'_X)\ M(\int \mathbf{U}) + 1$. □

Remark 3.5 *Another proof, more elegant, uses the characterization of functions with an asymptotic expansion of the Gevrey order* 1 *(Ramis-Sibuya theorem, [RSi89]). See exercises IV.*

3.3.2 Solutions

Now we can prove the main result of this subsection. As $(\tilde{z}(X,\varepsilon), \tilde{a}(\varepsilon))$ have $(\widehat{z}, \widehat{a})$ as an asymptotic expansion with Gevrey of order 1, it seems natural to choose $a^*(\varepsilon) = \tilde{a}(\varepsilon)$ and to try to prove that (3.7)

$$\varepsilon z \frac{dz}{dX} = -X(X+2)(z + \frac{1}{X+2}) + \tilde{a}(\varepsilon) - 1$$

with this choice of the parameter has an (exact) solution $(z^*(X,\varepsilon), a^*(\varepsilon))$, analytic for small X, small ε in a sector V, which is exponentially close to $(-\frac{1}{X+2} + \tilde{U}(X,\varepsilon), \tilde{a}(\varepsilon))$.

Theorem 3.10 *The equation*

$$\varepsilon z \frac{dz}{dX} = -X(X+2)(z + \frac{1}{X+2}) + \tilde{a}(\varepsilon) - 1$$

has an overstable solution $(z^*(X,\varepsilon), a^*(\varepsilon))$ *analytic for* $X \in \bar{D}_{\tilde{r}_0}(0),\ \tilde{r}_0 < 2$ *and* $\varepsilon \in V$. *Moreover,*
$\exists\ M > 0$ *such that* $\forall\ X \in \bar{D}_{\tilde{r}_0}(0),\ \forall\ \varepsilon \in V$

$$z^*(X,\varepsilon) = -\frac{1}{X+2} + \tilde{U}(X,\varepsilon) + Q(X,\varepsilon) \quad \textit{with} \quad |\ Q(X,\varepsilon)\ | \leq \exp(-M/|\ \varepsilon\ |)$$

Proof: $\tilde{U} = \tilde{z} + \frac{1}{2+X}$ satisfies the system

$$\text{(3.24)} \qquad \left\{ \begin{array}{rcl} \dot{X} &=& \tilde{U} - \frac{1}{2+X} \\ \varepsilon\ \dot{\tilde{U}} &=& -X(2+X)\tilde{U} + \tilde{a} - 1 + \exp(-T/\varepsilon)P(X,\varepsilon) \end{array} \right.$$

Let z^* a solution of Van der Pol equation in the Liénard plane for $a = \tilde{a}$:

$$\text{(3.25)} \qquad \left\{ \begin{array}{rcl} \dot{X} &=& z^* \\ \varepsilon\ \dot{z^*} &=& -X(2+X)(z^* + \frac{1}{2+X}) + \tilde{a} - 1 \end{array} \right.$$

We make the following transformation (an exponential split-off)

$$W = \varepsilon\ Log\Big(z^* + \frac{1}{2+X} - \tilde{U}(X,\varepsilon)\Big)$$

We remark that $z^* = -\frac{1}{2+X} + \tilde{U}(X,\varepsilon) + \exp(W/\varepsilon)$ and

$$\dot{W} = \frac{\varepsilon(\dot{z^*} - \dot{\tilde{U}})}{z^* - \tilde{U}} = \exp(-W/\varepsilon)(\varepsilon\dot{z^*} - \varepsilon\dot{\tilde{U}})$$

Thus, the system (3.25) can be rewritten in the variables (X, W)

$$\text{(3.26)} \qquad \left\{ \begin{array}{rcl} \dot{X} &=& z^* = -\frac{1}{2+X} + \tilde{U}(X,\varepsilon) + \exp(W/\varepsilon) \\ \varepsilon\ \dot{W} &=& -X(2+X) - \exp(-(T+W)/\varepsilon)\ P(X,\varepsilon) \end{array} \right.$$

With a Non Standard approach ([DR89]), we have the next results

Remark 3.6 *If* $T + W > 0,\ T + W \not\simeq 0$, *then* $|\ \exp(-\frac{T+W}{\varepsilon})\ P(X,\varepsilon)\ | \simeq 0$.
Moreover, if $W < 0,\ W \not\simeq 0$, *then* $z^*(X,\varepsilon) \simeq -\frac{1}{X+2}$.

Besides, $Y(X) = Y_0 + \int_0^X s(s+2)^2\ ds$ satisfies the equation

$$-\frac{1}{X+2}\,\frac{dY}{dX} = -X(X+2)$$

So if Y_0 is choosen such that $-T < Y_0 < 0,\ Y_0 \not\simeq -T,\ Y_0 \not\simeq 0$, there exists $\tilde{r}_0 > 0,\ \tilde{r}_0 < 2$ such that $\forall\ X \in D(0,\tilde{r}_0)$, $Y(X)$ is between $-T$ and 0, $Y(X) \not\simeq -T$, $Y(X) \not\simeq 0$.
A non standard Analysis result (*short shadow lemma*) claims that for $X \in D(0,\tilde{r}_0)$, $Y(X) \simeq W(X)$. Thus, $\forall\ X \in D(0,\tilde{r}_0),\ \forall\ \varepsilon \in V$, the function

$$z^*(X,\varepsilon) := -\frac{1}{X+2} + \tilde{U}(X,\varepsilon) + \exp(W/\varepsilon)$$

is a solution of Van der Pol equation that remains bounded. ($\forall\ \varepsilon \in V,\ \forall\ X \in D(0, \tilde{r}_0)$, $Y(X) \simeq W(X)$ and $-T < Y(X) < 0,\ Y(X) \not\simeq 0,\ Y(X) \not\simeq -T$. So there exists $M > 0$ such that

$$| \exp(W/\varepsilon) | \leq \exp(-M/| \varepsilon |)$$

z^* is a canard solution, moreover, the difference between $z^*(X, \varepsilon)$ and $-\frac{1}{X+2} + \tilde{U}(X, \varepsilon)$ is exponentially small of order 1 and $\tilde{U}(X, \varepsilon)$ has $\widehat{U}(X, \varepsilon)$ as asymptotic expansion with Gevrey of order 1. So z^* has $-\frac{1}{X+2} + \widehat{U}(X, \varepsilon)$ as asymptotic expansion with Gevrey of order 1, the first term being $-\frac{1}{X+2}$. □

Remark 3.7 *We can use Gronwall lemma [Har73] for the proof (see ([CRSS00])).*

3.4 General case

All these results about the Van der Pol equation can be generalized to singularly perturbed differential equations ([CRSS00]):

3.4.1 Preliminaries

Let us consider a system of n differential equations.

$$\varepsilon^D \frac{dy}{dx} = F(x, y, a, \varepsilon) \tag{3.27}$$

with a small parameter $\varepsilon \in \mathbb{C}$ and a vector a of m parameters, $a \in \mathbb{C}^m$. System (3.27) is called a system of *singularly perturbed differential equations*. We suppose

- D := diag$(\sigma_1, ..., \sigma_n)$, where σ_i are positive integers.
- The function F is an analytic function of the variables x, y and a in the open neighbourhood $\mathcal{D} := D_{r_1}(x_0) \times D_{r_2}(y_0) \times D_{r_3}(a_0) \subset \mathbb{C} \times \mathbb{C}^n \times \mathbb{C}^m$ of (x_0, y_0, a_0) where r_1, r_2, r_3 are positive. [26]
- Let $\sigma := \text{Min}\{\sigma_1, ..., \sigma_n\}$ and let V be an open sector of the complex plane whose vertex is at the origin. The function F is analytic for $\varepsilon \in V$ and is *asymptotic of Gevrey order* $1/\sigma$ *to the formal series* $\sum_{k\geq 0} f_k(x, y, a)\ \varepsilon^k$ *as* $V \ni \varepsilon \to 0$ *uniformly for* $(x, y, a) \in \mathcal{D}$.
- $F(x_0, y_0, a_0, 0) = 0$ for some point $(x_0, y_0, a_0) \in \mathcal{D}$.

Here and in the sequel, "*asymptotic of Gevrey order* $1/\sigma$" means that there are positive constants A, C such that for all $\varepsilon \in V$, $(x, y, a) \in \mathcal{D}$ and all $N \in \mathbb{N}^*$

$$\left| F(x, y, a, \varepsilon) - \sum_{k=0}^{N-1} f_k(x, y, a)\ \varepsilon^k \right| \leq C A^{N/\sigma}\ \Gamma(N/\sigma + 1)\ |\varepsilon|^N \quad .$$

The functions f_k are necessarily holomorphic on $\mathcal{D}$. Furthermore in this case, also the formal series $\sum_{k\geq 0} f_k(x, y, a)\ \varepsilon^k$ is of *Gevrey order* $1/\sigma$; this means that there are positive constants A, C such that for all $(x, y, a) \in \mathcal{D}$ and all $k \in \mathbb{N}$ one has $|f_k(x, y, a)| \leq C A^{k/\sigma} \Gamma(k/\sigma + 1)$. We define some geometric sets as in [Wall90].

[26] Here and in the sequel, $D_{r_1}(x_0)$ denotes the open disk of radius r_1 and center x_0 etc.

Definition 3.3 The set $\mathcal{L}_0 := \{(x,y) \in \mathbb{C} \times \mathbb{C}^n \;/\; F(x,y,a_0,0) = 0\}$ is called the *slow set* of equation (3.27).

Definition 3.4 A *slow curve* $\mathcal{C}_0$ of the equations (3.27) is a smooth subset of the slow set $\mathcal{L}_0$. It is the graph of a function ϕ_0 holomorphic on $D_{r_1'}(x_0), r_1' < r_1$.

In this case, we have $F(x, \phi_0(x), a_0, 0) = 0$ and $\phi_0(x_0) = y_0$.

Definition 3.5 An *overstable solution* of (3.27) is a couple $(y^*(x,\varepsilon), a^*(\varepsilon))$ of functions such that

- $y^*(x,\varepsilon)$ is holomorphic in x on $D_{r_1}(x_0)$,
- y^* and a^* are holomorphic in ε on a subsector W of V and satisfy

$$\varepsilon^D \frac{dy^*}{dx} = F(x, y^*, a^*, \varepsilon)$$

  for $x \in D_{r_1}(x_0)$ and $\varepsilon \in W$,
- $a^*(\varepsilon) \to a_0$ as $W \ni \varepsilon \to 0$,
- $y^*(x,\varepsilon) \to \phi_0(x)$ as $W \ni \varepsilon \to 0$, uniformly with respect to $x \in D_{r_1'}(x_0)$, $0 < r_1' < r$.

3.4.2 The hypothesis of transversality

We consider the linear part of equations (3.27), more precisely we denote by $A_0(x) = \frac{\partial F}{\partial y}(x, \phi_0(x), a_0, 0)$ the Jacobian of F. We suppose that $A_0(x)$ is invertible except at $x = x_0$ and hence

$$\det(A_0(x)) = (x - x_0)^m \; K(x)$$

where $m \in \mathbb{N}$ and $K(x)$ is analytic near x_0 with $K(x_0) \neq 0$.

For our method of proof, it will be important that the above m is the same as the number of components of the parameter vector a. The integer m was called *indice de fugacité* of ϕ_0 [Wall94]. This number is an invariant associated to the slow curve $\mathcal{C}_0$ and the point x_0. We recall the following characterisation of this number in the one dimensional case ($n = 1$) [Wall90]: The function $x \mapsto F(x, \phi_0(x) + \eta, a_0, 0)$ has m zeros close to x_0 for every $\eta \neq 0$, η sufficiently small.

The right hand side of (3.27) can be rewritten

$$F(x,y,a,\varepsilon) = A_0(x)\;(y - \phi_0(x)) + B_0(x)\;(a - a_0) + \mathcal{F}(x,y,a,\varepsilon)$$

where $B_0(x) = \frac{\partial F}{\partial a}(x, \phi_0(x), a_0, 0)$.

Definition 3.6 *We call 'hypothesis of transversality'* **(H)**

(H) The mapping $\mathcal{H}(D_r)^n \times \mathbb{C}^m \to \mathcal{H}(D_r)^n$ defined by $(y,a) \mapsto A_0 \cdot (y - \phi_0) + B_0 \cdot (a - a_0)$ is bijective for sufficiently small r, $0 < r < r_1$.

3.4.3 The main result

Theorem 3.11 *Consider the singularly perturbed ordinary differential equation (3.27) and suppose that the hypothesis below (3.27) are satisfied. Assume that* $\phi_0 : D_{r_1}(x_0) \to \mathbb{C}^n$ *is a slow curve corresponding to the parameter value* a_0 *with* $\phi_0(x_0) = y_0$ *and that* $A_0(x) = \frac{\partial F}{\partial y}(x, \phi_0(x), a_0, 0)$ *and* $B_0(x) = \frac{\partial F}{\partial a}(x, \phi_0(x), a_0, 0)$ *satisfy the hypothesis* **(H)** *of transversality.*

Then (3.27) has a unique formal solution

$$\hat{y}(x,\varepsilon) = \sum_{j=0}^{\infty} b_j(x)\varepsilon^j \ , \ \hat{a}(\varepsilon) = \sum_{j=0}^{\infty} c_j \varepsilon^j \ ,$$

where $b_j(x)$ *are analytic in* $D_{r_1}(x_0)$ *and* $b_0 = \phi_0$, $c_0 = a_0$. *Moreover, these formal series are of Gevrey order* $1/\sigma$.

For $\gamma \in \mathbb{R}$ *and sufficiently small* $\tilde{r}_0, \varepsilon_0, \delta_0 > 0$ *such that* $W = \{\varepsilon \mid |\arg\varepsilon - \gamma| < \delta_0, \ |\varepsilon| < \varepsilon_0\}$ *defines a proper subsector* W *of* V *there is an overstable solution* (y^*, a^*), $a^* : W \to \mathbb{C}^m$, $y^* : D_{\tilde{r}_0}(x_0) \times W \to \mathbb{C}^n$ *of (3.27) having* $(\hat{y}, \hat{a})$ *as asymptotic expansion of Gevrey order* $1/\sigma$ *uniformly for* $x \in D_{\tilde{r}_0}(x_0)$.

Remark 3.8 *For the Van der Pol equation,* $A_0(X) = -X(2+X)$ *so* $m = 1$ *and* $B_0(X) = -(1+X)^2$ *is equal to* $-4 \neq 0$ *for* $X = 0$, *so the hypothesis of transversality* **(H)** *is verified.*

Remark 3.9 *The case* $A_0(X)$ *invertible was already known ([Sib58, Was65, Sib90-1]).*

References

[Bal94] W. Balser, *From divergent power series to analytic functions*, Springer Lecture Notes in Math. **1582** (1994).

[BBRS91] W. Balser, B.L.J. Braaksma, J.-P. Ramis et Y. Sibuya, *Multisummability of formal power series solutions of linear ordinary differential equations*, Asymp. Anal. **5** (1991), 27–4.

[BCDD81] E. Benoît, J.-L. Callot, F. Diener, M. Diener, *Chasse au canard*, Collect. Math. **32** 1–3 (1981), 37–119.

[Bor28] E. Borel, *Leçons sur les séries divergentes*, Deuxième édition, Gauthier-Villars, Paris (1928).

[Bor99] E. Borel, *Mémoire sur les séries divergentes*, Ann. Ecole Norm. Sup., 3ème série, XVI (1899), 2–136.

[Bra91] B.L.J. Braaksma, *Multisummability and Stokes multipliers of linear meromorphic differential equations*, J. Diff. Equations **92** (1991), 45–75.

[Bra92] B.L.J. Braaksma, *Multisummability and Stokes multipliers of non-linear meromorphic differential equations*, Ann. Inst. Fourier **4** Fascicule **3** (1992), 517–540.

[Ca91] M. Canalis–Durand, *Formal expansions of van der Pol equation canard solutions are Gevrey*, in Dynamic bifurcations, E. Benoît Ed., Lecture Notes in Math. **1493** (1991), 29–39 ou C. R. Acad. Sci. série I **311** (1990), 27–30.

[Ca93] M. Canalis–Durand, *Solution formelle Gevrey d'une équation singulièrement perturbée: Le cas multidimensionnel*, Ann. Inst. Fourier **43** Fascicule **2** (1993), 469–483.

[Ca94] M. Canalis–Durand, *Solution formelle Gevrey d'une équation singulièrement perturbée*, Asymp. Anal. **8** (1994), 185–216.

[Cal93] J.-L. Callot, *Champs lents-rapides complexes à une dimension lente,* Ann. Sci. Ec. Norm. Sup. 4ème série t. **26** (1993), 149-73.

[Can89] B. Candelpergher, *Une introduction à la résurgence*, Gazette des Mathématiciens **42** (1989), 31–64.

[Can95] B. Candelpergher, *Fonctions d'une variable complexe*, Armand Colin Editeur, Paris, (1995).

[CDD90] B. Candelpergher, F. Diener, M. Diener, *Retard à la bifurcation : du local au global*, in Bifurcations of planar vector fields, J.-P. Françoise et R. Roussarie Ed., Springer, (1990), 1–19.

[CDG89] M. Canalis–Durand, F. Diener, M. Gaetano, *Calcul des valeurs à canard à l'aide de Macsyma*, dans Mathématiques Finitaires et Analyse non Standard, Luminy 1985, M. Diener, G. Wallet Eds., Preprint URA 212, U.F.R. de Math., Univ. Paris 7, 4 Place Jussieu, 75251 Paris cedex, France (1989), 149–163.

[CL55] E.A. Coddington, N. Levinson, *Theory of Ordinary Differential Equations*, Mc Graw-Hill, (1955).

[CMT01] M. Canalis–Durand, F. Michel, M. Teisseyre, *Algorithms for Formal Reduction of Vector Fields Singularities*, Journal on Dynamical and Control Systems, vol. 7, No. 1, (2001), 101–125.

[CRSS00] M. Canalis–Durand, J.-P. Ramis, R. Schäfke, Y. Sibuya, *Gevrey Solutions of Singularly perturbed differential equations*, Fuer Die Reine Und Angewandte Mathematik Crelles Journal **518** (2000), 95–129.

[Din73] R.B. Dingle, *Asymptotic Expansions : their Derivation and Interpretation*, Academic Press, (1973).

[DR89] F. Diener, G. Reeb, *Analyse Non Standard*, Collection Enseignement des Sciences, Hermann, Paris, (1989).

[Dum87] D. Dumont, *Sur les séries divergentes*, Preprint, (1987).

[Eca93] J. Ecalle, *Introduction aux fonctions analysables et preuve constructive de la conjecture de Dulac*, Acta Math., Hermann, (1993).

[FS96-1] A. Fruchard, R. Schäfke, *Exceptional complex solutions of the forced van der Pol equation*, Prépublication IRMA, Strasbourg (1996).

[FS96-2] A. Fruchard, R. Schäfke, *On the Borel transform*, C. R. Acad. Sci. série I **323** (1996), 999–1004.

[Gev18] E. Gevrey, *Sur la nature analytique des solutions des équations aux dérivées partielles*, Ann. Sci. Ec. Norm. Sup. 3ème série **35** (1918), 129–90 (o.c.p 243–304).

[Har73] P. Hartmann, *Ordinary Differential Equations*, J. Wiley, Baltimore, (1973).

[Ler00] E. Leroy, *Sur les séries divergentes et les fonctions définies par un développement de Taylor*, Ann. Fac. Université de Toulouse, (1900), 317–430.

[Lie28] A. Liénard, *Etude des oscillations entretenues*, Rev. Gén. Electr. **23** (22), (1928), 901–954.

[LR90] M. Loday-Richaud, *Introduction à la multisommabilité*, Gazette des Mathématiciens, (Soc. Math. de France) **44** (avril 1990).

[LR95] M. Loday-Richaud, *Séries formelles provenant de systèmes différentiels linéaires méromorphes*, Expo. Math. **13** (1995), 116–162.

[Mai03] E. Maillet, *Sur les séries divergentes et les équations différentielles*, Ann. ENT, (1903), 487–518.

[Mal89] B. Malgrange, *Sur le théorème de Maillet*, Asymp. Anal. **2** (1989), 1–4.

[Mal91] B. Malgrange, *Equations différentielles à coefficients polynomiaux*, Progress in Math., Birkhaüser, (1991).

[Mal95] B. Malgrange, *Sommation des séries divergentes*, Expo. Math. **13** (1995), 163–222.

[MalR92] B. Malgrange, J.-P. Ramis, *Fonctions Multisommables*, Ann. Inst. Fourier **42** Fascicule 1 et 2, (1992), 353–368.

[MR88] J. Martinet, J.-P. Ramis, *Théorie de Galois différentielle et resommation*, in : E. Tournier (ed.) Computer Algebra and differential equations. London : Academic Press, (1988), 117–214.

[MR91] J. Martinet, J.-P. Ramis, *Elementary acceleration and multisummability*, Ann. Inst. Henri Poincaré, Physique Théorique **54** (1991), 331–401.

[Nev19] F. Nevanlinna, *Zur Theorie der Asymptotischen Potenzreihen*, Ann. Acad. Scient. Fennicae, ser. A, from XII (1919), 1–81.

[Poi81] H. Poincaré, *Mémoire sur les courbes définies par une équation différentielle*, J. Math. Pures et Appl. **3** (1881), 375–422.

[Poi90] H. Poincaré, *Les Méthodes nouvelles de la Mécanique Céleste*, T2 (1890), New York: Dover, (1957).

[Poi92] H. Poincaré, *Sur le problème des trois corps et les équations de la dynamique*, Acta Math **13** (1892), 1–271.

[Ram78] J.-P. Ramis, *Dévissage Gevrey*, Astérisque (Soc. Math. France) **59-60** (1978), 173–204.

[Ram80] J.-P. Ramis, *Les séries k-sommables et leurs applications,* Analysis, Microlocal Calcul and Relativistic Quantum Theory, Proceedings "Les houches" 1979, Springer Lecture Notes in Physics **126** (1980), 178-199.

[Ram93] J.-P. Ramis, *Séries divergentes et théories asymptotiques*, Panoramas et synthèses I (Soc. Math. de France), (1993).

[RS96] J.-P. Ramis, R. Schäfke, *Gevrey separation of fast and slow variables*, Nonlinearity **9** (1996), 353–84.

[RSi89] J.P. Ramis, Y. Sibuya, *Hukuhara's domains and fundamental existence and uniqueness theorems for asymptotic solutions of Gevrey type*, Asymp. Anal. **2** (1989), 39–94.

[ShZv84] M.A. Shubin, A.K. Zvonkin, *Nonstandard analysis and singular perturbations of ordinary differential equations*, Russian Math. Surveys **39** (1984), 69–131.

[Sib58] Y. Sibuya, *Sur réduction analytique d'un système d'équations différentielles ordinaires linéaires contenant un paramètre*, Jour. of Fac. Sci., Univ. of Tokio, Sec. I **7** (1958), 527–540.

[Sib81] Y. Sibuya, *A theorem concerning uniform simplification at a transition point and the problem of resonance*, SIAM J. Math. Anal. **2** (1981), 653–668.

[Sib89] Y. Sibuya, *Gevrey property of formal solutions in a parameter*, Preprint School of Mathematics, University of Minnesota, Minneapolis, (1989).

[Sib90-1] Y. Sibuya, *Linear differential equations in the complex domain, problems of analytic continuation*, Trans. Math. Monographs vol **82** (Providence, RI: Ann. Math. Soc.), (1990).

[Sib90-2] Y. Sibuya, *Gevrey property of formal solutions in a parameter*, Asymptotic and Computational Analysis, (R. Wong ed.), Marcel Dekker, Lecture Notes in Applied Math. **124** (1990), 393–401.

[Sib92] Y. Sibuya, *Gevrey Asymptotics*, Ordinary and Delay Differential Equations, J. Wiener, J.K. Hale eds, Pitman Research Notes in Math. **272** (1992), 205–217.

[Tou89] J.C. Tougeron, *An introduction to the theory of Gevrey expansions and to the Borel-Laplace transform with some applications*, Cours à Rennes et Toronto, (1989-90).

[VdP26] B. Van der Pol, *Relaxation oscillations*, Philos. Mag.Vol. **2-7** (1926), 978–992.

[Wall90] G. Wallet, *Surstabilité pour une équation différentielle analytique en dimension un*, Ann. Inst. Fourier **40, 3** (1990), 557–95.

[Wall94] G. Wallet, *Singularité analytique et perturbation singulière en dimension 2*, Bull. Soc. Math. France **122** (1994), 185–208.

[Was65] W. Wasow, *Asymptotic expansions for ordinary differential equations,* Interscience, New York, (1965).

[Was85] W. Wasow, *Linear Turning Point Theory*, Springer, New York, (1985).

[Wat12] G. N. Watson, *The transformation of an asymptotic series into a convergent series of inverse factoriels*, Rend. Circ. Mat., Palermo **34** (1912), 41-88.