Sliding Control Modes.

2. Variable Structure Systems and SCM.
Geometric methods.

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Let us consider a single input dynamical system given by

$$\dot{x} = f(x) + ug(x)$$

where $x \in U$, an open set of $\mathbb{R}^n$, f and g are smooth vector fields on U with $g(x) \neq 0$ everywhere, and $u : U \longrightarrow \mathbb{R}$ is the control input. Let S be a submanifold in U defined by a smooth function $s : U \longrightarrow \mathbb{R}$, namely

$$S = \{x \in U \mid s(x) = 0\}$$

where $(\text{grad } s)(x) \neq 0$, $\forall x \in U$ and $S \cap U \neq \emptyset$ are assumed.

As for the input, let us take u defined by

$$u = \begin{cases} u^+(x) & \text{if } H(s(x), x) > 0 \\ u^-(x) & \text{if } H(s(x), x) < 0 \end{cases}$$

where both u^+ and u^- are smooth functions on x, and H, in turn, is a function on $(s(x), x, t)$.

Finally, let $\phi(x,t)$ be the trajectory of the dynamical system $x(0) = x_0$.

Definition: S is said to be a sliding surface if there exists θ, an open set in U containing S, in such a way that $\forall x \in \theta \setminus S$, one of the following conditions holds.

1. there exists a finite time $t_s > 0$ such that

$$s(\phi(x,t)) \neq 0 \quad 0 \leq t < t_s \quad \text{and} \quad s(\phi(x,t)) = 0 \quad t \geq t_s$$

2. there exist t_s and $\hat{t}_s$, $0 < t_s < \hat{t}_s < \infty$ such that

$$s(\phi(x,t)) \neq 0 \quad 0 \leq t < t_s \quad \text{and} \quad s(\phi(x,t)) = 0 \quad t_s \leq t < \hat{t}_s$$

   and $\phi(x,\hat{t}_s) \in \partial(S \cap U)$

Questions:

1. **Existence.** Which conditions on f, g, u, σ and S, if any, guarantee that S be a sliding surface?
2. **Ideal sliding dynamics.** The dynamics is not defined on S; however, if S is a sliding surface for this dynamics, which vector field governs the system on S?

Let us define the equivalent control as the control law, $u_{eq} : U \to \mathbb{R}$, which makes S an invariant manifold, that is to say, u_{eq} is such that the vector field $f + gu_{eq}$ is tangent to S. This results in

$$\langle \text{grad } s, f + g\, u_{eq} \rangle = 0$$

where $\langle \cdot, \cdot \rangle$ denotes the standard scalar product, and thus

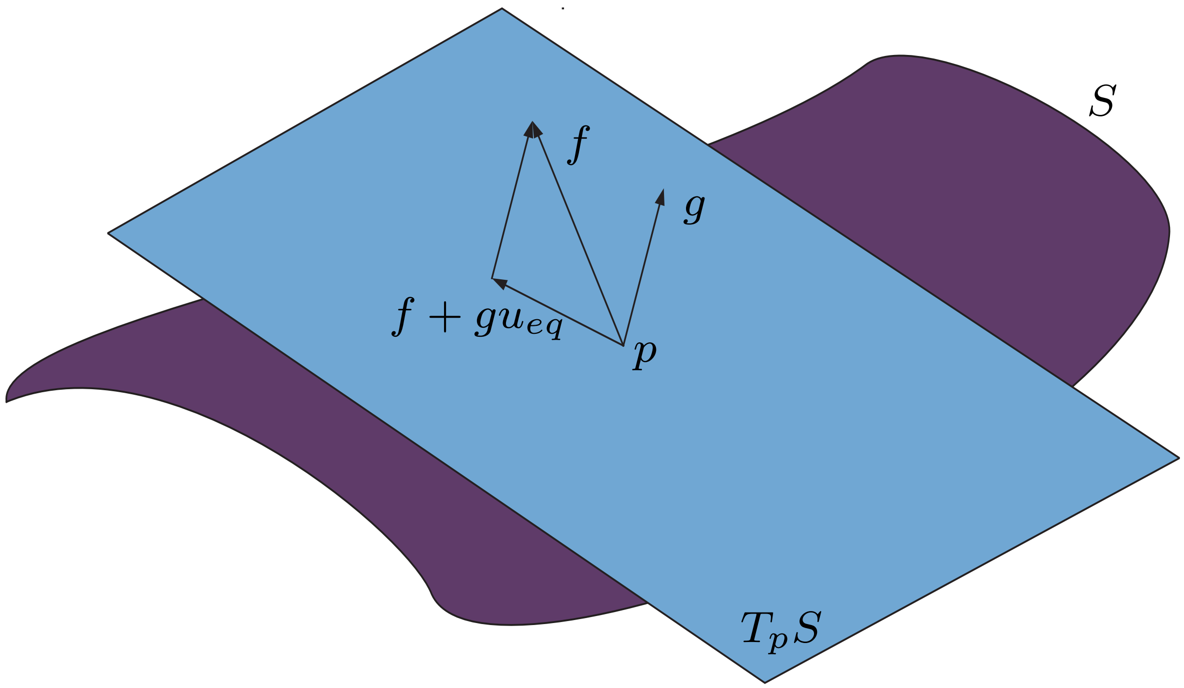
$$u_{eq} = -\frac{\langle \text{grad } s, f \rangle}{\langle \text{grad } s, g \rangle}$$

As it is proved in a paper by Filipov on differential equations with discontinuous right hand side, the ideal sliding dynamics, i.e. the dynamics on S, is governed by the vector field

$$f(x) + g(x)\, u_{eq}(x)$$

Notice that a necessary condition for the existence of equivalent control is $\langle \text{grad } s, g \rangle \neq 0$

- Sliding dynamics results in an projection operator.
- Robustness. Matching condition.



Proposition S is a sliding surface for the dynamical system if and only if there exists θ, a neighbourhood of S, such that

$$\frac{d}{dt} s(\phi(x,t)) < 0 \quad \text{if } s(\phi(x,t)) > 0$$

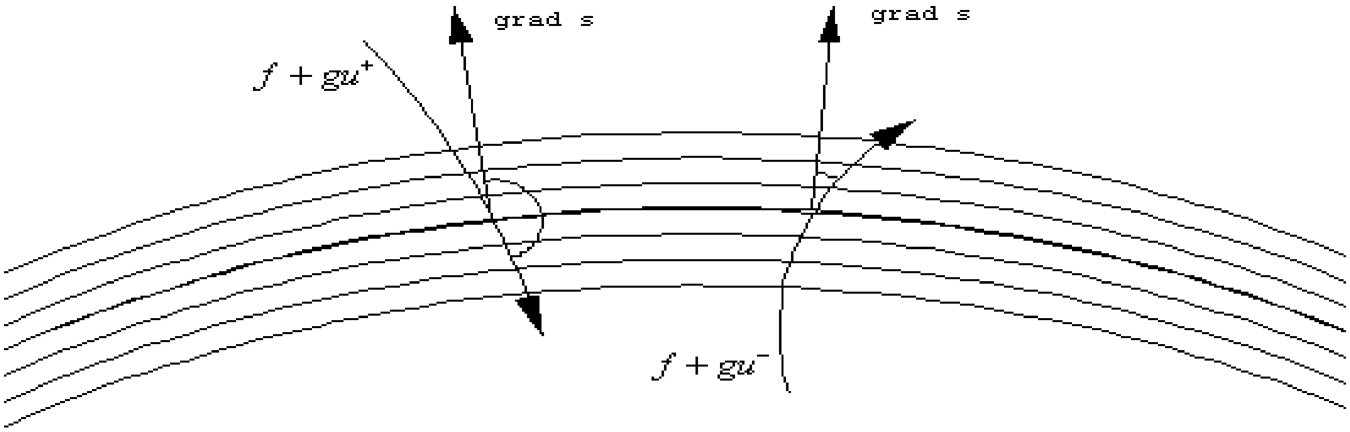
$$\frac{d}{dt} s(\phi(x,t)) > 0 \quad \text{if } s(\phi(x,t)) < 0$$

These conditions can also be written as

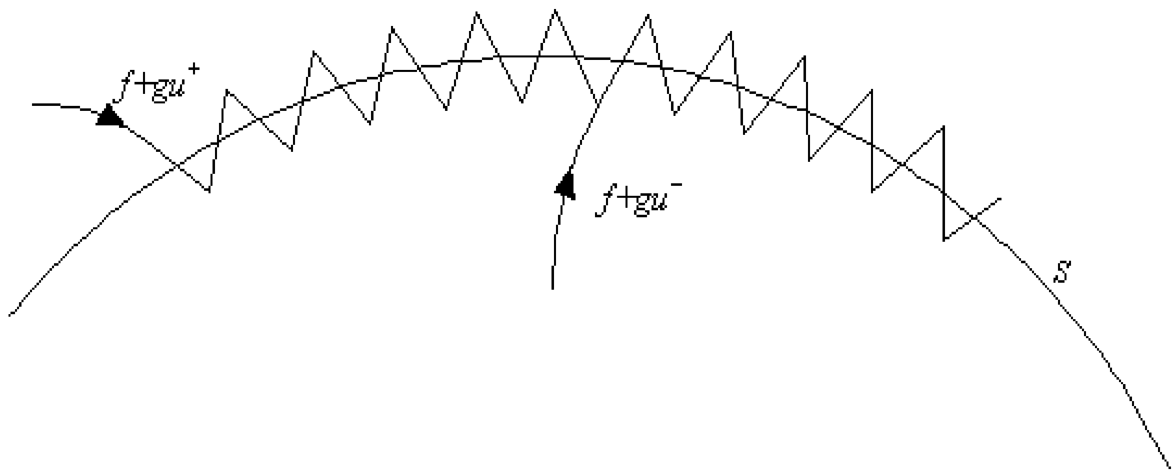
$$\left.\begin{array}{l} \lim\limits_{s \to 0^+} L_{f+gu^+} s(x) < 0 \\ \\ \lim\limits_{s \to 0^-} L_{f+gu^-} s(x) > 0 \end{array}\right\}$$

They are equivalent to

$$\left.\begin{array}{l} \lim\limits_{s\to 0^+} \langle \text{grad } s, f+gu^+\rangle < 0 \\ \\ \lim\limits_{s\to 0^-} \langle \text{grad } s, f+gu^-\rangle > 0 \end{array}\right\}$$



In practice, sliding motion is not attainable; imperfections such as hysteresis, delays, sampling and unmodelled dynamics will result in a chattering motion in a neighbourhood of the sliding surface.



In case the control functions u^+ and u^- can be designed arbitrarily, the next proposition gives a very simple condition for S to be a sliding surface.

Proposition. A necessary and sufficient condition for the existence of control functions u^+ and u^- that makes S be a sliding surface is

$$\langle \text{grad } s, g \rangle \neq 0$$

which is known as the transversality condition.

Proposition. Otherwise, sliding mode exists on the submanifold of $s = 0$ defined by

$$\min\{u^-, u^+\} \leq u_{eq} \leq \max\{u^-, u^+\}$$

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