# Sliding Control Modes.

*1. Variable Structure Systems and SCM. Introduction.*

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# History

- The term “Variable Structure System” (VSS) appeared in the USSR in the late fifties.
- The primitive examples:
  - Vibrational control of aircraft D.C. (Kulebakin, V. 1932)
  - On automatic stability of a ship on a given course (Nikolski, G. 1934)
- First books:
  - Discontinuous automatic control by Irmgard Flügge-Lotz, Princeton university Press, 1953.
  - V. Tsypkin, 1955
- First steps: three papers published in the late fifties by S. Emel’yanov.
  The novelty of the approach was that the feedback gains could take several values depending on the system state. Each of the systems consists of a set of linear structures and was supplied with a switching logic. (VSS). The author observed that, due to altering the structure in the course of control process, the properties could be attained which were not inherent in any of the structures.

## Example

$$\ddot{x} = u \qquad u = -\eta x, \quad \eta = \begin{cases} \omega_1^2 & \text{if } x\dot{x} > 0 \\ \omega_2^2 & \text{if } x\dot{x} < 0 \end{cases} \qquad \omega_1^2 > \omega_2^2$$

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$$u = -k_1|x|\text{sign}(s), \qquad s = cx + \dot{x}, \quad k_1, c > 0$$

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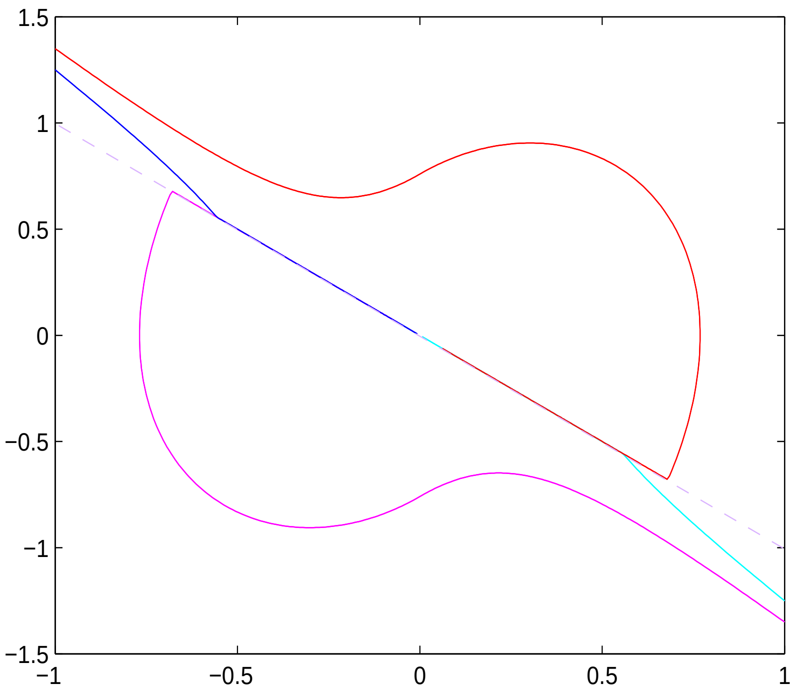


### Characteristics

- the order of the motion equation is reduced,
- although the original system is governed by a non-linear second equation, the motion equation of sliding mode is linear,
- sliding mode does not depend on the plant dynamics and is determined by parameter $c$ selected by a designer.

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- The uniqueness problem of unambiguous description of sliding modes in discontinuous systems was approached by regularisation techniques.
- The universal approach to regularisation consists in introducing a boundary layer $\|s\| < \Delta$ around the manifold $s = 0$, where an ideal discontinuous control is replaced by a real one such that state trajectories run inside the layer. If, with $\Delta$ tending to zero, the limit of the solution exists, it is taken as a solution of the ideal sliding dynamics.

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# Mathematical remarks (I)

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- Ideal sliding motion is regarded as a result of limiting procedure with all non-idealities tending to zero.
- (Utkin) Ideal Sliding Mode (uniqueness) exits for systems affine in the control.

## Example

$$
\begin{aligned}
\dot{x}_1 &= 0.3x_2 + ux_1 \\
\dot{x}_2 &= -0.7x_1 + 4u^3x_1, \\
u &= -\text{sign}(x_1 s), \\
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- **sign** function obtained saturating a linear function.
- **sign** function obtained as an hysteresis.

Uniqueness problem.

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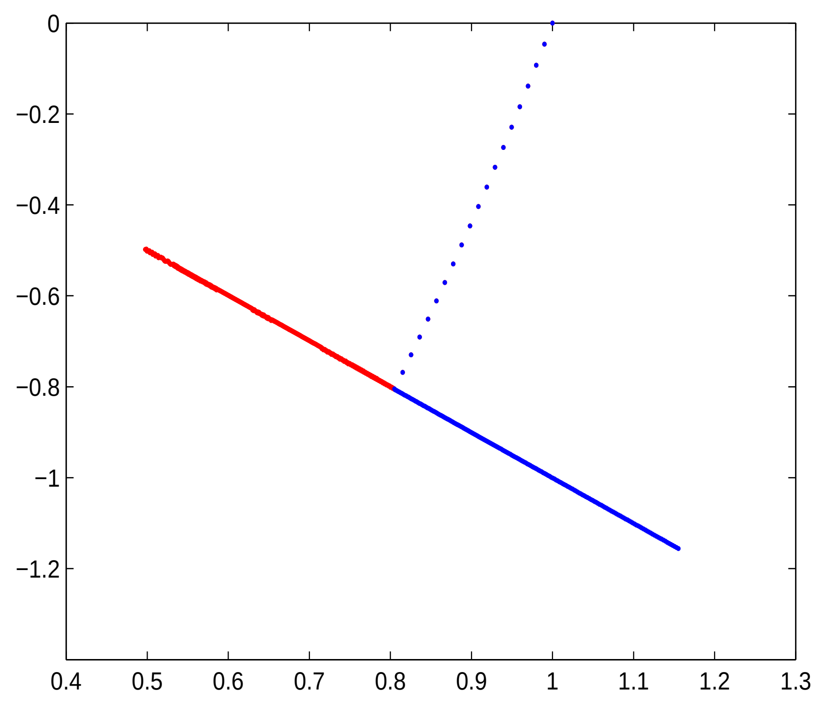


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We deal with systems as

$$\dot{x} = f(t, x, u), \qquad x(t_0) = a$$

where $u = u(x)$ is discontinuous on $s(x) = 0$.
Equivalently, we are given a multifunction (set-valued mapping)

$$G : \Omega \longrightarrow \mathbb{R}^n$$

$G(x) = f(t, x, u)$ where $f$ is continuous, $G(x)$ is the convex hull generated by $f(t, x, u_i)$.
The previous initial-value problem results in

$$\dot{x} \in G(x), \qquad x(t:0) = a$$

A solution to the initial-value problem is a function $y : [t_0, t_0 + T) \longrightarrow \Omega$ for some positive $T \leq +\infty$ such that its derivative exists for almost all $t \in (t_0, t_0 + T)$, it is locally integrable for every $(a, b) \subset (t_0, t_0 + T)$ and $\dot{y}(t) \in G(y(t))$ for almost all $t \in (t_0, t_0 + T)$.

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For the simple case

$$\dot{x} = f(x,u) \qquad u = \begin{cases} u^+ & \text{if } s(x) > 0 \\ u^- & \text{if } s(x) < 0 \end{cases}$$

where $\partial_x s \neq 0$ on $s(x) = 0$, the Filipov dynamics on $s(x) = 0$ is given by

$$\dot{y}(t) = \frac{[(\partial_x s f^-) f^+ - (\partial_x s f^+) f^-]}{\partial_x s (f^- - f^+)}$$

The equivalent control dynamics is obtained by

$$f(x, u_{eq}) \in T_x S$$

where $S := \{x \mid s(x) = 0\}$

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$$f(t,x,u) = f_1(t,x) + \sum u_i(t,x) g_i(t,x)$$

- The sliding dynamics obtained by using the equivalent control agree with Filipov dynamics in the particular case of control systems affine in the control input.

- The design procedure may be illustrated easily for systems represented in the regular form

$$\dot{\mathbf{x}}_1 = f_1(\mathbf{x}_1, \mathbf{x}_2, t),$$
$$\dot{\mathbf{x}}_2 = f_2(\mathbf{x}_1, \mathbf{x}_2, t) + B_2(\mathbf{x}_1, \mathbf{x}_2, t)u, \quad \text{det}(B_2) \neq 0$$

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- The state vector $\mathbf{x}_2$ is handled as a fictitious control in the first equation and selected as a function of $\mathbf{x}_1$ to provide the desired dynamics in the first subsystem: $\mathbf{x}_2 = -s_0(\mathbf{x}_1)$

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$$s(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_2 + s_0(\mathbf{x}_1) = 0$$

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- Note that the motion is of a reduced order and neither depends on function $f_2(\mathbf{x}_1, \mathbf{x}_2, t)$ nor on function $B_2(\mathbf{x}_1, \mathbf{x}_2, t)$ in the second equation of the original system.

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