Sliding Control Modes.

4. Variable Structure Systems and SCM.
Multiple input non-linear systems.

Enric Fossas Colet

Institut d'Organització i Control de Sistemes Industrials
UNIVERSITAT POLITÈCNICA DE CATALUNYA

Let us consider a multiple input dynamical system given by

$$\dot{x} = f(x) + \sum_{k=1}^{m} u_k g_k(x)$$

where $x \in U$, an open set of $\mathbb{R}^n$, f and g_k are smooth vector fields on U with $g_k(x) \neq 0$ everywhere, and $u_k : U \longrightarrow \mathbb{R}$ are the control inputs. Let S be a m-dimension sub-manifold in U defined by smooth functions $s_k : U \longrightarrow \mathbb{R}$, namely

$$S = \{x \in U \mid s_k(x) = 0 \; k = 1, \ldots, m\}$$

As for the input, let us assume u_k is defined by

$$u_k = \begin{cases} u_k^+(x) & \text{if } H_k(s_1(x), \ldots, s_m(x), x) > 0 \\ u_k^-(x) & \text{if } H_k(s_1(x), \ldots, s_m(x), x) < 0 \end{cases}$$

where both u_k^+ and u_k^- are smooth functions on x, and H_k, in turn, is a function of $(s_1, \ldots, s_m, x)$.

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Finally, let $\phi(x,t)$ be the trajectory of this dynamical system with initial conditions $x(0)=x_0$.

Definition: S is said to be a sliding surface if there exists θ, an open set in U containing S, in such a way that $\forall x \in \theta \setminus S$, one of the following conditions holds.

1. there exists a finite time $t_s > 0$ such that

$$s(\phi(x,t)) \neq 0 \quad 0 \leq t < t_s \quad \text{and} \quad s(\phi(x,t)) = 0 \quad t \geq t_s$$

2. there exist t_s and $\hat{t}_s$, $0 < t_s < \hat{t}_s < \infty$ such that

$$s(\phi(x,t)) \neq 0 \quad 0 \leq t < t_s \quad \text{and} \quad s(\phi(x,t)) = 0 \quad t_s \leq t < \hat{t}_s$$

   and $\phi(x,\hat{t}_s) \in \partial(S \cap U)$

Remark Compare ideal sliding with real sliding mode.

Questions:

1. **Existence.** Which conditions on f, g_k, u_k, σ and S, if any, guarantee that S is a sliding surface?
2. **Ideal sliding dynamics.** The dynamics is not defined on S; however, if S is a sliding surface for this dynamics, which vector field governs the system on S?

Let us define the equivalent control as the control law, $u_{eq} : U \to \mathbb{R}$, which makes S an invariant manifold, that is to say, $u_{k_{eq}}$ is such that the vector field $f + \sum_{k=1}^{m} u_{k_{eq}} g_k$ is tangent to S. For a trajectory sliding on S, this results in

$$\frac{ds_k(\phi(x,t))}{dt} = 0$$

but,

$$\frac{d}{dt}\begin{pmatrix} s_1 \\ \vdots \\ s_m \end{pmatrix} = \begin{pmatrix} \frac{\partial s_1}{\partial x_1} & \cdots & \frac{\partial s_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial s_m}{\partial x_1} & \cdots & \frac{\partial s_m}{\partial x_n} \end{pmatrix} \left[\begin{pmatrix} f_1 \\ \vdots \\ f_n \end{pmatrix} + \begin{pmatrix} g_{11} & \cdots & g_{m1} \\ \vdots & \ddots & \vdots \\ g_{1n} & \cdots & g_{mn} \end{pmatrix} \begin{pmatrix} u_1 \\ \vdots \\ u_m \end{pmatrix} \right]$$

and

$$\mathbf{u_{eq}} = -\left(\frac{\partial s}{\partial x} G\right)^{-1} \left(\frac{\partial s}{\partial x} f\right)$$

provided that $\left(\frac{\partial s}{\partial x} G\right)$ is invertible.

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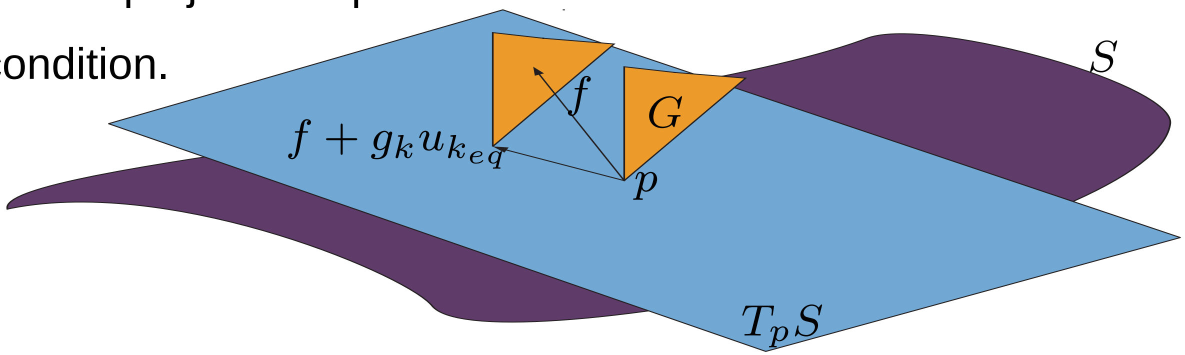
- As it is proved in a paper by Filipov on differential equations with discontinuous right hand side, the ideal sliding dynamics, i.e. the dynamics on S, is governed by the vector field

$$f(x) + \sum_{k=1}^{m} u_{k_{eq}}(x) g_k(x)$$

- Sliding dynamics results in an projection operator.
- Robustness. Matching condition.



Proposition For the domain S on the intersection of discontinuity surfaces $s_1 = 0, \ldots, s_m = 0$ to be a stable sliding domain, it is sufficient that for all x belonging to this domain, there exists, in a certain region Ω of the space $(s_1, \ldots, s_m)$ containing the origin, a function $v(s, x, t)$ continuously differentiable with respect to all of its arguments such that:

1. $v(s, x, t)$ is definite positive with respect to s, i.e. if $s \neq 0$, $v(s, x, t) \neq 0$, $\forall x, t$; $v(0, x, t) = 0$, $\forall x, t$. On the sphere $\|s\| = R$, let

$$\inf_{\{\|s\|=R\}} v(s, x, t) = h_R, \qquad \sup_{\{\|s\|=R\}} v(s, x, t) = H_R, \qquad R \neq 0,$$

   where h_R and H_R are some positive quantities depending only on R and

$$\begin{aligned} \lim_{R \to 0} H_R &= 0, \\ \lim_{R \to \infty} h_R &= \infty \end{aligned}$$

2. the total derivative of $v(s, x, t)$ is negative everywhere except on the discontinuity surfaces where this function is not defined.

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- Example of an unstable dynamics with a piece-wise Lyapunov function.

$$\begin{aligned} \frac{ds_1}{st} &= -2\text{sign}(s_1) - \text{sign}(s_2) \\ \frac{ds_2}{dt} &= -2\text{sign}(s_1) + \text{sign}(s_2) \end{aligned}$$

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$$\frac{d\nu}{dt} = 4\text{sign}(s_1)\frac{ds_1}{dt} + \text{sign}(s_2)\frac{ds_2}{dt} = \ldots = \begin{cases} -13 & \text{if sign}(s_1 s_2) > 0 \\ -1 & \text{if sign}(s_1 s_2) < 0 \end{cases}$$

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- The dynamics shows sliding mode on $s_1 = 0$. The ideal sliding dynamics is given by

$$\frac{ds_2}{dt} = -2\left(\text{sign}(s_1)\right)_{eq} + \text{sign}(s_2)$$

where $\left(\text{sign}(s_1)\right)_{eq}$ fulfils $-2\left(\text{sign}(s_1)\right)_{eq} - \text{sign}(s_2) = 0$

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- Thus $(\text{sign}(s_1))_{eq} = -0.5$ and the origin is unstable because of

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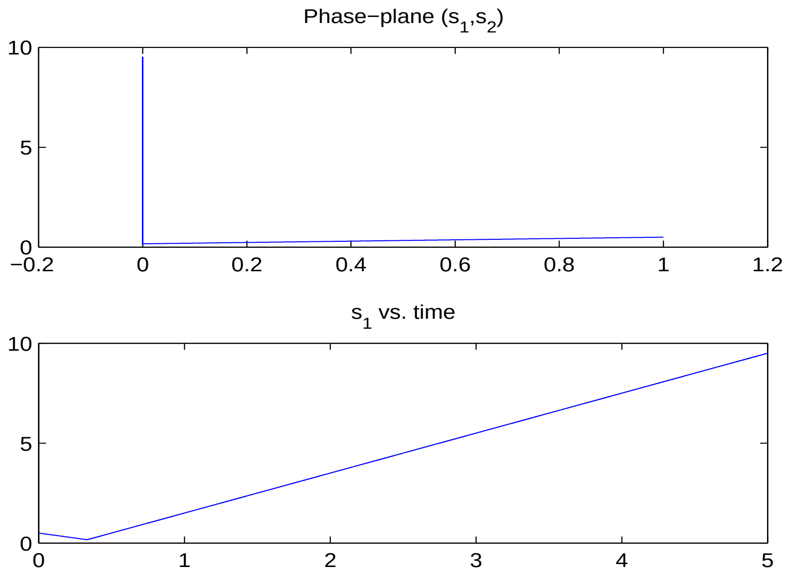

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- Use $s^T s$ as Lyapunov function and try to solve the $(s_1, \ldots, s_m) - (u_1, \ldots, u_m)$ decoupling problem in order to easily design u_k-gains as in Single Input systems.

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- Let us take $\mathbf{u} = M\hat{\mathbf{u}}$, then

$$\frac{d\mathbf{s}}{dt} = \frac{\partial \mathbf{s}}{\partial x}\mathbf{f} + \frac{\partial \mathbf{s}}{\partial x}G\mathbf{u} = \frac{\partial \mathbf{s}}{\partial x}\mathbf{f} + \frac{\partial \mathbf{s}}{\partial x}GM\hat{\mathbf{u}}$$

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- Let us take now $\hat{\mathbf{s}} = M(x,t)\mathbf{s}$, then

$$\frac{d\hat{\mathbf{s}}}{dt} = \frac{dM}{dt}\mathbf{s} + M\left[\frac{\partial \mathbf{s}}{\partial x}\mathbf{f} + \frac{\partial \mathbf{s}}{\partial x}G\mathbf{u}\right]$$

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- Then, $M = \left(\frac{\partial \mathbf{s}}{\partial x}G\right)^{-1}$ solves the decoupling problem provided that M neither depends on $\mathbf{x}$, nor on t.

Following the paper *Design considerations in Sliding-mode controlled parallel connected inverters* by Biel et al. published in Proceedings if ISCAS-2002, design appropriate control gains in order to fulfil the following specifications:

- Output voltage tracking of a pure sinusoidal signal.
- Equilibrating currents.