Recurrence relations with reflection

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What is a recurrence relation with reflection?

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They are similar to differential equations with reflection such as

$$x'(t) + m\, x(-t) = 0, \quad t \in \mathbb{R}.$$

Previous work

- Cabada, A., T., F.A.F.: *Green's functions for reducible functional differential equations*. Bull. Malays. Math. Sci. Soc. pp. 1-22 (2016)
- Cabada, A., T., F.A.F.: *On linear differential equations and systems with reflection*. Appl. Math. Comput. **305**, 84-102 (2017)

Differential Equations + Algebraic Structure

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Differential Equations + Algebraic Structure

Differential Equations: Homogeneous linear differential equations with reflection and constant coefficients:

$$Tu(t) := \sum_{k=0}^{n} a_k u^{(k)}(t) + \sum_{k=0}^{n} b_k u^{(k)}(-t) = 0.$$

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$$Tu(t) := \sum_{k=0}^{n} a_k u^{(k)}(t) + \sum_{k=0}^{n} b_k u^{(k)}(-t) = 0.$$

T is a composition of the usual *differential operator* $\widetilde{D}$ and the *pullback by the reflection* function $\widetilde{\varphi}(t) = -t$, that is, the operator $\widetilde{\varphi}^*$ such that $(\widetilde{\varphi}^* f)(t) = f(-t)$.

Objective, notation and preliminaries

Objective: To obtain analogous results as the ones known for the case of linear recurrence equations and systems with reflection.

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Let V be vector space, $\mathcal{S}$ the space of $\mathbb{Z}$-sequences in V. We define the *right shift operator* D as

$$\begin{aligned} \mathcal{S} &\xrightarrow{\quad D \quad} \mathcal{S} \\ (x_k)_{k\in\mathbb{Z}} &\longmapsto (x_{k+1})_{k\in\mathbb{Z}} \end{aligned}$$

D is bijective and will play the role the differential operator.

An *order n linear recurrence relation* is

$$x_{k+n}=\sum_{j=0}^{n-1}a_jx_{k+j}+c_k,\ k\in\mathbb{N};\quad x_k=\xi_k,\ k=1,\dots,n,\tag{2.1}$$

where $\xi_k\in\mathbb{F},\ k=1,\dots,n;a_j\in\mathbb{F},\ j=0,\dots,n-1;a_0\neq 0$ and $c=(c_k)_{k\in\mathbb{N}}$.

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where $\xi_k \in \mathbb{F},\ k = 1, \dots, n; a_j \in \mathbb{F},\ j = 0, \dots, n-1; a_0 \neq 0$ and $c = (c_k)_{k \in \mathbb{N}}$.

We can write (2.1) as

$$\left(D^n - \sum_{j=0}^{n-1} a_j D^j\right) x = c; \quad x_k = \xi_k,\ k = 1, \dots, n,$$

where $x = (x_k)_{k \in \mathbb{N}}$.

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where $x=(x_k)_{k\in\mathbb{N}}$. Hence, we study equations of the kind

$$Ux:=\sum_{j=0}^{n}a_jD^jx=c;\quad x_k=\xi_k,\ k=1,\dots,n, \tag{2.2}$$

where $a_0a_n\neq 0$. $U\in\mathbb{F}[D]$, the *algebra of polynomials* on D with coefficients in $\mathbb{F}$.

Recurrence relations with reflection

Let $\varphi : \mathbb{Z} \to \mathbb{Z}$ be such that $\varphi(t) = -t$. Define the *pullback* by φ, φ^*, as

$$\begin{aligned} \mathcal{S} &\xrightarrow{\varphi^*} \mathcal{S} \\ (x_k)_{k\in\mathbb{Z}} &\longmapsto (x_{-k})_{k\in\mathbb{Z}} \end{aligned}$$

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Consider

$$Lx := \sum_{j=-n}^{n} \left(a_j + b_j\varphi^*\right) D^j x = c, \tag{2.3}$$

where $x, c \in \mathcal{S}$; $a_j, b_j \in \mathbb{F}$ for $j = 0, \dots, n$ and $D^{-j} = (D^{-1})^j$ for $j \in \mathbb{N}$. We say L belongs to the *operator algebra* $\mathbb{F}[D, D^{-1}, \varphi^*]$ with the composition operation.

Reduction

Theorem

Let $L=\varphi^*P+Q$ *with* $P,Q\in\mathbb{F}[D,D^{-1}]$. *Then* $\widetilde{R}:=\varphi^*P-\varphi^*(Q)\in\mathbb{F}[D,\varphi^*]$ *satisfies* $\widetilde{R}L=L\widetilde{R}\in\mathbb{F}[D,D^{-1}]$.

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There exists a least $k\in\{0,1,2,\dots\}$ such that $L\widetilde{R}D^k\in\mathbb{F}[D]$. From now on we will write $\overline{R} :=\widetilde{R}D^k$.

Example 1

The first differential equation with reflection of which a Green's function was obtained was $x'(t) + m\,x(-t) = 0$ for some $m \in \mathbb{R}$. This operator is a square root of the harmonic oscillator.

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Substitute $\widetilde{D}$ by *forward difference operator* $\Delta = D - \text{Id}$ and $\widetilde{\varphi}$ by φ and we get $L = \Delta + m\varphi^* = D - \text{Id} + m\varphi^*$, that is,

$$x_{n+1} - x_n + m\,x_{-n} = 0,\ n \in \mathbb{Z}.$$

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We have that $\widetilde{R} = \mathrm{Id} - D^{-1} + m\varphi^*$. Thus,

$$L\widetilde{R} = \widetilde{R}L = (D - \mathrm{Id} + m\varphi^*)(\mathrm{Id} - D^{-1} + m\varphi^*) = D + D^{-1} + (m^2 - 2)\,\mathrm{Id}\,.$$

Hence, if $Lx = 0$ holds, so does $DRLx = 0$ and we get the equation

$$(D^2 + (m^2 - 2)D + \mathrm{Id})x = 0,$$

that is, $x_{n+2} + (m^2 - 2)x_{n+1} + x_n = 0$ for $n \in \mathbb{Z}$.

Case Example : $|m| > 2$. Solutions are are of the form

$$x_n = c_1 2^{-n}\left(-m^2 + |m|\sqrt{m^2-4}+2\right)^n + c_2 2^{-n}\left(-m^2 - |m|\sqrt{m^2-4}+2\right)^n$$

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with $c_1, c_2 \in \mathbb{R}$.

In any case, $Lx=0$ has to hold, so we deduce that

$$c_2 = \frac{1}{2}\left(\frac{|m|}{m}\sqrt{m^2-4}+m\right)c_1,$$

and all solutions of $Lx=0$ are expressed as

$$\begin{aligned} x_n =& c_1\left[2^{-n}\left(-m^2+|m|\sqrt{m^2-4}+2\right)^n + \right.\\ & \left.\frac{1}{2}\left(\frac{|m|}{m}\sqrt{m^2-4}+m\right)2^{-n}\left(-m^2-|m|\sqrt{m^2-4}+2\right)^n\right], \end{aligned}$$

for some $c_1 \in \mathbb{R}$.

Example 2

Now substitute $\widetilde{D}$ by D, that is, $L = D + m\varphi^*$ and

$$x_{n+1} + m\, x_{n-1} = 0, n \in \mathbb{Z}.$$

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$$\widetilde{R}L = L\widetilde{R} = (D + m\varphi^*)(-D^{-1} + m\varphi^*) = (m^2 - 1)\,\mathrm{Id}\,.$$

If the equation $(D + m\varphi^*)x = 0$ holds for some nontrivial $x \in \mathcal{S}$, so does $(m^2 - 1)x = 0$, which is only satisfied if $m = \pm 1$.

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In the case $L = D - \varphi^*$, take $(v_n)_{n\in\mathbb{N}} \subset \mathbb{F}$ arbitrarily and define $x_n = v_n$ if $n \in \mathbb{N}$ and $x_n = x_{1-n}$ if $n \leqslant 0$. x satisfies $Lx = 0$.

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The exponential map

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We can compute $e^{a\widetilde{\varphi}^*}$ for $a \in \mathbb{C}$ taking into account that $\widetilde{\varphi}|_{\mathbb{Z}} = \varphi$.

$$e^{a\widetilde{\varphi}^*} = \sum_{n=0}^{\infty} \frac{(a\widetilde{\varphi}^*)^n}{n!} = \sum_{n=0}^{\infty} \frac{a\,\mathrm{Id}}{(2n)!} + \sum_{n=0}^{\infty} \frac{a\widetilde{\varphi}^*}{(2n+1)!} = \cosh(a)\,\mathrm{Id} + \sinh(a)\varphi^*.$$

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Analogously, we obtain *Euler's formula*:

$$e^{\widetilde{\varphi}^*\widetilde{D}} = \sum_{n=0}^{\infty} \frac{(\widetilde{\varphi}^*\widetilde{D})^n}{n!} = \cos(\widetilde{D}) + \widetilde{\varphi}^* \sin(\widetilde{D}).$$

Given a vector space V we denote by V^* its algebraic dual. Let

$$\mathcal{T}_n=\left\{\left(\sum_{j=1}^{p}\alpha_j k^{n_j} z_j^k\right)_{k\in\mathbb{Z}}\in\mathcal{S}\ :\ z_j\in\overline{\mathbb{F}},\ n_j\in\{0,1,\dots,n\},\ \alpha_j\in\mathbb{F};\ p\in\mathbb{N}\right\}.$$

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For every $L \in \mathbb{F}[D, D^{-1}, \varphi^*]$, we have that $L(f) \in \mathcal{T}_n\ \forall f \in \mathcal{T}_n$.

Theorem

Let $W \in (\mathcal{T}_n^*)^n$*. Consider the problem*

$$Lx = c,\ Wx = h. \tag{3.1}$$

Then, there exists $\overline{R} \in \mathbb{F}[D, \varphi^*]$ *such that* $L\overline{R} \in \mathbb{F}[D]$ *and a solution of problem* (3.1) *is given by*

$$u := \Phi\,(W\Phi)^{-1}h + \left(\overline{R}H - \Phi(W\Phi)^{-1}W\overline{R}H\right)c$$

where H *is a Green's function associated to the problem*

$$L\overline{R}x = c,\ Wx = W\overline{R}x = 0, \tag{3.2}$$

assuming it exists, $W\overline{R}Hc$ *is well defined,* Φ *is the general solution of* $L\overline{R}x = 0$ *and* $W\Phi$ *is invertible.*

Systems of linear recurrence

$$(Ju)_k := Fx_{k+1} + Gx_{-k-1} + Ax_k + Bx_{-k} = 0,\ k \in \mathbb{Z}, \tag{4.1}$$

where $x_k \in \mathbb{F}^n$, $n \in \mathbb{N}$, $A, B, F, G \in \mathcal{M}_n(\mathbb{F})$ and $u \in \mathcal{F}(\mathbb{Z}, \mathbb{F}^n)$.

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We say that $M \in \mathcal{F}(\mathbb{Z}, M_n(\mathbb{F}))$ is a *fundamental matrix* of problem (4.1) if $(u_k)_{k\in\mathbb{Z}} = (M(k)\, u_0))_{k\in\mathbb{Z}}$ is a solution of equation (4.1) for every $u_0 \in \mathbb{F}^n$, that is

$$FM(k+1) + GM(-k-1) + AM(k) + BM(-k) = 0,\ k \in \mathbb{Z}.$$

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If M is a block matrix of the form

$$M = \left(\begin{array}{c|c} M_1 & M_2 \\ \hline M_3 & M_4 \end{array}\right),$$

where $M_k \in \mathcal{M}_n(\mathbb{F})$, we define $M_{(k)} := M_k$.

Fundamental matrix

Theorem

Assume that

$$\left(\begin{array}{c|c} F & G \\ \hline B & A \end{array}\right) \text{ and } \left(\begin{array}{c|c} A & B \\ \hline G & F \end{array}\right)$$

are invertible. Then

$$M := \left(\left[-\left(\begin{array}{c|c} F & G \\ \hline B & A \end{array}\right)^{-1} \left(\begin{array}{c|c} A & B \\ \hline G & F \end{array}\right) \right]^k_{(1)} + \left[-\left(\begin{array}{c|c} F & G \\ \hline B & A \end{array}\right)^{-1} \left(\begin{array}{c|c} A & B \\ \hline G & F \end{array}\right) \right]^k_{(2)} \right)_{k\in\mathbb{Z}}.$$

is a fundamental matrix of problem (4.1). *Furthermore, problem* (4.1) *equipped with the initial condition* $x_0 = u_0 \in \mathbb{F}^n$ *has a unique solution given by* $(u_k)_{k\in\mathbb{Z}} = (M(k)\,u_0))_{k\in\mathbb{Z}}$.

Theorem

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are invertible. Consider the problem

$$Jx = c, \quad Wx = h. \tag{4.2}$$

Then the sequence given by

$$u = \pi_1\left(XZ^{-1}\left[\begin{pmatrix} h \\ h \end{pmatrix} - \begin{pmatrix} W \\ W\varphi^* \end{pmatrix} Y\right] + Y\right),$$

where

$$X := \left(\left[-\left(\begin{array}{c|c} F & G \\ \hline B & A \end{array}\right)^{-1}\left(\begin{array}{c|c} A & B \\ \hline G & F \end{array}\right)\right]^k\right)_{k\in\mathbb{Z}}, \; Y := \overline{H}\left(\begin{array}{c|c} F & G \\ \hline B & A \end{array}\right)^{-1}\begin{pmatrix} c \\ \varphi^* c \end{pmatrix}, \; Z := \begin{pmatrix} W \\ W\varphi^* \end{pmatrix} X,$$

and $\pi_1 : \mathbb{F}^n \times \mathbb{F}^n \to \mathbb{F}^n$ *is such that* $\pi_1(x, y) = x$, *is the unique solution of problem* (4.2).

Open problems

There are some clear ways in which the theory could be extended. We point out here some of them.

- Non-constant coefficients.
- General involutions (order n).
- Partial difference equations.

More Information:

T., F.A.F.: *Green's functions of recurrence relations with reflection*. J. Math. Anal. Appl. 477(2). 2019, pp. 1463-1485.

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Thank you for your attention!