Limit cycles for real and complex planar fewnomial vector fields

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Talk based on the papers:

- M.J. Álvarez, B. Coll, A. Gasull and R. Prohens. *More limit cycles for complex differential equations with three monomials*. Preprint 2024.
- A. Gasull and P. Santana. *On a variant of Hilbert's 16th problem*. Preprint 2024.

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Outline of the talk

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Motivation

In Hilbert's 16th problem the main goal is to get (good) upper and lower bounds, for the number of limit cycles of a planar polynomial differential system in terms of its degree.

This problems turns out to be extremely difficult.

In this talk we try to face the same question but with a different point of view:

> *We want to control the number of limit cycles in terms of the*
> *number of monomials*
> *of the planar polynomial differential equation.*

This number of monomials can be counted either in the usual real notation or in the complex notation.

As we will see, both points of view are, not at all, equivalent.

Real and complex notations

In the real notation we simply count the total number of monomials in each component of the differential equation. For instance, the ODE

$$\begin{cases} \dot{x} = ay + by^3 + cx^5 + dx^3y^7 + exy^{23}, \\ \dot{y} = fx + gy + hxy^{23}, \end{cases}$$

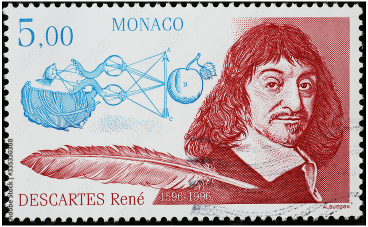
with $a, b, c, d, e, f, g, h \in \mathbb{R}$, has 8 monomials.

In the complex notation we write the planar polynomial ODE as $\dot{z} = F(z, \bar{z})$, and again simply count the number of monomials. For instance, the ODE

$$\dot{z} = Az^2\bar{z}^3 + Bz^3\bar{z}^6 + C\bar{z}^{22},$$

with $A, B, C \in \mathbb{C}$ has 3 monomials.

Motivation: Descartes rule



From Descartes' rule it follows that given a polynomial with real coefficients:

- its number of complex roots is given by its degree, but
- its number of real roots is at most:

  $$2\times \text{(its number of monomials)}+1,$$

  and it is independent of its degree.

Outline of the talk

Known results about the Hilbert monomial number

Let $\mathcal{M}_m$ be the family of real planar polynomial vector fields with m monomials. We define the *Hilbert monomial number*,

$$\mathcal{H}^M(m) = \sup\{\text{number of limit cycles of } X : \ X \in \mathcal{M}_m\}.$$

So far very little is known about $\mathcal{H}^M(m)$.

- $\mathcal{H}^M(m) = 0$ for $m \in \{1, 2, 3\}$,
- $\mathcal{H}^M(m) \geqslant m - 3$ for $m \geqslant 4$, and
- there is a sequence $(m_k) \subset \mathbb{N}$, with $m_k \to \infty$, such that $\mathcal{H}^M(m_k) \geqslant N(m_k)$, with $N(m)$ of order $O(m \ln m)$.

C. A. Buzzi, Y. R. Carvalho, A. Gasull *Limit cycles for some families of smooth and non-smooth planar systems*. Nonlinear Anal., 207, 112298, 2021.

New results about the Hilbert monomial number

Our main result on this problem is an improvement of these general lower bounds. Our first main result is:

Theorem

If $m \geqslant 9$, then $\mathcal{H}^M(m) \geqslant \frac{1}{2}m^2 - 3m - 8$.

In fact, more concretely, we prove

$$\mathcal{H}^M(m) \geqslant \frac{1}{2}m^2 - 3m - 8 + \frac{9}{4}(1 - (-1)^m) \geqslant \frac{1}{2}m^2 - 3m - 8.$$

Notice that this lower bound is higher when m is even.

We also have some better lower bounds for small values of m like for instance

$$\mathcal{H}^M(9) \geqslant 24, \quad \mathcal{H}^M(10) \geqslant 32.$$

but we will skip the details for the sake of shortness.

A fundamental result

Theorem

For any n, $r \in \mathbb{Z}_{\geqslant 0}$, there is a planar polynomial vector field with $n+r+4$ monomials and at least $2n(r+1)+n\big(1+(-1)^r\big)$ limit cycles.

Notice that the above result, by taking $m = n+r+4$, implies

$$\mathcal{H}^M(m) \geqslant 2(m-r-4)(r+1).$$

From it we prove our main results:

- By choosing $r = \frac{1}{2}m$ we get that

$$\mathcal{H}^M(m) \geqslant \frac{1}{2}m^2 - 3m - 8.$$

- For instance, by taking $r = 2$ and $n = 3$ and 4 in the theorem,

$$\mathcal{H}^M(9) \geqslant 24, \quad \mathcal{H}^M(10) \geqslant 32.$$

Outline of the talk

Known results in complex notation

For ODE with 2 monomials $\dot{z} = Az^k\bar{z}^l + Bz^m\bar{z}^n$, with k, l, m, n fixed non-negative integers and $A, B \in \mathbb{C}$ their maximum number of limit cycles is 1 and it can only exist when the total degree is odd.

M. J. Álvarez, A. Gasull, R. Prohens. *Uniqueness of limit cycles for complex differential equations with two monomials*. J. Math. Anal. Appl., 518, 126663, 2023.

For ODE with 3 monomials:

$$\dot{z} = Az^k\bar{z}^l + Bz^m\bar{z}^n + Cz^p\bar{z}^q,$$

with k, l, m, n, p, q fixed non-negative integers and $A, B, C \in \mathbb{C}$ it was proved that in general there is no upper bound for its number of limit cycles.

A. Gasull, C. Li, J. Torregrosa. *Limit cycles for 3-monomial differential equations*. J. Math. Anal. Appl., **428**, 735–749, 2015.

Known results in complex notation

We rewrite with more detail the above known results by using the following notation:

- $N = \max(k+l, m+n, p+q)$,
- $H_j(N) \in \mathbb{N} \cup \{\infty\}$ denotes the maximum number of limit cycles of the systems of the above type, with j monomials.

Theorem (AGP)

For $N = 1$, or N even, $H_2(N) = 0$ and for $N \geq 3$ odd, $H_2(N) = 1$.

Theorem (GLT)

For $N \geq 3$ odd, $H_3(N) \geq \dfrac{N+3}{2}$.

The second aim of this talk is:

- Improve the lower bound of $H_3(N)$.
- Study $H_3(2)$.

New results about $H_3(N)$

Recall that it is known that for N odd, $H_3(N) \geq (N+3)/2$.

We prove:

Theorem

For $N \geq 4$, $H_3(N) \geq N - 3$.

Results about $H_3(2)$

$$\dot{z} = A + Bz + C\bar{z} + Dz^2 + Ez\bar{z} + F\bar{z}^2.$$

There are $\binom{6}{3} = 20$ families of QS with 3 monomials. Among them it is well-known that the linear systems,

$$\dot{z} = A + Bz + C\bar{z},$$

and the homogenous QS,

$$\dot{z} = Az^2 + Bz\bar{z} + C\bar{z}^2$$

do not have limit cycles.

Hence it remains to study 18 families of QS, 9 of them with exactly one non-linear term and 9 with exactly two non-linear terms.

Our results about their number of limit cycles are resumed in next theorem.

Results about $H_3(2)$

Theorem

Consider the differential equation

$$\dot{z} = AM_1 + BM_2 + CM_3,$$

with $A, B, C \in \mathbb{C}$ *and* M_1, M_2 *and* M_3, *are 3 different fixed monomials* $M_j \in \{1, z, \bar{z}, z^2, z\bar{z}, \bar{z}^2\}$, *corresponding each one of the 18 families described above. Then its number of limit cycles is given in next tables.*

Results about $H_3(2)$

Theorem

Monomials	$1, z$	$1, \bar{z}$	$z, \bar{z}$
z^2	0	≥ 1	≥ 1
$z\bar{z}$	1	1	1
$\bar{z}^2$	0	0	0

Monomials	$z^2, z\bar{z}$	$z^2, \bar{z}^2$	$z\bar{z}, \bar{z}^2$
1	$1+1$	$1+1$	$1+1$
z	≥ 1	≥ 2	≥ 1
$\bar{z}$	≥ 1	≥ 1	≥ 1

The $1+1$ means that the family has at most 2 limit cycles, that when they exist they are not nested.

Outline of the talk

Recall our fundamental result

Theorem

For any $n,\ r \in \mathbb{Z}_{\geqslant 0}$, there is a planar polynomial vector field with $n + r + 4$ monomials and at least $2n(r+1) + n\big(1 + (-1)^r\big)$ limit cycles.

We will give some ideas of its proof. We will use the Poincaré–Pontryagin Theorem and the study of some Abelian integrals.

The Poincaré–Pontryagin Theorem: an extended version

Consider a smooth perturbation of a smooth Hamiltonian system

$$\dot{x} = -\frac{\partial H}{\partial y}(x,y) + \varepsilon f(x,y), \quad \dot{y} = \frac{\partial H}{\partial x}(x,y) + \varepsilon g(x,y),$$

with a continuum of periodic orbits $\gamma_h, h \in (a,b)$. Define

$$I(h) = \int_{\gamma_h} f\, dy - g\, dx = \iint_{\Gamma_h} \frac{\partial f}{\partial x} + \frac{\partial g}{\partial y}\, dxdy,$$

where $\Gamma_h \subset \mathbb{R}^2$ is the interior region bounded by γ_h. Then

- If $I(h^*) = 0$ and $I'(h^*) \neq 0$, for $|\varepsilon| > 0$ small enough it has a limit cycle that tends to γ_{h^*} when ε tends to zero.
- Let $(h_1, h_2) \subset (a,b)$ such that $I(h_1)I(h_2) < 0$. Then, for $|\varepsilon| > 0$ small enough it has at least one limit cycle between γ_{h_1} and γ_{h_2}.

Fundamental result: a more detailed statement

Theorem

Given $n \geqslant 1$*, there is a polynomial* $R_n \colon \mathbb{R}^2 \to \mathbb{R}$ *with* $n+1$ *monomials and* $\varepsilon_0 > 0$ *such that the perturbed system* $X_{n,r} = (P_n, Q_r)$ *given by*

$$\dot{x} = P_n(x,y) = y - y^3 + \varepsilon R_n(x,y), \quad \dot{y} = Q_r(x) = \prod_{k=-r}^{r} (x-k),$$

has at least

$$2n(r+1) + n\big(1 + (-1)^r\big)$$

limit cycles, for $0 < |\varepsilon| < \varepsilon_0$*. In particular,* $X_{n,r}$ *has* $n+r+4$ *monomials.*

For the sake of simplicity we will only give the idea of the proof of the existence of

$$2n(r+1)$$

limit cycles.

Proof of the fundamental result

PROOF: For $\varepsilon = 0$, consider the unperturbed Hamiltonian system

$$\dot{x} = y - y^3, \quad \dot{y} = x \prod_{k=-r}^{r} (x - k).$$

It has

- $r + 3$ monomials,
- $3(2r + 1)$ equilibria,
- $3r + 1$ saddles, and
- $3r + 2$ centers: $r + 1$ on each of the lines $y = \pm 1$ and r on the line $y = 0$.

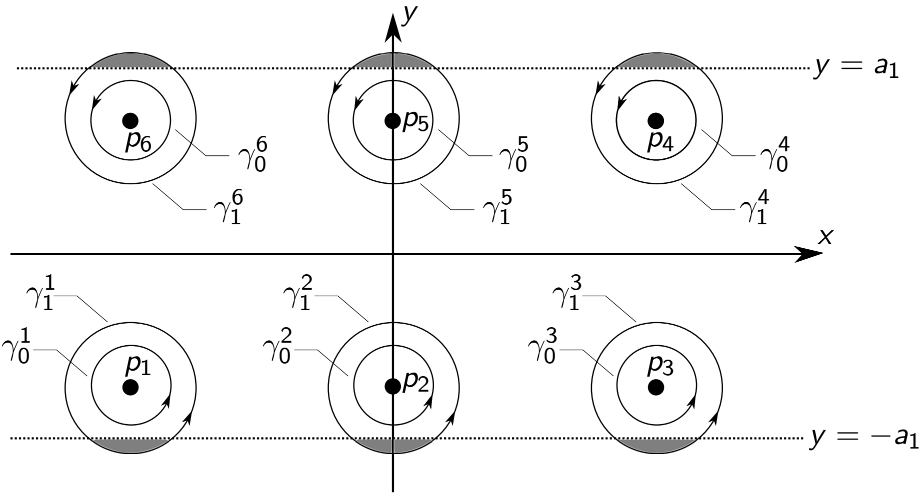


Figure: Illustration for $r = 2$ and $n = 1$ of the unperturbed system. The total number of centers is 8 and the total number of saddles 7.

Proof of the fundamental result

Given a polynomial $R\colon \mathbb{R}^2 \to \mathbb{R}$ and a periodic orbit γ of the unperturbed vector field, set

$$I(R,\gamma) = \iint_\Gamma \frac{\partial R}{\partial x}(x,y)\, dxdy,$$

where Γ is the interior region bounded by γ.

Recall that it is the Abelian integral appearing in the Poincaré-Pontryagin Theorem associated to

$$\dot{x} = P(y) + \varepsilon R(x,y), \quad \dot{y} = Q_r(x),$$

associated to γ.

Proof of the fundamental result

$$I(R,\gamma) = \iint_{\Gamma} \frac{\partial R}{\partial x}(x,y)\, dxdy.$$

Let $R_0(x,y) = x^{2n+1}$ and observe that:

$$I(R_0,\gamma_i^k) > 0 \quad \text{because} \quad \frac{\partial R_0}{\partial x}(x,y) = (2n+1)x^{2n} \geq 0.$$

Given $m_1 \geqslant 1$ set,

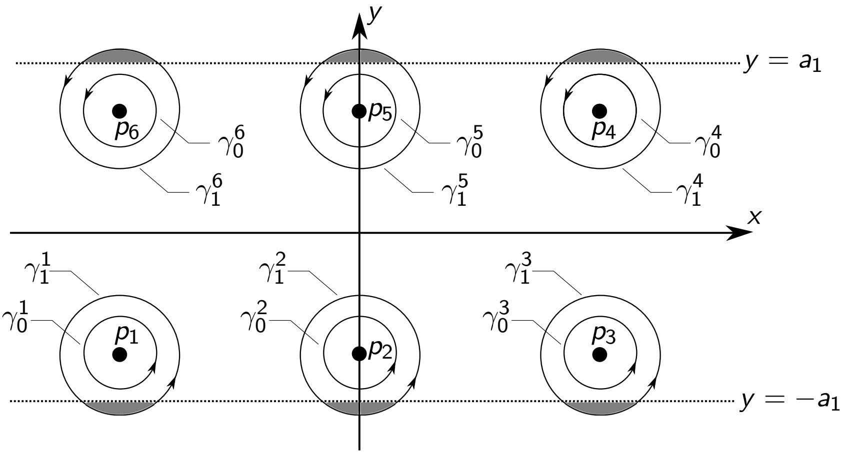
$$R_1(x,y) = R_1(x,y;m_1) = x^{2n+1} - x^{2n-1}\left(\frac{y}{a_1}\right)^{2m_1}.$$

We claim that there is $m_1 \geqslant 1$ big enough such that

$$I(R_1,\gamma_0^k) > 0 \quad \text{and} \quad I(R_1,\gamma_1^k) < 0,$$

for every $k \in \{1,\ldots,2r+2\}$.

Proof that $I(R_1, \gamma_0^k) > 0$ and $I(R_1, \gamma_1^k) < 0$



$$I(R_1, \gamma) = (2n+1) \iint_\Gamma x^{2n}\, dxdy - (2n-1) \iint_\Gamma x^{2n-2} \left(\frac{y}{a_1}\right)^{2m_1} dxdy$$

Proof of the fundamental result

Similarly, we can continue this process and obtain another family of $2(r+1)$ cycles by considering,

$$R_2(x, y; m_1, m_2) = x^{2n+1} - x^{2n-1}\left(\frac{y}{a_1}\right)^{2m_1} + x^{2n-3}\left(\frac{y}{a_2}\right)^{2m_2}.$$

Then, for this vector field we have obtained $4(r+1)$ limit cycles.

More precisely, once obtained R_1, we can take $m_2 > m_1$ big enough such that none of the previous Abelian integrals changes sign at the same time that $I(R_2, \gamma_2^k) > 0$, $k \in \{1, \ldots, 2r+2\}$.

Proof of the fundamental result

Continuing this process, we can obtain a perturbation of the form

$$R_n(x,y) = \sum_{k=0}^{n} (-1)^k x^{2(n-k)+1} \left(\frac{y}{a_k}\right)^{2m_k},$$

with $a_0 = 1$, $m_0 = 0$ and $m_k \gg m_{k-1}$, $k \in \{1, \ldots, n\}$, such that the perturbed vector field has $n + r + 4$ monomials and at least $2n(r+1)$ limit cycles, for $|\varepsilon| > 0$ small enough. □

Recall that in the theorem we state that the obtained number of limit cycles is:

$$2n(r+1) + n\big(1 + (-1)^r\big).$$

The remaining $n\big(1 + (-1)^r\big)$ cycles can be created from the two centers on the y-axis, that only exist when r is even.

Outline of the talk

Proofs of the new results with complex monomials

Theorem

For $N \geq 4$, $H_3(N) \geq N - 3$.

PROOF: For each integer $n \geq 1$, let us consider the differential equation of degree $N = n + 3 \geq 4$,

$$\dot{z} = (A + B)z - Az^{n+1} - Bz^{n+2}\bar{z} = Az(1 - z^n) + Bz(1 - \bar{z}z^{n+1}),$$

being $A = n + 1 + a + i, B = -n + i$. The critical points of this equation are $z = 0$ and the points $z = w_j$ such that $w_j^n = 1$ for $j = 1, \ldots, n$.

Observe that this equation is invariant by the change of the dependent variable $u = w_j^{n-1} z$ for all $j = 1, \ldots, n$. By this change, the critical point w_j of the original equation is transformed into the critical point $u = 1$. Hence, varying j we get that all the critical points w_j of the original equation have the same character and stability as $z = 1$.

Let us study this critical point.

It holds that

$$\begin{aligned} \operatorname{div}(X)_{z=1} &= -2na, \\ \det(dX)_{z=1} &= n|A|^2 + n|B|^2 + n(n+1)|A||B| > 0. \end{aligned}$$

Hence, if $a = 0$ the point $z = 1$ is a weak focus.

Let us compute its first Lyapunov quantity L_1 and prove that $L_1 \neq 0$.

We perform the translation $w = z - 1$ to move the critical point to the origin. We arrive to the differential equation

$$\dot{w} = (A+B)(w+1) - A(w+1)^{n+1} - B(w+1)^{n+2}(\bar{w}+1).$$

After some tedious computation we obtain that

$$L_1 = -\frac{(5+2n-n^2)n^3}{9n^2+8n+3}.$$

Notice that $L_1 < 0$ for $n = 1, 2, 3$ and $L_1 > 0$ for $n \geq 4$. Hence, the point $z = 1$ of the initial equation is an attractor when $n \leq 3$ and a repellor otherwise.

Finally it is easy to see that an Andronov-Hopf bifurcation undergoes, moving slightly the parameter a and taking it with the suitable sign.

One gets a hyperbolic limit cycle born from the critical point $(1, 0)$ of the original differential equation.

From the symmetries of the initial differential equation, from each one of the n non-zero critical points of the system a limit cycle is born at the same time. Thus, the system has at least $n = N - 3$ hyperbolic limit cycles.

The limit cycles exist for $|a|$ small enough and $a < 0$ when $n = 1, 2, 3$ and are stable and also for $|a|$ small enough and $a > 0$ when $n \geq 4$ and are unstable. □

Color artistic illustration of the distribution of the 8 limit cycles of our example of vector field with 3 complex monomials and degree 11

Some other results about $H_3(N)$

We have proved:

Theorem

For $N \geq 4$, $H_3(N) \geq N - 3$.

Some more new results that we have obtained are:

Theorem

For $N \geq 4j - 1$ and $j \geq 1$, $H_3(N) \geq N + 1$.

Proposition

For $N = 3j - 1, j \geq 1$ there are equations with three monomials and $2j$ limit cycles (then $H_3(N) \geq \frac{2(N+1)}{3}$). The limit cycles are formed by j couples of two nested limit cycles surrounding, where each couple surrounds a single critical point.

Study of some quadratic cases

- All examples with limit cycles (the lower bounds) of both tables are realized via Andronov-Hopf type bifurcations. We skip the details.
- Let us prove for instance the red cases of next tables:

Monomials	$1, z$	$1, \bar{z}$	$z, \bar{z}$
z^2	0	≥ 1	≥ 1
$z\bar{z}$	1	1	1
$\bar{z}^2$	0	0	0

Monomials	$z^2, z\bar{z}$	$z^2, \bar{z}^2$	$z\bar{z}, \bar{z}^2$
1	$1+1$	$1+1$	$1+1$
z	≥ 1	≥ 2	≥ 1
$\bar{z}$	≥ 1	≥ 1	≥ 1

Study of some quadratic cases

Monomials	$1, z$	$1, \bar{z}$	$z, \bar{z}$
$\bar{z}^2$	0	0	0

This result is a straightforward consequence of Bendixson–Dulac criterion because if $\dot{z} = F(z, \bar{z})$ the divergence of the associated vector field is

$$2\,\mathrm{Re}\left(\frac{\partial}{\partial z} F(z, \bar{z})\right),$$

and for the differential equations

$$\dot{z} = A + Bz + C\bar{z}^2, \quad \dot{z} = A + B\bar{z} + C\bar{z}^2, \quad \dot{z} = Az + B\bar{z} + C\bar{z}^2,$$

the respective divergences are $2\,\mathrm{Re}(B), 0$ and $2\,\mathrm{Re}(A)$. Because they do not change sign, the differential equations do not have limit cycles.

Study of some quadratic cases

Monomials	$z^2, z\bar{z}$	$z^2, \bar{z}^2$	$z\bar{z}, \bar{z}^2$
1	$1+1$	$1+1$	$1+1$

The proof is based on next result, proved in 1981 by Suo Guangjian and published in Chinese. We end the talk with a proof inspired by the one of the original paper.

Theorem (Suo Guangjian)

The system

$$\dot{z} = A + Bz^2 + Cz\bar{z} + D\bar{z}^2$$

either does not have limit cycles or it has exactly two limit cycles, γ and $-\gamma$. Moreover, in this latter case they are hyperbolic, with different stabilities and each one of them surrounds a different critical point.

A preliminary result

The following theorem is a well-known result on QS. We state next version due to Coppel:

Theorem (Coppel)

Suppose a QS satisfies one of the following conditions:

- *it has an invariant straight line,*
- *the highest degree terms are proportional,*

Then, the QS has at most one limit cycle and when it exists it is hyperbolic.

A scheme of the proof of Guangjian's result

$$\dot z = A + Bz^2 + Cz\bar z + D\bar z^2.$$

$$\Downarrow$$

$$\begin{cases} \dot x = a + a_{2,0}x^2 + a_{1,1}xy + a_{0,2}y^2, \\ \dot y = b + b_{2,0}x^2 + b_{1,1}xy + b_{0,2}y^2. \end{cases}$$

$$\Downarrow \quad (x_0, y_0) \longrightarrow (1, 0)$$

$$\begin{cases} \dot x = a - ax^2 + a_{1,1}xy + a_{0,2}y^2, \\ \dot y = b - bx^2 + b_{1,1}xy + b_{0,2}y^2. \end{cases}$$

$$\Downarrow \quad \text{A ``rotation''}$$

$$\begin{cases} \dot x = a - ax^2 + a_{1,1}xy, \\ \dot y = b - bx^2 + b_{1,1}xy + b_{0,2}y^2. \end{cases}$$

$$\Downarrow$$

A scheme of the proof of Guangjian's result

$$\begin{cases} \dot{x} = a - ax^2 + a_{1,1}xy, \\ \dot{y} = b - bx^2 + b_{1,1}xy + b_{0,2}y^2. \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{x} = 1 - x^2 + a_{1,1}xy, \\ \dot{y} = b - bx^2 + b_{1,1}xy + b_{0,2}y^2. \end{cases}$$

$$\Downarrow$$

$$\begin{cases} \dot{x} = 1 - x^2 + xy, \\ \dot{y} = b - bx^2 + b_{1,1}xy + b_{0,2}y^2. \end{cases}$$

$$\Downarrow \quad (X = x^2, \quad Y = 1 - x^2 + xy)$$

$$\begin{cases} X' = & 2XY, \\ Y' = & b_{0,2} + (b - 2\,b_{0,2} - b_{1,1})\,X - (2\,b_{0,2} + 1)\,Y + (-b + b_{0,2} + b_{1,1})\,X^2 \\ & + (2\,b_{0,2} + b_{1,1} - 1)\,XY + (b_{0,2} + 1)\,Y^2. \end{cases}$$

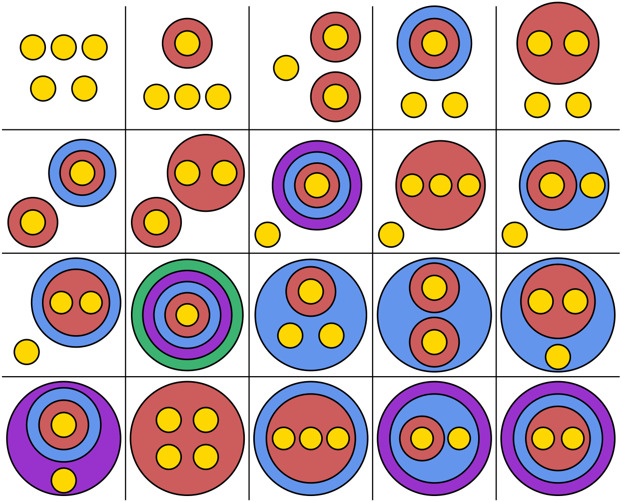


Figure from: *Counting configurations of limit cycles and centers*, A. Gasull, A. Guillamon, V. Mañosa, 2023.

Thank you very much for your attention