Invariants and reversibility in polynomial systems of ODEs

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Outline:

- Symmetries and polynomial invariants
- Two-dimensional polynomial systems with $1 : -1$ resonant singularity at the origin
- Three-dimensional polynomial systems with $1 : \zeta : \zeta^2$ resonant singularity at the origin

https://arxiv.org/abs/2309.01817

Consider the n-dimensional system of ordinary differential equations

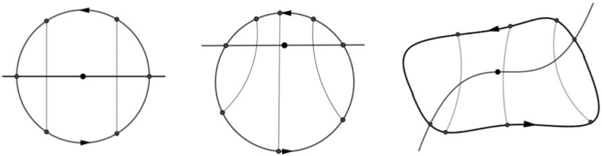
$$\dot{x} = F(x),$$

where $F(x)$ is an n-dimensional vector of smooth functions defined on some domain Ω of $\mathbb{R}^n$ or $\mathbb{C}^n$.

Definition.

It is said that system $\dot{x} = F(x)$ is *time-reversible* on Ω if there exists an involution Ψ defined on Ω such that

$$D_{\Psi}^{-1} \cdot F \circ \Psi = -F.$$



Picture from [Bastos, J. L. R., Buzzi, C. A., Torregrosa, J. CPAA, (2021)].

Motivation: symmetries

In case of a real autonomous two-dimensional system of ODEs a straight line L is *an axis of symmetry* if the orbits of the system are symmetric with respect to the line L.

- Mirror symmetry: when the phase portrait remains unchanged after it is reflected over the line L.
- Time-reversible symmetry: when the phase portrait remains unchanged after it is reflected over the line L **and** the sense of every trajectory is reversed (corresponding to a reversal of time).

Motivation: invariants and normal forms

For $\zeta = (-1)^{2/3}$ let

$$\begin{aligned}\dot{x} &= x + a_{100}x^2 + a_{001}xz + a_{101}x^2z,\\ \dot{y} &= \zeta y + b_{010}y^2 + b_{100}xy + b_{110}xy^2,\\ \dot{z} &= \zeta^2 z + c_{001}z^2 + c_{010}yz + c_{011}yz^2\end{aligned}$$

Normal form up to order 4:

$$\begin{aligned}\dot{x} &= x + \frac{1}{3}a_{001}a_{100}c_{010}x^2yz + \frac{2}{3}(-1)^{1/3}a_{001}a_{100}c_{010}x^2yz + \frac{1}{3}(-1)^{2/3}a_{001}a_{100}c_{010}x^2yz - (-1)^{1/3}a_{101}c_{010}x^2yz + 1/3a_{001}b_{100}\\ \dot{y} &= \zeta y - \frac{2}{3}a_{001}b_{010}b_{100}xy^2z - \frac{1}{3}(-1)^{1/3}a_{001}b_{010}b_{100}xy^2z + \frac{1}{3}(-1)^{2/3}a_{001}b_{010}b_{100}xy^2z + (-1)^{2/3}a_{001}b_{110}xy^2z + \frac{1}{3}a_{001}b_{100}\\ \dot{z} &= \zeta^2 z + -\frac{2}{3}a_{001}b_{100}c_{010}xyz^2 - \frac{1}{3}(-1)^{1/3}a_{001}b_{100}c_{010}xyz^2 + \frac{1}{3}(-1)^{2/3}a_{001}b_{100}c_{010}xyz^2 + \frac{1}{3}b_{100}c_{001}c_{010}xyz^2 - \frac{1}{3}(-1)^{1/3}\end{aligned}$$

- The coefficients of the normal form are invariants of a certain Lie group action
- We will give an algorithm to compute a basis of the subalgebra of invariants
- We will show an interconnection of the invariants and integrability
- Similar properties in 2-dim case have been studied by Yirong Liu and Jibin Li [Y. Liu and J. Li. Theory of values of singular point in complex autonomous differential systems. *Sci. China Ser. A* **33** (1990) 10–23.]

Motivation: integrability

[Y. N. Bibikov, Local Theory of Nonlinear Analytic Ordinary Differential Equations, LNM, Vol. 702, 1979]:

if the system

$$\dot{\mathbf{x}} = A\mathbf{x} + X(\mathbf{x}), \qquad (\mathbf{x} \in \mathbb{C}^{\mathbf{n}}) \tag{1}$$

is time-reversibile with respect to a certain linear involution, $A = \mathrm{diag}[1, -1, \lambda_3, \ldots, \lambda_n]$, then the system has at least one analytic first integral $\Psi(\mathbf{x}) = x_1 x_2 + h.o.t.$ in a neighborhood of the origin.

[J. Llibre, C. Pantazi, S. Walcher, Bull. Sci. Math. **136** (2012), 342–359], [S. Walcher, J. Math. Anal. Appl. **180** (1993), 617–632]:

If for the vector field $\mathcal{X}$ of system (1) under a linear transformation it holds

$$D_\varphi \cdot \mathcal{X} = \zeta \mathcal{X} \circ \varphi$$

where ζ is a primitive root of unity, then under certain conditions all first integrals of

$$\dot{\mathbf{x}} = A\mathbf{x} \tag{2}$$

are conserved. That is, if

$$\Psi(\mathbf{x}) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n}$$

is a first integral of (2) then

$$\widetilde{\Psi}(\mathbf{x}) = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_n^{\alpha_n} + h.o.t.$$

is a first integral of (1).

The two-dimensional case.

We consider a family of two-dimensional systems of ODEs of the form

$$\begin{aligned} \dot{x} &= x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q, \\ \dot{y} &= -y + \sum_{(p,q)\in S} b_{qp} x^q y^{p+1}, \end{aligned} \qquad (3)$$

where the coefficients a_{pq}, b_{qp} are complex numbers, and
$S = \{(p_j, q_j) \mid p_j + q_j \geq 1, j = 1, \ldots, \ell\} \subset \mathbb{N}_{-1} \times \mathbb{N}_0$.

The interconnection of time-reversibility and a one-parameter group action

$$x \mapsto e^{-i\varphi} x, \quad y \mapsto e^{i\varphi} y \quad (\varphi \in \mathbb{C} \text{ or } \mathbb{R})$$

on the phase space, by the means of polynomial invariants of corresponding group action on the space of parameters, was studied by K. Sibirsky and by Yirong Liu and Jibin Li.

Polynomial invariants

Let k be a field and G a subgroup of $GL_n(k)$. For a matrix $A \in G$ and $x \in k^n$ let $A \cdot x$ denote the usual action of G on k^n (multiplication).

Definition.

A polynomial $f \in k[x_1, \dots, x_n]$ is *invariant under* G (or *an invariant of* G) if

$$f(x) = f(A \cdot x) \text{ for all } A \in G \text{ and all } x \in k^n.$$

$$x' = e^{-\varphi} x, \quad y' = e^{\varphi} y \quad (\varphi \in \mathbb{C}) \tag{4}$$

$$\dot{x}' = x' - \sum_{(p,q) \in S} a(\varphi)_{pq} x'^{p+1} y'^{q}, \quad \dot{y}' = -y' + \sum_{(p,q) \in S} b(\varphi)_{qp} x'^{q} y'^{p+1},$$
$$a(\varphi)_{pq} = a_{pq} e^{(p-q)\varphi}, \quad b(\varphi)_{qp} = b_{qp} e^{-(p-q)\varphi}, \tag{5}$$

Equations (5) define a representation of group (4) in $\mathbb{C}^{2\ell}$, the *parameter space* of system (3).

System (3):

$$\dot{x} = x - \sum_{(p,q)\in S} a_{pq} x^{p+1} y^q$$
$$\dot{y} = -y + \sum_{(p,q)\in S} b_{qp} x^q y^{p+1}$$

Let ν denote the 2ℓ-tuple

$$\nu = (\nu_1, \ldots, \nu_{2\ell}) \in \mathbb{N}_0^{2\ell}.$$

For a given (ordered) set $S = \{(p_j, q_j) \mid j = 1, \ldots \ell\}$ let $L : \mathbb{N}_0^{2\ell} \to \mathbb{Z}^2$ be a homomorphism of the additive monoids, defined as

$$L(\nu) = \begin{pmatrix} p_1 \\ q_1 \end{pmatrix} \nu_1 + \ldots + \begin{pmatrix} p_\ell \\ q_\ell \end{pmatrix} \nu_\ell + \begin{pmatrix} q_\ell \\ p_\ell \end{pmatrix} \nu_{\ell+1} + \ldots + \begin{pmatrix} q_1 \\ p_1 \end{pmatrix} \nu_{2\ell}$$

and let $\mathcal{M}$ denote the set

$$\mathcal{M} = \left\{ \nu \in \mathbb{N}^{2\ell} \mid L(\nu) = \begin{pmatrix} k \\ k \end{pmatrix} \text{ for some } k \in \mathbb{N}_0 \right\}.$$

Obviously, $\mathcal{M}$ is an Abelian monoid.

For $\nu = (\nu_1, \dots, \nu_{2\ell}) \in \mathbb{N}_0^{2\ell}$ and the ordered vector of indeterminates $(a_{p_1 q_1} \dots a_{p_\ell q_\ell}, b_{q_\ell p_\ell} \dots b_{q_1 p_1})$, we denote by $[\nu]$ the monomial

$$[\nu] = a_{p_1 q_1}^{\nu_1} \cdots a_{p_\ell q_\ell}^{\nu_\ell} b_{q_\ell p_\ell}^{\nu_{\ell+1}} \cdots b_{q_1 p_1}^{\nu_{2\ell}} \in \mathbb{C}[a, b].$$

a monomial $[\nu]$ is an invariant of group (4) $\iff$ $\nu \in \mathcal{M}$.

For ν in $\mathcal{M}$ let $\hat{\nu}$ denote the involution of the vector ν,

$$\hat{\nu} = (\nu_{2\ell}, \nu_{2\ell-1}, \dots, \nu_1).$$

$$\mathcal{I}_S = \langle [\nu] - [\hat{\nu}] \mid \nu \in \mathcal{M} \rangle \subset \mathbb{C}[a, b].$$

The ideal $\mathcal{I}_S$ is called the *Sibirsky ideal* of system (3).

It was proved in [V. R. Open Syst. & Inform. Dyn. 2008] that the variety $\mathbf{V}(\mathcal{I}_S)$ of the ideal $\mathcal{I}_S$ is the Zariski closure of the set of systems which are time-reversible with respect to the linear transformations

$$x \mapsto \alpha y, \;\; y \mapsto \alpha^{-1} x$$

for some $\alpha \in \mathbb{C} \setminus \{0\}$. It was also proved that all systems (3) whose parameters belong to $\mathbf{V}(\mathcal{I}_S)$ are **locally analytically integrable** in a neighborhood of the origin.

$$x' = e^{-\phi}x, \quad y' = e^{\phi}y \tag{4}$$

Let $\alpha \in \mathbb{C} \setminus \{0\}$ and denote

$$x \mapsto \alpha y, \; y \mapsto \alpha^{-1}x \tag{6}$$

Transformation (6) is a composition of an orthogonal transformation and the permutation with the matrix

$$T_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

The diagonalization of T_2 is the matrix

$$E_2 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, the Lie group $e^{E_2\varphi}$ with the infinitesimal generator E_2 gives rise to the same group of transformations as (4).

Algorithm for computing the Hilbert basis

Let $A = (a_{ij})$ be an $n \times d$-matrix, $n \geq d$, with integer elements and rank d. Let $t_1, \ldots, t_d, x_1, \ldots, x_n, y_1, \ldots, y_n$ be variables and fix any elimination monomial order with $\{t_1, \ldots, t_d\} \succ \{x_1, \ldots, x_n\} \succ \{y_1, \ldots, y_n\}$. Consider $\mathbb{C}$-algebra homomorphism

$$\mathbb{C}[x_1, \ldots, x_n, y_1, \ldots, y_n] \longrightarrow \mathbb{C}[t_1, \ldots, t_d, t_1^{-1}, \ldots, t_d^{-1}, y_1, \ldots, y_n]$$
$$x_i \mapsto y_i \prod_{j=1}^{d} t_j^{a_{ij}}, \quad y_i \mapsto y_i \quad \text{for all} \quad i = 1, \ldots, n.$$

The Hilbert basis H of monoid $\mathcal{M}_A = \{\nu \in \mathbb{N}_0^n \ : \ \nu \cdot A = 0\}$ can be obtained using Algorithm 1.4.5 in [B. Sturmfels, *Algorithms in Invariant Theory*]:

1. Compute the reduced Gröbner basis $\mathcal{G}$ with respect to $\succ$ for the ideal
$$\left\langle x_i - y_i \prod_{j=1}^{d} t_j^{a_{ij}} \ : \ i = 1, \ldots, n \right\rangle .$$
2. The Hilbert basis H of $\mathcal{M}_A$ consists of all vectors ν such that $\mathbf{x}^\nu - \mathbf{y}^\nu$ appears in $\mathcal{G}$. (Here: for $\mathbf{x} = (x_1, \ldots, x_n)$ we denote by $\mathbf{x}^\nu$ the monomial $x_1^{\nu_1} x_2^{\nu_2} \ldots x_n^{\nu_n}$.)

Recall:

$$L(\nu) = \binom{p_1}{q_1}\nu_1 + \ldots + \binom{p_\ell}{q_\ell}\nu_\ell + \binom{q_\ell}{p_\ell}\nu_{\ell+1} + \ldots + \binom{q_1}{p_1}\nu_{2\ell}$$

$$\mathcal{M} = \left\{ \nu \in \mathbb{N}^{2\ell} \mid L(\nu) = \binom{k}{k} \text{ for some } k \in \mathbb{N}_0 \right\}.$$

The set of all nonegative integer vector solutions ν of equation

$$(p_1 - q_1)\nu_1 + \cdots + (p_\ell - q_\ell)\nu_\ell + (q_\ell - p_\ell)\nu_{\ell+1} + \cdots + (q_1 - p_1)\nu_{2\ell} = 0$$

coincides with monoid $\mathcal{M}$. Therefore, to obtain a Hilbert basis of $\mathcal{M}$ with the algorithm presented in the previous slide, the matrix

$$\left[(p_1 - q_1) \ \ldots \ (p_\ell - q_\ell) \ (q_\ell - p_\ell) \ \ldots \ (q_1 - p_1)\right]^\top \in \mathbb{Z}^{2\ell\times 1}.$$

can be used.

For $\mathbf{y} = (y_1, \ldots, y_{2\ell})$ we denote by $\mathbf{y}^\nu$ the monomial $y_1^{\nu_1} y_2^{\nu_2} \ldots y_{2\ell}^{\nu_{2\ell}}$.

Theorem

Let $Y = \langle [\nu] - \mathbf{y}^\nu \mid \nu \in H\rangle$ be the ideal returned by the above algorithm (so H is a Hilbert basis of $\mathcal{M}$). Then $B = \{[\nu] - [\hat{\nu}] \mid \nu \in H, \nu \neq \hat{\nu}\}$ is a Gröbner basis of $\mathcal{I}_S$.

Some algebraic properties of ideal $\mathcal{I}_S$

$$\mathcal{I}_S = \langle [\nu] - [\hat{\nu}] \mid \nu \in \mathcal{M} \rangle \subset \mathbb{C}[a, b].$$

Let $\mathfrak{A} = [\mathfrak{a}_1, \dots, \mathfrak{a}_n]$ be a $d \times n$-matrix of integers, $\mathbf{t} = (t_1, \dots, t_d)$, and let $k[\mathbf{t}, \mathbf{t}^{-1}] := k[t_1^{\pm}, \dots, t_d^{\pm}]$ be the Laurent polynomial ring over k in the variables $t_1, \dots t_d$. We define a k-algebra homomorphism

$$\pi : k[\mathbf{x}] \to k[\mathbf{t}, \mathbf{t}^{-1}] \qquad \text{by} \qquad x_i \to \mathbf{t}^{\mathfrak{a}_i}.$$

The image of π is the k-subalgebra of $k[\mathbf{t}, \mathbf{t}^{-1}]$ denoted by $k[\mathfrak{A}]$ and called the *toric ring* of matrix $\mathfrak{A}$. The kernel of π is denoted $I_{\mathfrak{A}}$ and called the *toric ideal* of $\mathfrak{A}$.

Denote by $\mathfrak{M}$ the $1 \times 2\ell$ matrix

$$\mathfrak{M} = [(p_1 - q_1) \ \dots \ (p_\ell - q_\ell) \ (q_\ell - p_\ell) \ \dots \ (q_1 - p_1)]. \tag{7}$$

Then $I_{\mathfrak{M}}$ is the corresponding toric ideal.

For a field k we denote by $k[\mathbf{x}] := k[x_1, \ldots, x_n]$ the polynomial ring in $x_1, \ldots, x_n$. A subgroup L of $\mathbb{Z}^n$ is called a lattice. For $\theta \in \mathbb{Z}^n$ we write

$$\theta = \theta^+ - \theta^-,$$

where $\theta^+, \theta^- \in \mathbb{N}_0^n$, and denote by f_θ the binomial $\mathbf{x}^{\theta^+} - \mathbf{x}^{\theta^-}$. The ideal of $k[\mathbf{x}]$ generated by all binomials f_θ, where $\theta \in \mathrm{L}$, is the so-called *lattice ideal* of L.

In our case $n = 2\ell$, the polynomial ring is $\mathbb{C}[a, b]$. Let L be the set consisting of all elements $\nu - \hat{\nu}$, where $\nu \in \mathcal{M}$. Since $\mathcal{M}$ is a monoid closed under the action of involution on vectors, L is a subgroup of $\mathbb{Z}^{2\ell}$. By I_{L} we denote the lattice ideal of L.

Theorem

The Sibirsky ideal $\mathcal{I}_S$ is a lattice ideal. More precisely, $\mathcal{I}_S = I_{\mathrm{L}}$ for a lattice L *consisting of all elements $\nu - \hat{\nu}$, where $\nu \in \mathcal{M}$.*

Ideal $\mathcal{I}_S$ and time-reversibility

Conditions of time-reversibility:

$$b_{q_s p_s} = \alpha^{p_s - q_s} a_{p_s q_s}, \qquad a_{p_s q_s} = b_{q_s p_s} \alpha^{q_s - p_s} \quad : \qquad s = 1, \ldots, \ell.$$

Thus, the set $\mathcal{R}$ of all time-reversible systems of family (3) in the space of parameters $E(a, b) = \mathbb{C}^{2\ell}$ is subset of the variety of of the first elimination ideal $\mathcal{I}$ of the ideal

$$J = \left\langle a_{p_s q_s} - b_{q_s p_s} \alpha^{q_s - p_s}, \; b_{q_s p_s} - \alpha^{p_s - q_s} a_{p_s q_s} \; : \; s = 1, \ldots, \ell \right\rangle.$$

in $\mathbb{C}[\alpha, a, b]$,

$$\mathcal{I} = J \cap \mathbb{C}[a, b].$$

Theorem

Let $\mathbf{y} = (y_1, \ldots, y_{2\ell})$. *Then* $\mathcal{I}$ *is the kernel of the polynomial map* ϕ *defined by*

$$a_{p_s q_s} \mapsto y_s t^{q_s - p_s}, \quad b_{q_s p_s} \mapsto y_{2\ell - s + 1} t^{p_s - q_s}, \quad y_{2\ell - s + 1} \mapsto y_s$$

for $s = 1, \ldots, \ell$.

Note that the kernel of the map

$$a_{p_s q_s} \mapsto y_s t^{q_s - p_s}, \quad b_{q_s p_s} \mapsto y_{2\ell - s + 1} t^{p_s - q_s}, \quad y_i \mapsto y_i,$$

for $s = 1, \ldots, \ell$ and $i = 1, \ldots, 2\ell$, is the toric ideal of the Lawrence lifting

$$\Lambda(\mathfrak{M}) = \begin{pmatrix} \mathfrak{M} & 0 \\ I_{2\ell} & I_{2\ell} \end{pmatrix}.$$

The kernel of map introduced in previous theorem is the toric ideal of the matrix

$$B = \begin{pmatrix} \mathfrak{M} \\ I_{2\ell} \mid \widetilde{I}_{2\ell} \end{pmatrix},$$

where $\widetilde{I}_{2\ell}$ is the $2\ell \times 2\ell$ matrix having 1 on the secondary diagonal and the other elements equal to 0.

Theorem

$\mathcal{I}_S = \mathcal{I} = I_B.$

A similar statement does not hold in the higher dimensional case!

The three-dimensional case (n-dim case is similar)

We investigate similar problems for the three-dimensional polynomial systems of the form

$$\begin{aligned}\dot{x} &= x\left(1 + \sum_{(p,q,r)\in S} a_{pqr} x^p y^q z^r\right)\\ \dot{y} &= y\left(\zeta + \sum_{(p,q,r)\in S} b_{rpq} x^r y^p z^q\right)\\ \dot{z} &= z\left(\zeta^2 + \sum_{(p,q,r)\in S} c_{qrp} x^q y^r z^p\right),\end{aligned} \tag{8}$$

where $\zeta^3 = 1$ the coefficients of the system are complex numbers and S is the finite set of triples
$S = \{(p_j, q_j, r_j) \mid p_j + q_j + r_j \geq 1, j = 1, \ldots, \ell\} \subset \mathbb{N}_{-1} \times \mathbb{N}_0 \times \mathbb{N}_0$.
• System (8) is a simplest generalization of the 2-dim case to the higher dimensional case.

We first study polynomial invariants of the action of the group

$$\begin{aligned} x_1 &= e^{\psi} x \\ y_1 &= e^{\psi\zeta} y \\ z_1 &= e^{\psi\zeta^2} z, \end{aligned} \tag{9}$$

where ψ is a real or complex parameter, on system (8) and show how to compute a basis of the subalgebra of invariants.

Then we study properties of system (8) related to reversibility and their connection to the invariants. It is shown that the arising binomial ideals in the two-dimensional case and in the three dimensional case have essential differences.

For the permutation matrix

$$T_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

the diagonalization is the matrix

$$E_3 = \mathrm{diag}(1, \zeta, \zeta^2),$$

where $\zeta^3 = 1,\ \zeta \neq 1$. Applying the Lie group $e^{E_3 \psi}$ with the infinitesimal generator E_3 to the phase space of system (8) we have transformation (9). It is easy to see that there are no systems in family (8) which are time-reversible with respect to a linear involution Ψ.

The following generalization of time-reversibility was already considered in [J. Llibre, C. Pantazi, and S. Walcher. *Bull. Sci. Math.*, **136** (2012) 342–359]

Definition

Let $F(x,y,z)$ be the right hand side of system (8). We say that system (8) is ζ-reversible if

$$T_3^{-1} \cdot F \circ T_3 = \zeta F \tag{10}$$

with $\zeta^3 = 1$, $\zeta \neq 1$. If (10) holds with $\zeta = 1$, we say that system (8) is T_3-equivariant.

By the result of J. Llibre, C. Pantazi, and S. Walcher all ζ-**reversible systems are locally integrable**.

We look for ζ-reversibile and T_3-equivariant systems in family (8) and discuss a connection of such systems and invariants of group (9).

Invariants

The map $L : \mathbb{N}_0^{3\ell} \to \mathbb{Z}^3$ defined as

$$L(\nu) = \sum_{j=1}^{\ell} \left(\begin{pmatrix} p_j \\ q_j \\ r_j \end{pmatrix} \nu_{3j-2} + \begin{pmatrix} r_j \\ p_j \\ q_j \end{pmatrix} \nu_{3j-1} + \begin{pmatrix} q_j \\ r_j \\ p_j \end{pmatrix} \nu_{3j} \right)$$

is a homomorphism of additive monoids.

Let $\mathcal{M} = \left\{ \nu \in \mathbb{N}_0^{3\ell} \mid L(\nu) = \begin{pmatrix} k \\ k \\ k \end{pmatrix} \text{ for some } k = 0, 1, 2, \ldots \right\}.$

$$[\nu] = a_{p_1 q_1 r_1}^{\nu_1} b_{r_1 p_1 q_1}^{\nu_2} c_{q_1 r_1 p_1}^{\nu_3} a_{p_2 q_2 r_2}^{\nu_4} \cdots a_{p_\ell q_\ell r_\ell}^{\nu_{3\ell-2}} b_{r_\ell p_\ell q_\ell}^{\nu_{3\ell-1}} c_{q_\ell r_\ell p_\ell}^{\nu_{3\ell}},$$

Theorem

A monomial $[\nu]$ is an invariant of the group (9) if and only if $\nu \in \mathcal{M}$.

For $\nu \in \mathbb{C}^{3\ell}$ let $\widetilde{\nu} = (\nu_3, \nu_1, \nu_2, \nu_6, \dots, \nu_{3\ell}, \nu_{3\ell-2}, \nu_{3\ell-1})$ and $\bar{\nu} = (\nu_2, \nu_3, \nu_1, \nu_5, \dots, \nu_{3\ell-1}, \nu_{3\ell}, \nu_{3\ell-2})$.

$$\widetilde{\nu} = \mathrm{diag}[T_3, \dots, T_3]\nu, \quad \bar{\nu} = \mathrm{diag}[T_3^{-1}, \dots, T_3^{-1}]\nu.$$

Definition

A monomial $[\hat{\nu}]$ is conjugate to the monomial $[\nu]$ if $\hat{\nu} \in \{\widetilde{\nu}, \bar{\nu}\}$.

If $\nu \in \mathcal{M}$, then $\hat{\nu} \in \mathcal{M}$.

We call the ideal $\mathcal{J}_S = \langle [\nu] - [\hat{\nu}] \ : \ \nu \in \mathcal{M} \rangle$ the Sibirsky ideal of system (8).

Theorem

Let $Y = \langle [\nu] - \mathbf{y}^\nu \mid \nu \in H \rangle$ be the ideal returned by Algorithm (thus H is a Hilbert basis of $\mathcal{M}$). Then $B = \{[\nu] - [\hat{\nu}] \mid \nu \in H, \nu \neq \hat{\nu}\}$ is a Gröbner basis of $\mathcal{J}_S$.

Invariants and normal forms

Let $\Phi = x_1 x_2 x_3$.
For system (8) the Stanley decomposition of the normal form module in $\mathcal{V}^3$ is

$$\ker \pounds = \oplus_{i=1}^{3} \mathbb{Q}[[\Phi]] x_i \mathbf{e}_i .$$

We can show that the decomposition of the normal form module can be written as

$$\ker \pounds = \oplus_{i=1}^{3} \mathbb{Q}[H][[\Phi]] x_i \mathbf{e}_i .$$

Reversibility

We obtain the following facts about ideal $\mathcal{I}_\zeta$, that corresponds to all ζ-reversible systems, and ideal $\mathcal{I}_E$, representing T_3-equivariance.
The condition of ζ-reversibility is equivalent to

$$\begin{aligned} \zeta\, a_{pqr}\, \alpha^p \beta^q \gamma^r &= b_{rpq} \\ \zeta\, b_{rpq}\, \alpha^r \beta^p \gamma^q &= c_{qrp} \\ \zeta\, c_{qrp}\, \alpha^q \beta^r \gamma^p &= a_{pqr} \end{aligned}$$

for all $(p, q, r) \in S$. Using the Elimination Theorem we have the following statement.

Proposition

The Zariski closure of the set of ζ-reversible systems is the variety of the ideal

$$\mathcal{I}_\zeta = I \cap k[a, b, c],$$

where $I = \langle 1 - w\,\alpha\beta\gamma, 1 - v(\zeta - 1), \zeta^3 - 1, \zeta\, a_{pqr}\, \alpha^p \beta^q \gamma^r - b_{rpq}, \zeta\, b_{rpq}\, \alpha^r \beta^p \gamma^q - c_{qrp}, \zeta\, c_{qrp}\, \alpha^q \beta^r \gamma^p - a_{pqr} \rangle \subset \mathbb{Q}[\alpha, \beta, \gamma, w, v, a, b, c]$.

Theorem

If the parameters of system (8) belong to the variety of the ideal $\mathcal{I}_\zeta$, then the corresponding system has a local analytic first integral in a neighborhood of the origin.

Denote by $a \cdot b \cdot c$ the product of all parameters of system (8).

The main result of our work:

Theorem A

$\mathcal{J}_S : (a \cdot b \cdot c)^\infty = \mathcal{I}_\zeta.$

- Similar results hold in the n-dim case.

Proof: https://arxiv.org/abs/2309.01817

Example: a cubic system

$$\begin{aligned}\dot{x} &= x + a_{100}x^2 + a_{001}xz + a_{101}x^2z,\\ \dot{y} &= \zeta y + b_{010}y^2 + b_{100}xy + b_{110}xy^2+,\\ \dot{z} &= \zeta^2 z + c_{001}z^2 + c_{010}yz + c_{011}yz^2\end{aligned} \tag{11}$$

In this case the corresponding matrix $\mathcal{A}$ is

$$\mathfrak{M} = \begin{pmatrix} 1 & -1 & 0 & 0 & 1 & -1 & 1 & 0 & -1\\ 0 & 1 & -1 & -1 & 0 & 1 & -1 & 1 & 0\end{pmatrix}$$

The corresponding toric ideal $I_{\mathfrak{M}}$ is

$$I_{\mathfrak{M}} = \langle -1 + a_{101}b_{110}c_{011}, c_{010} - b_{110}c_{011}, c_{001} - a_{101}c_{011}, b_{100} - a_{101}b_{110}, b_{010} - b_{110}c_{011}, a_{100} - a_{101}b_{110}, a_{001} - a_{101}c_{011}\rangle$$

To compute the Hilbert basis we use the Algorithm and consider the ideal

$$\langle a_{100} - t_1y_1, b_{010} - (t_2y_2)/t_1, c_{001} - y_3/t_2, a_{001} - y_4/t_2, b_{100} - t_1y_5, c_{010} - (t_2y_6)/t_1, a_{101} - (t_1y_7)/t_2, b_{110} - t_2y_8, c_{011} - y_9/t_1\rangle.$$

Polynomials from the Groebner basis which do not depend on t_1 and t_2 are

$$\{a_{101}b_{110}c_{011} - y_7y_8y_9, -b_{110}c_{011}y_6 + c_{010}y_8y_9, a_{101}c_{010} - y_6y_7, -a_{101}b_{110}y_5 + b_{100}y_7y_8, b_{100}c_{011} - y_5y_9, -b_{110}y_5y_6 + b_{100}c_{010}y_8, a_{001}b_{110} - y_4y_8, -a_{101}c_{011}y_4 + a_{001}y_7y_9, -c_{011}y_4y_6 + a_{001}c_{010}y_9, -a_{101}y_4y_5 + a_{001}b_{100}y_7, a_{001}b_{100}c_{010} - y_4y_5y_6, b_{110}c_{001} - y_3y_8, -a_{101}c_{011}y_3 + c_{001}y_7y_9, -a_{001}y_3 + c_{001}y_4, -c_{011}y_3y_6 + c_{001}c_{010}y_9, -a_{101}y_3y_5 + b_{100}c_{001}y_7, b_{100}c_{001}c_{010} - y_3y_5y_6, -b_{110}c_{011}y_2 + b_{010}y_8y_9, -c_{010}y_2 + b_{010}y_6, a_{101}b_{010} - y_2y_7, -b_{110}y_2y_5 + b_{010}b_{100}y_8, -c_{011}y_2y_4 + a_{001}b_{010}y_9, a_{001}b_{010}b_{100} - y_2y_4y_5, -c_{011}y_2y_3 + b_{010}c_{001}y_9, b_{010}b_{100}c_{001} - y_2y_3y_5, -a_{101}b_{110}y_1 + a_{100}y_7y_8, -b_{100}y_1 + a_{100}y_5, a_{100}c_{011} - y_1y_9, -b_{110}y_1y_6 + a_{100}c_{010}y_8, -a_{101}y_1y_4 + a_{001}a_{100}y_7, a_{001}a_{100}c_{010} - y_1y_4y_6, -a_{101}y_1y_3 + a_{100}c_{001}y_7, a_{100}c_{001}c_{010} - y_1y_3y_6, -b_{110}y_1y_2 + a_{100}b_{010}y_8, a_{001}a_{100}b_{010} - y_1y_2y_4, a_{100}b_{010}c_{001} - y_1y_2y_3\}.$$

The Hilbert basis of the monoid $\mathcal{M}$ is

$$H = \{(0,0,0,0,0,0,1,1,1),(0,0,0,0,0,1,1,0,0),(0,0,0,0,1,0,0,0,1),(0,0,0,1,0,0,0,1,0),\\ (0,0,0,1,1,1,0,0,0),(0,0,1,0,0,0,0,1,0),(0,0,1,0,1,1,0,0,0),(0,1,0,0,0,0,1,0,0),\\ (0,1,0,1,1,0,0,0,0),(0,1,1,0,1,0,0,0,0),(1,0,0,0,0,0,0,0,1),(1,0,0,1,0,1,0,0,0),\\ (1,0,1,0,0,1,0,0,0),(1,1,0,1,0,0,0,0,0),(1,1,1,0,0,0,0,0,0)\}$$

and the Sibirsky ideal J_S of system (11) is

$$J_S = \langle a_{101}c_{010} - a_{001}b_{110}, b_{100}c_{011} - c_{010}a_{101}, a_{001}b_{110} - b_{100}c_{011}, b_{110}c_{001} - a_{100}c_{011},\\ b_{100}c_{001}c_{010} - a_{100}a_{001}c_{010}, a_{101}b_{010} - c_{001}b_{110}, a_{001}b_{010}b_{100} - c_{001}b_{100}c_{010},\\ b_{010}b_{100}c_{001} - a_{100}c_{001}c_{010}, a_{100}c_{011} - b_{010}a_{101}, a_{001}a_{100}c_{010} - b_{010}a_{001}b_{100},\\ a_{100}c_{001}c_{010} - a_{100}b_{010}a_{001}, a_{001}a_{100}b_{010} - b_{010}c_{001}b_{100}, \rangle.$$

The ideal $\mathcal{I}_\zeta$ is computed as the third elimination ideal of

$$\langle 1 - w\alpha\beta\gamma, a_{100}\alpha - b_{010}, a_{001}\gamma - b_{100}, a_{101}\alpha\gamma - b_{110}, b_{010}\beta - c_{001}, b_{100}\alpha - c_{010},\\ b_{110}\alpha\beta - c_{011}, c_{001}\gamma - a_{100}, c_{010}\beta - a_{001}, c_{011}\beta\gamma - a_{101}\rangle$$

and is

$$\mathcal{I}_\zeta = \langle -a_{101}c_{010} + b_{100}c_{011}, a_{001}b_{110} - a_{101}c_{010}, a_{101}b_{010} - b_{110}c_{001}, a_{001}b_{010} - c_{001}c_{010},\\ -b_{110}c_{001} + a_{100}c_{011}, -b_{010}b_{100} + a_{100}c_{010}, a_{001}a_{100} - b_{100}c_{001}\rangle.$$

Computations with `Singular` give

$$J_S : (a \cdot b \cdot c)^\infty = \mathcal{I}_\zeta,$$

which agrees with Theorem A.

Thank you for your attention.