EJECTION-COLLISION SOLUTIONS FROM KAM TORI IN RESTRICTED N-BODY PROBLEMS

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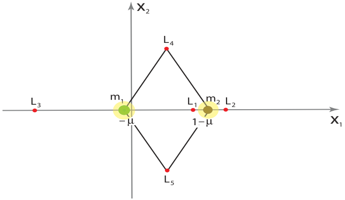
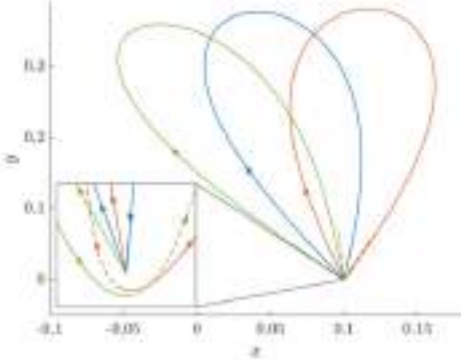
Ejection-Collision Trajectories





Source: T.-M. Seara, M. Ollé, Ó. Rodríguez, J. Soler: Journal of Nonlinear Science (2023).

Goal: Prove the existence of ejection-collision (EC) orbits in the restricted circular three-body problem applying a different approach.

Contents

Purpose and Techniques

What: Provide a proof on the existence of EC solutions in the planar restricted circular three-body problem relating these orbits to the KAM 2-tori of "rectilinear" type.

How: Applying regularisation, normalisation, symplectic reduction and a special KAM theorem.

Other Cases

1. Ejection-collision orbits from one primary body to the other (M.J. Capiński, S. Kepley, J.D. Mireles James).
2. Orbits of infinitesimal body which are asymptotic to L_4 in backward time and collide with a primary in forward time (M.J. Capiński, S. Kepley, J.D. Mireles James).
3. Parabolic ejection-collision orbits for the restricted planar circular three body problem (M. Guardia, J. Lamas, T.-M. Seara): go arbitrarily far away for small values of the mass ratio.
4. ...

This is a classic problem, treated mainly using numerical approaches but also applying analytical tools.

EC Solutions Related to Quasi-Periodic Motions of Rectilinear Type

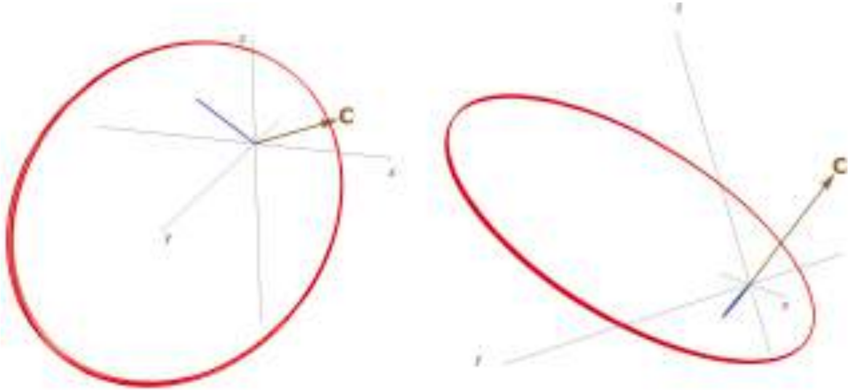
It seems rather natural to link these solutions with the invariant tori leading to motions near rectilinear in different classes of N-body problems:

1. Sitnikov restricted problems;
2. planar and spatial restricted circular three-body problems;
3. planar and spatial three-body problems;
4. extension to N-body problems.

Common feature: One particle (the infinitesimal in the restricted cases) is near to another mass).

This is what is called Lunar regime

Quasi-Periodic Motions in the Spatial Three-Body Problem



Source: J.F. P., F. Sayas, P. Yanguas: Journal of Differential Equations (2024)

Contents

Hamiltonian Functions

- For the planar RCTBP the Hamiltonian of the infinitesimal particle is

$$H_3 = \frac{1}{2}(y_1^2 + y_2^2)\,\mathcal{H}\,(x_1 y_2\,\mathcal{H}\,x_2 y_1)\,\mathcal{H}\,\frac{1\,\mathcal{H}\,\mu}{\sqrt{(\mu + x_1)^2 + x_2^2}}\,\mathcal{H}\,\frac{\mu}{\sqrt{(\mu + x_1\,\mathcal{H}\,1)^2 + x_2^2}}.$$

- For the (restricted) N-body case, if (a_k, m_k) is a central configuration, it is

$$H_N = \frac{1}{2}(y_1^2 + y_2^2)\,\mathcal{H}\,(x_1 y_2\,\mathcal{H}\,x_2 y_1)\,\mathcal{H}\sum_{k=1}^{N-1} \frac{m_k}{\sqrt{(x_1\,\mathcal{H}\,a_{k1})^2 + (x_2\,\mathcal{H}\,a_{k2})^2}}.$$

The central configuration can be the Lagrangian triangle, colineal, rhomb, kite, N-polygonal, etc.

Lunar Regime

The infinitesimal particle moves in the neighbourhood of one primary: ε is introduced as

$$(x_{1,p}, x_{2,p}) \to \varepsilon^2 m_1^{1/3}(x_{1,p}, x_{2,p}), \qquad (y_{1,p}, y_{2,p}) \to \varepsilon^{-1} m_1^{1/3}(y_{1,p}, y_{2,p}).$$

Resulting Hamiltonian functions:

$$H_3 = \tfrac{1}{2}(y_{1,p}^2 + y_{2,p}^2)\,\mathcal{H}\,\frac{1}{\sqrt{x_{1,p}^2 + x_{2,p}^2}} + \varepsilon^3(x_{2,p}y_{1,p}\mathcal{H}x_{1,p}y_{2,p}) + \tfrac{1}{2}\varepsilon^6\mu\left(x_{2,p}^2\,\mathcal{H}\,2x_{1,p}^2\right) + \mathcal{O}(\varepsilon^8),$$

$$H_N = \tfrac{1}{2}(y_{1,p}^2 + y_{2,p}^2)\,\mathcal{H}\,\frac{1}{\sqrt{x_{1,p}^2 + x_{2,p}^2}} + \varepsilon^3(x_{2,p}y_{1,p}\mathcal{H}x_{1,p}y_{2,p}) + \varepsilon^4(c_1x_{1,p} + c_2x_{2,p}) + \varepsilon^6(d_1x_{1,p}^2 + d_2x_{1,p}x_{2,p} + d_3x_{2,p}^2) + \mathcal{O}(\varepsilon^8),$$

where c_1, c_2 and d_1, d_2, d_3 are constant terms that depend on a_k and m_k.

Levi-Civita Regularisation

- We apply a parabolic symplectic transformation:

$$x_{1,p} = u^2 \,\mathcal{H}\, v^2, \quad x_{2,p} = 2uv, \quad y_{1,p} = \frac{uU \,\mathcal{H}\, vV}{2(u^2 + v^2)}, \quad y_{2,p} = \frac{uV + vU}{2(u^2 + v^2)},$$

together with

$$\frac{dt}{ds} = u^2 + v^2.$$

- We fix an energy level $h < 0$ and the frequency $w = \sqrt{\mathcal{H}8h}$.
- After an appropriate scaling the dominant term becomes

$$\frac{1}{2}(U^2 + V^2 + u^2 + v^2).$$

Higher-order terms are a polynomial perturbation of the harmonic oscillator.

Action-Angle Coordinates

We put the Hamiltonian in Lissajous action-angle variables (φ_i, Φ_i):

$$u = \sqrt{(\Phi_1 + \Phi_2)/2}\,\cos(\varphi_1 + \varphi_2)\,\mathcal{H}\,\sqrt{(\Phi_1\,\mathcal{H}\,\Phi_2)/2}\,\cos(\varphi_1\,\mathcal{H}\,\varphi_2),$$

$$v = \sqrt{(\Phi_1 + \Phi_2)/2}\,\sin(\varphi_1 + \varphi_2) + \sqrt{(\Phi_1\,\mathcal{H}\,\Phi_2)/2}\,\sin(\varphi_1\,\mathcal{H}\,\varphi_2),$$

$$U = \mathcal{H}\sqrt{(\Phi_1 + \Phi_2)/2}\,\sin(\varphi_1 + \varphi_2) + \sqrt{(\Phi_1\,\mathcal{H}\,\Phi_2)/2}\,\sin(\varphi_1\,\mathcal{H}\,\varphi_2),$$

$$V = \sqrt{(\Phi_1 + \Phi_2)/2}\,\cos(\varphi_1 + \varphi_2) + \sqrt{(\Phi_1\,\mathcal{H}\,\Phi_2)/2}\,\cos(\varphi_1\,\mathcal{H}\,\varphi_2).$$

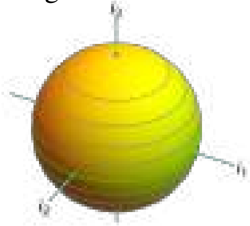
Then we average with respect to φ_1 first and φ_2 later, so that we push them to higher orders.

Symplectic Reduction

After truncating remainder and fixing $\Phi_1 = L$, we rewrite the Hamiltonians in terms of invariants:

$$i_1 = \frac{1}{2}\sqrt{L^2\,\mathcal{H}\,\Phi_2^2}\cos\varphi_2, \quad i_2 = \frac{1}{2}\sqrt{L^2\,\mathcal{H}\,\Phi_2^2}\sin\varphi_2, \quad i_3 = \frac{\Phi_2}{2}.$$

1. The poles correspond to circular solutions with opposite orientation.
2. The circles around the poles are KAM $2D$-tori.
3. When moving towards the equator, the eccentricity of the quasi-periodic orbits increases until arriving at rectilinear motions ($i_3 = 0$).



Normalised Hamiltonian Functions

We scale the angular momentum Φ_2 to $\varepsilon^6\Phi_2$ (three-body and N-body if $c_i = 0$) or to $\varepsilon^4\Phi_2$ (N-body if some $c_i \neq 0$), as the orbits are near rectilinear:

$$H_3 = \Phi_1 \,\mathcal{H}\, \varepsilon^6 \frac{5\mu\Phi_1^3}{2w^4} \,\mathcal{H}\, \varepsilon^9 \frac{2\Phi_1\Phi_2}{w^2} + \mathcal{O}(\varepsilon^{10}),$$

$$H_N = \Phi_1 + \varepsilon^6 \frac{5(d_1+d_3)\Phi_1^3}{w^4} \,\mathcal{H}\, \varepsilon^7 \frac{2\Phi_1\Phi_2}{w^2} + \mathcal{O}(\varepsilon^8) \quad \text{(if some } c_i \neq 0\text{)},$$

$$H_N = \Phi_1 + \varepsilon^6 \frac{5(d_1+d_3)\Phi_1^3}{w^4} \,\mathcal{H}\, \varepsilon^9 \frac{2\Phi_1\Phi_2}{w^2} + \mathcal{O}(\varepsilon^{18}) \quad \text{(if } c_1 = c_2 = 0\text{)}.$$

As the actions appear in three different scales, the Hamiltonians are very degenerate.

Invariant Tori in Hamiltonian Systems with High-Order Proper Degeneracy

Han, Li, Yi (2010)

Let

$$h(I,\varphi,\varepsilon) = h_0(I^{n_0}) + \varepsilon^{\beta_1} h_1(I^{n_1}) + \ldots + \varepsilon^{\beta_a} h_a(I^{n_a}) + \varepsilon^{\beta_a+1} p(I,\varphi,\varepsilon),$$

such that the intermediate Hamiltonian admits a family of invariant n-tori.

Let

- $\bar{I}^{n_i} = (I_{n_{i-1}+1}, \ldots, I_{n_i})$, with $n_{-1} = 0$ and $\bar{I}^{n_0} = I^{n_0}$.
- $\Omega = \left(\nabla_{\bar{I}^{n_0}} h_0(I^{n_0}), \ldots, \nabla_{\bar{I}^{n_a}} h_{n_a}(I^{n_a})\right)$, $i = 0, 1, \ldots, a$.
- Condition to be satisfied: $\mathrm{Rank}\left\{\partial_I^{\alpha}\,\Omega(I) \;:\; 0 \leq |\alpha| \leq s\right\} = n.$

There exists a Cantor family of real analytic invariant n-tori with excluding measure for the existence of invariant tori $\mathcal{O}(\varepsilon^{\delta/s})$ with $0 < \delta < 1/5$.

Application of Han-Li-Yi Theorem

It is enough to build the matrix

$$\Omega = \left(\frac{\partial h_0}{\partial \Phi_1}, \ \frac{\partial h_1}{\partial \Phi_1}, \ \frac{\partial h_2}{\partial \Phi_2} \right),$$

where h_0, h_1, h_2 are the first three terms of the Hamiltonians and the matrix

$$A = \left(\Omega, \quad \frac{\partial \Omega}{\partial \Phi_1}, \quad \frac{\partial \Omega}{\partial \Phi_2} \right).$$

• Since $\mathcal{H}1/(2n^{2/3})$ is a minor of order two then $\mathrm{Rank}(A) = 2$ and we can ensure the persistence of invariant $2D$-tori of rectilinear type.

• The excluding measure for the persistence of these KAM tori is improved to be of an order ε^7 or ε^9.

Next Steps

1. On the 2-sphere, we solve the variational equations of the reduced system order by order to determine the quasi-periodic solutions of rectilinear type.

2. Return to the initial rectangular coordinates, i.e. undo the successive transformations: (i) normalisation procedure; (ii) from Lissajous to LC coordinates; (iii) from LC to rectangular coordinates.

The (approximate) quasi-periodic solutions of rectilinear type we have determined depend upon a parameter.

How to Get the EC Orbits?

To characterise EC orbits we use the idea of T.-M. Seara *et al.*:

Assume ε small enough: an ejection orbit is an EC orbit if and only if it satisfies that at a minimum in the distance (with the primary) the angular momentum $M = uV \,\mathcal{H}\, vU = 0$.

We compute the time t_0 such that the infinitesimal particle gets the n-th minimum in the distance, so for $\theta_0 \in [0, \pi)$ we get $M(t_0)$ as

$$
\begin{aligned}
M_3(t_0) &= \mathcal{H}\frac{15\pi}{n^{4/3}}\varepsilon^6 \mu \sin(4\theta_0) + \mathcal{O}(\varepsilon^8), \\
M_N(t_0) &= \frac{6\pi}{n^{2/3}}\varepsilon^4 \left(c_2 \cos(2\theta_0) \,\mathcal{H}\, c_1 \sin(2\theta_0)\right) \\
&\quad \mathcal{H}\frac{10\pi}{n^{4/3}}\varepsilon^6 \left((d_3 \,\mathcal{H}\, d_1)\sin(4\theta_0) + d_2 \cos(4\theta_0)\right) + \mathcal{O}(\varepsilon^7).
\end{aligned}
$$

Main Result

Theorem

In the planar restricted N-body problem, when the particle is in a sufficiently small neighbourhood of one of the primaries, for any $n \in \mathbb{N}$, there exist either two or four families of n-EC orbits depending on the configuration of the primaries.

If $c_1 c_2 \neq 0$ there are two families of n-EC, if $c_1 c_2 = 0$ and $(d_3\, \mathcal{H}\, d_1) d_2 \neq 0$ there are four families.

Some Cases

• All configurations we have checked: equilateral Lagrangian triangle, colinear with $N \mathcal{H} 1$ masses (either evenly distributed or with the same masses), rhomb, kite, etc. satisfy $c_1 = c_2 = 0$ and there are four n-EC solutions.

• The exceptions are the regular $N \mathcal{H} 1$-polygon central configurations (with or without) the infinitesimal body around one vertex as then $c_1 \neq 0$ and there are two n-EC solutions.

A Picture for the Restricted Circular Three-Body Problem

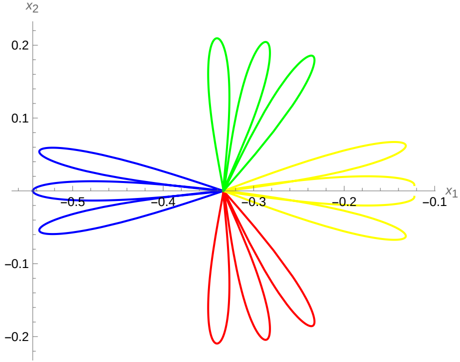


Figure: n-EC orbits to the big primary of the CRTBP in (x_1, x_2)-coordinates for $n = 3, \mu = \frac{1}{3}, \varepsilon = 0.5$. Blue: $\theta_0 = 0$. Red: $\theta_0 = \frac{\pi}{4}$. Yellow: $\theta_0 = \frac{\pi}{2}$. Green: $\theta_0 = \frac{3\pi}{4}$.

Regular $N \mathcal{H} 2$-Polygon with a Central Mass

• When the infinitesimal particle is in the neighbourhood of the central mass there are $2N \mathcal{H} 2$ EC solutions.

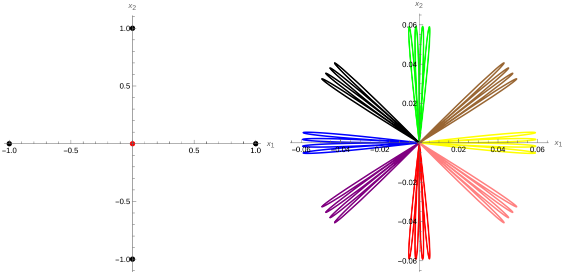


Figure: (Left) Position of the primaries in the rotating coordinates system. In red the primary in the neighborhood of with moves the particle. (Right) n-EC orbits with $N = 6, n = 4$ and $\varepsilon = 1/3$. Blue: $\theta_0 = 0$. Red: $\theta_0 = \pi/4$. Yellow: $\theta_0 = \pi/2$. Green: $\theta_0 = 3\pi/4$. Purple: $\theta_0 = \pi/8$. Pink: $\theta_0 = 3\pi/8$. Brown: $\theta_0 = 5\pi/8$. Black: $\theta_0 = 7\pi/8$.

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Hamiltonian Function

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$$H_3 = \frac{1}{2}(y_1^2 + y_2^2 + y_3^2)\,\mathcal{H}\,(x_1 y_2\,\mathcal{H}\,x_2 y_1) \\ \mathcal{H}\frac{1\,\mathcal{H}\,\mu}{\sqrt{(\mu + x_1)^2 + x_2^2 + x_3^2}}\,\mathcal{H}\,\frac{\mu}{\sqrt{(\mu + x_1\,\mathcal{H}\,1)^2 + x_2^2 + x_3^2}}.$$

1. 3 DOF Hamiltonian system.
2. There are Lagrangian 3-KAM tori in different regions of phase space.
3. Some near the coplanar plane and some near vertical to them.

Main Idea

We use a similar approach to the planar case with some remarks:

1. We apply Moser regularisation of the Kepler problem, valid for n dimensions.
2. The reduced space is the manifold $S^2 \times S^2$.
3. We apply a theory that combines normalisation with reduction, to get the quasi-periodic solutions related to nearly vertical and rectilinear motions.