Rigid systems in the plane. Overview and new results

M.J. Álvarez (Universitat de les Illes Balears)

Based on a work done in collaboration with J.L. Bravo (UEX) and L.A. Calderón (UIB).

The authors are supported by grant number PID2020-1187626GB-I00 funded by MCIN/ AEI /10.13039/501100011033.

Contents

Contents

Rigid planar systems

We are going to deal with rigid planar polynomial systems.

Characterization of rigid systems

- The origin is its only finite critical point and
- the solutions rotate around the origin with constant angular velocity.

Rigid planar analytic systems can be written (after a change of time and variables if necesary) as [Conti]

$$\begin{cases} x' = -y + xF(x,y), \\ y' = \quad x + yF(x,y), \end{cases} \qquad (1)$$

for some analytic function $F(x,y)$.

Rigid planar systems

We are going to deal with rigid planar polynomial systems.

Characterization of rigid systems

- The origin is its only finite critical point and
- the solutions rotate around the origin with constant angular velocity.

Rigid planar analytic systems can be written (after a change of time and variables if necesary) as [Conti]

$$\begin{cases} x' = -y + xF(x,y), \\ y' = \quad x + yF(x,y), \end{cases} \tag{1}$$

for some analytic function $F(x,y)$.

Basic definitions

- A center is a critical point having a neighborhood such that all the solutions are periodic.
- A center is isochronous if all the periodic orbits around it take the same time in doing a complete revolution.
- A focus is a critical point such that the orbits spiral towards or from it in positive time.
- A periodic orbit $\gamma(t)$ is a solution for which there exists $T \in \mathbb{R}^+$ such that $\gamma(T+t) = \gamma(t)$, for all t.
- A limit cycle is an isolated periodic orbit.

Rigid systems

$$\begin{cases} x' = -y + xF(x,y), \\ y' = \quad x + yF(x,y). \end{cases} \qquad (1)$$

System (1) is monodromic if $F(0,0) = 0$.

In the monodromic case, system (1) in polar coordinates writes as

$$\begin{cases} r' = rF(r\cos\theta, r\sin\theta), \\ \theta' = 1, \end{cases} \qquad (2) \qquad \Longleftrightarrow \qquad \frac{dr}{d\theta} = rF(r\cos\theta, r\sin\theta). \qquad (3)$$

Rigid systems

$$\begin{cases} x' = -y + xF(x,y), \\ y' = \quad x + yF(x,y). \end{cases} \qquad (1)$$

System (1) is monodromic if $F(0,0) = 0$.

In the monodromic case, system (1) in polar coordinates writes as

$$\begin{cases} r' = rF(r\cos\theta, r\sin\theta), \\ \theta' = 1, \end{cases} \qquad (2) \qquad \Longleftrightarrow \qquad \frac{dr}{d\theta} = rF(r\cos\theta, r\sin\theta). \qquad (3)$$

Isochronous centers

From the form in polar coordinates (2), we can deduce that if the origin is a center, then it is isochronous.

The centers with constant angular velocity are called uniformly isochronous centers.

Isochronous $\neq$ uniformly isochronous

Loud QS:

$$\begin{cases} x' = -y + xy, \\ y' = \ \ x + \dfrac{1}{4}y^2, \end{cases} \iff \begin{cases} r' = r^2 \dfrac{\sin\theta}{4}(3\cos^2\theta + 1), \\ \theta' = 1 - \dfrac{3}{4} r \cos\theta \sin^2\theta, \end{cases}$$

it has an isochronous center at the origin that it is not uniformly isochronous.

Isochronous centers

From the form in polar coordinates (2), we can deduce that if the origin is a center, then it is isochronous.

The centers with constant angular velocity are called uniformly isochronous centers.

Isochronous $\neq$ uniformly isochronous

Loud QS:

$$\begin{cases} x' = -y + xy, \\ y' = \ \ x + \frac{1}{4}y^2, \end{cases} \iff \begin{cases} r' = r^2 \frac{\sin\theta}{4}(3\cos^2\theta + 1), \\ \theta' = 1 - \frac{3}{4} r\cos\theta\sin^2\theta, \end{cases}$$

it has an isochronous center at the origin that it is not uniformly isochronous.

Why are rigid systems interesting?

- Any potential limit cycles, if they exist, have to be nested around the origin. All limit cycles are periodic orbits of the associated generalized Abel equation (3).
- The center problem is equivalent to the isochronous problem.
- Rudenok proved that any polynomial system with linear part $(-y, x)^t$ has an isochronous center if and only it can be transformed into a rigid system. Moreover, the change is of type

$$(x \to x + P(y^2), y \to y + Q(x, y))$$

Why are rigid systems interesting?

- Any potential limit cycles, if they exist, have to be nested around the origin. All limit cycles are periodic orbits of the associated generalized Abel equation (3).
- The center problem is equivalent to the isochronous problem.
- Rudenok proved that any polynomial system with linear part $(-y, x)^t$ has an isochronous center if and only it can be transformed into a rigid system. Moreover, the change is of type

$$(x \to x + P(y^2), y \to y + Q(x, y))$$

Why are rigid systems interesting?

- Any potential limit cycles, if they exist, have to be nested around the origin. All limit cycles are periodic orbits of the associated generalized Abel equation (3).
- The center problem is equivalent to the isochronous problem.
- Rudenok proved that any polynomial system with linear part $(-y, x)^t$ has an isochronous center if and only it can be transformed into a rigid system. Moreover, the change is of type

$$(x \to x + P(y^2), y \to y + Q(x, y))$$

Contents

Main problems for rigid planar system

Two main problems:

- Center-focus problem: distinguishing between a center and a focus.
- Number of limit cycles (lower and upper bounds).

REMARK: Rigid planar polynomial systems are equivalent to generalized polynomial Abel equations.

- For concrete systems, the center-focus problem is very difficult.
- The number of limit cycles can be as large as desired (Lins-Neto).

Main problems for rigid planar system

Two main problems:

- Center-focus problem: distinguishing between a center and a focus.
- Number of limit cycles (lower and upper bounds).

REMARK: Rigid planar polynomial systems are equivalent to generalized polynomial Abel equations.

- For concrete systems, the center-focus problem is very difficult.
- The number of limit cycles can be as large as desired (Lins-Neto).

Main problems for rigid planar system

Two main problems:

- Center-focus problem: distinguishing between a center and a focus.
- Number of limit cycles (lower and upper bounds).

REMARK: Rigid planar polynomial systems are equivalent to generalized polynomial Abel equations.

- For concrete systems, the center-focus problem is very difficult.
- The number of limit cycles can be as large as desired (Lins-Neto).

Center-focus problem

$$\begin{cases} x' = -y + xF_n(x,y), \\ y' = \quad x + yF_n(x,y). \end{cases}$$

Theorem (Conti)

Consider the previous system being $F_n(x,y)$ a homogeneous polynomial of degree $n \geq 1$.
The origin of the system is a center if and only if either n is odd or n is even and

$$\int_0^{2\pi} F_n(\cos\theta, \sin\theta) = 0.$$

Center-focus problem

$$\begin{cases} x' = -y + x(F_1(x,y) + F_n(x,y)), \\ y' = \quad x + y(F_1(x,y) + F_n(x,y)). \end{cases}$$

$$\begin{cases} x' = -y + x(F_2(x,y) + F_{2n}(x,y)), \\ y' = \quad x + y(F_2(x,y) + F_{2n}(x,y)). \end{cases}$$

Theorem (Algaba-Reyes)

Consider the previous systems being $F_k(x,y)$ a homogeneous polynomial of degree k and with $n > 1$.
The origin of the any of both system is a center if and only if it is reversible.

Lower bounds for the number of limit cycles

$$\begin{cases} x' = -y + x(F_1(x,y) + F_n(x,y)), \\ y' = \quad x + y(F_1(x,y) + F_n(x,y)). \end{cases}$$

Theorem (Algaba-Reyes)

There are systems in the family with at least $\left[\frac{n}{2}\right]$ limit cycles.

$$\begin{cases} x' = -y + x(F_2(x,y) + F_{2n}(x,y)), \\ y' = \quad x + y(F_2(x,y) + F_{2n}(x,y)). \end{cases}$$

Theorem (Algaba-Reyes)

There are systems in the family with at least n limit cycles.

Lower bounds for the number of limit cycles

$$\begin{cases} x' = -y + x(F_1(x,y) + F_n(x,y)), \\ y' = \quad x + y(F_1(x,y) + F_n(x,y)). \end{cases}$$

Theorem (Algaba-Reyes)

There are systems in the family with at least $[\frac{n}{2}]$ limit cycles.

$$\begin{cases} x' = -y + x(F_2(x,y) + F_{2n}(x,y)), \\ y' = \quad x + y(F_2(x,y) + F_{2n}(x,y)). \end{cases}$$

Theorem (Algaba-Reyes)

There are systems in the family with at least n limit cycles.

Lower bounds for the number of limit cycles

$$\begin{cases} x' = -y + x(F_0(x,y) + F_m(x,y) + F_n(x,y)), \\ y' = \quad x + y(F_0(x,y) + F_m(x,y) + F_n(x,y)). \end{cases}$$

Theorem (Gasull-Torregrosa)

A lower bound for the number of limit cycles of the previous system is given in the following table, depending on the degrees m, n :

m / n	1	2	3	4
0	0	1	0	1
1	-	2	2	3
2	-	-	4	4

Rigid systems with $F(x,y)$ having an even degree monomial, usually have a rotatory parameter.

Upper bounds

$$\begin{cases} x' = -y + x(a + f_n(x,y)), \\ y' = \quad x + y(a + f_n(x,y)). \end{cases}$$

Theorem (Gasull-Torregrosa)

Consider the previous system being f_n a homogeneous polynomial of degree n and define $B = \int_0^{2\pi} f_n(\cos\theta, \sin\theta)\, d\theta$.

- *If $B = 0$ and $a = 0$ then it has a center at the origin and has no limit cycles.*
- *If $a^2 + B^2 \neq 0$ and $aB \geq 0$ then it has no periodic orbits.*
- *If $aB < 0$ then it has at most a periodic orbit, which is, whenever exists, a hyperbolic limit cycle.*

Upper bounds

$$\begin{cases} x' = -y + xF(x,y), \\ y' = \quad x + yF(x,y). \end{cases} \qquad (1)$$

Theorem (Gasull-Giacomini)

Given the general rigid system (1) with F of class $\mathcal{C}^2$, consider the function

$$H(x,y) = F_{xx}F_{yy} - F_{xy}^2.$$

If $H \geq 0$ and it vanishes on a null measure set, then the rigid system (1) has at most $L_F(V)$ limit cycles, where

$$V = (x^2+y^2)(xFF_x + yFF_y + xF_y - yF_x - 1 - F^2)$$

and $L_F(V)$ is the sum of the number of holes of the connected component of the set $\mathbb{R}^2 \setminus \{V=0\}$ plus the number of limit cycles contained in $\{V=0\}$.

Contents

Studied family

We are going to study the polynomial rigid family

$$\begin{cases} x' = -y + x(F_1(x,y) + F_3(x,y)), \\ y' = \quad x + y(F_1(x,y) + F_3(x,y)), \end{cases} \quad (4) \iff r' = B(\theta)r^2 + A(\theta)r^4,$$

where

$$F_1(x,y) = b_1 x + b_2 y,$$
$$F_3(x,y) = a_1 x^3 + a_2 x^2 y + a_3 x y^2 + a_4 y^3,$$
$$B(\theta) = b_1 \cos\theta + b_2 \sin\theta,$$
$$A(\theta) = a_1 \cos^3\theta + a_2 \cos^2\theta \sin\theta + a_3 \cos\theta \sin^2\theta + a_4 \sin^3\theta.$$

It is the simplest family for which none of its parameters is rotatory.

It is possible to consider the parameter $a_4 = 0$.

Center conditions

Theorem

The origin of system (4) *is a center if and only if*

$$3a_1b_2 - a_2b_1 + a_3b_2 = 0 \text{ and } b_2(-3a_3b_1^2 + 2a_2b_1b_2 + a_3b_2^2) = 0.$$

Proof.

Necessity: Lyapunov constants.
Sufficiency:

1. If $b_1 = b_2 = 0$, the system is integrable, and one first integral is

$$H(x,y) = \frac{-1 + a_2x^3 - 3a_1x^2y - (2a_1 + a_3)y^3}{3\sqrt{(x^2+y^2)^3}}.$$

2. If $b_1^2 + b_2^2 \neq 0$, the system is reversible with respect to the straight line $b_1x + b_2y = 0$.

□

Center conditions

Theorem

The origin of system (4) *is a center if and only if*

$$3a_1b_2 - a_2b_1 + a_3b_2 = 0 \text{ and } b_2(-3a_3b_1^2 + 2a_2b_1b_2 + a_3b_2^2) = 0.$$

Proof.

Necessity: Lyapunov constants.
Sufficiency:

1. If $b_1 = b_2 = 0$, the system is integrable, and one first integral is

$$H(x,y) = \frac{-1 + a_2x^3 - 3a_1x^2y - (2a_1 + a_3)y^3}{3\sqrt{(x^2+y^2)^3}}.$$

2. If $b_1^2 + b_2^2 \neq 0$, the system is reversible with respect to the straight line $b_1x + b_2y = 0$.

□

Lower bound for the number of limit cycles

Proposition

There are systems inside the family (4) *having at least one limit cycle.*

Proof.

Consider next system $(\star)$

$$\begin{cases} x' = -y + x\left(5x + y + \dfrac{1+120a_2\pi - 82\varepsilon}{74\pi}x^3 + a_2x^2y + \dfrac{-3+10a_2\pi+98\varepsilon}{74\pi}xy^2\right), \\ y' = \quad x + y\left(5x + y + \dfrac{1+120a_2\pi - 82\varepsilon}{74\pi}x^3 + a_2x^2y + \dfrac{-3+10a_2\pi+98\varepsilon}{74\pi}xy^2\right). \end{cases}$$

Its Lyapunov constants are

$$l_2 = \varepsilon, \quad l_3 = 1.$$

Choosing $\varepsilon < 0$ small enough, one limit cycle is born from the origin by a degenerate Hopf bifurcation. □

Non-existence of limit cycles

Theorem

Consider the family (4) *for which it is not restrictive to assume* $b_1 = 0$, *that is, the family*

$$\begin{cases} x' = -y + x(b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2 + a_4 y^3), \\ y' = \quad x + y(b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2 + a_4 y^3). \end{cases}$$

If $a_1 a_3 \geq 0$ *then the system has no limit cycles.*

Proof.

We transform the system into the Abel equation and apply a criterium by [Bravo-Torregrosa]. □

Non-existence of limit cycles

Theorem

Consider the family (4) *for which it is not restrictive to assume* $b_1 = 0$, *that is, the family*

$$\begin{cases} x' = -y + x(b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2 + a_4 y^3), \\ y' = \quad x + y(b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2 + a_4 y^3). \end{cases}$$

If $a_1 a_3 \geq 0$ *then the system has no limit cycles.*

Proof.

We transform the system into the Abel equation and apply a criterium by [Bravo-Torregrosa]. □

Vector field on the Poincaré sphere

Proposition

The vector field (4) *with* $a_4 = 0$, *is topologically equivalent to the restriction to the northern hemisphere of the system*

$$\begin{cases} z_1' &= z_3\left(-z_2z_3 + b_1z_1^2z_3^2 + b_2z_1z_2z_3^2 + a_1z_1^4 + a_2z_1^3z_2 + a_3z_1^2z_2^2\right), \\ z_2' &= z_3\left(z_1z_3 + b_1z_1z_2z_3^2 + b_2z_2^2z_3^2 + a_1z_1^3z_2 + a_2z_1^2z_2^2 + a_3z_1z_2^3\right), \\ z_3' &= \left(z_3^2 - 1\right)\left(b_1z_1z_3^2 + b_2z_2z_3^2 + a_1z_1^3 + a_2z_1^2z_2 + a_3z_1z_2^2\right). \end{cases} \tag{5}$$

Remark

By the classical compactification, the infinity is full of critical points. We reparametrize the system and now the equator is no longer invariant.

Critical points at infinity (simple ones)

$$\begin{cases} z_1' = z_3\left(-z_2z_3 + b_1z_1^2z_3^2 + b_2z_1z_2z_3^2 + a_1z_1^4 + a_2z_1^3z_2 + a_3z_1^2z_2^2\right), \\ z_2' = z_3\left(z_1z_3 + b_1z_1z_2z_3^2 + b_2z_2^2z_3^2 + a_1z_1^3z_2 + a_2z_1^2z_2^2 + a_3z_1z_2^3\right), \\ z_3' = \left(z_3^2 - 1\right)\left(b_1z_1z_3^2 + b_2z_2z_3^2 + a_1z_1^3 + a_2z_1^2z_2 + a_3z_1z_2^2\right). \end{cases} \tag{5}$$

Proposition

The simple critical points of system (5) *at infinity are cusps.*
More concretely, denoting $D = a_2^2 - 4a_1a_3$*, system* (5) *has only simple critical points if and only if* $(a_1^2 + a_2^2)\, a_3\, D \neq 0$*. Moreover,*

- *If* $D > 0$*, there are three infinite critical points in the chart* $\mathcal{U}_2$*, which are cusps.*
- *If* $D < 0$*, there is only one infinite critical point in the chart* $\mathcal{U}_2$*, which is a cusp.*

Critical points at infinity (simple ones)

$$\begin{cases} z_1' = z_3\left(-z_2z_3 + b_1z_1^2z_3^2 + b_2z_1z_2z_3^2 + a_1z_1^4 + a_2z_1^3z_2 + a_3z_1^2z_2^2\right), \\ z_2' = z_3\left(z_1z_3 + b_1z_1z_2z_3^2 + b_2z_2^2z_3^2 + a_1z_1^3z_2 + a_2z_1^2z_2^2 + a_3z_1z_2^3\right), \\ z_3' = \left(z_3^2 - 1\right)\left(b_1z_1z_3^2 + b_2z_2z_3^2 + a_1z_1^3 + a_2z_1^2z_2 + a_3z_1z_2^2\right). \end{cases} \tag{5}$$

Proposition

The simple critical points of system (5) *at infinity are cusps.*
More concretely, denoting $D = a_2^2 - 4a_1a_3$, *system* (5) *has only simple critical points if and only if* $(a_1^2 + a_2^2)\, a_3\, D \neq 0$. *Moreover,*

- *If* $D > 0$, *there are three infinite critical points in the chart* $\mathcal{U}_2$, *which are cusps.*
- *If* $D < 0$, *there is only one infinite critical point in the chart* $\mathcal{U}_2$, *which is a cusp.*

Critical points at infinity (multiple ones)

Proposition

The multiple infinite critical points at infinity of system (5) *are either cusps, the union of two hyperbolic sectors, the union of two hyperbolic and one parabolic sector, or the union of two hyperbolic and two parabolic sectors.*

More concretely, assuming $u = 0$ is the multiple critical point, then:

- *If $a_1 \neq 0$:*
  - *If $a_3 = 0$, $a_2 \neq 0$, the system in the chart $\mathcal{U}_2$ has a simple critical point, which is a cusp, and a double one that has two hyperbolic and two parabolic sectors or it has only two hyperbolic sectors.*
  - *If $a_2 = a_3 = 0$, the system in the chart $\mathcal{U}_2$ only has a triple critical point, which has two hyperbolic and one parabolic sectors when $b_2 \neq 0$, and is a cusp when $b_2 = 0$.*
- *If $a_1 = 0$ (in which case the only possibility is $a_3 = 0$, $a_2 \neq 0$) the system in the chart $\mathcal{U}_2$ has a double critical point, which has two hyperbolic and two parabolic sectors, and a simple one that is a cusp.*

Critical points at infinity (multiple ones)

Proposition

The multiple infinite critical points at infinity of system (5) *are either cusps, the union of two hyperbolic sectors, the union of two hyperbolic and one parabolic sector, or the union of two hyperbolic and two parabolic sectors. More concretely, assuming* $u = 0$ *is the multiple critical point, then:*

- *If* $a_1 \neq 0$*:*
  - *If* $a_3 = 0$, $a_2 \neq 0$, *the system in the chart* $\mathcal{U}_2$ *has a simple critical point, which is a cusp, and a double one that has two hyperbolic and two parabolic sectors or it has only two hyperbolic sectors.*
  - *If* $a_2 = a_3 = 0$, *the system in the chart* $\mathcal{U}_2$ *only has a triple critical point, which has two hyperbolic and one parabolic sectors when* $b_2 \neq 0$, *and is a cusp when* $b_2 = 0$.
- *If* $a_1 = 0$ *(in which case the only possibility is* $a_3 = 0$, $a_2 \neq 0$*) the system in the chart* $\mathcal{U}_2$ *has a double critical point, which has two hyperbolic and two parabolic sectors, and a simple one that is a cusp.*

Centers in the sphere

$$\begin{cases} x' = -y + x(b_1 x + b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2), \\ y' = \quad x + y(b_1 x + b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2). \end{cases} \tag{4}$$

$$\begin{cases} z_1' = z_3\left(-z_2 z_3 + b_1 z_1^2 z_3^2 + b_2 z_1 z_2 z_3^2 + a_1 z_1^4 + a_2 z_1^3 z_2 + a_3 z_1^2 z_2^2\right), \\ z_2' = z_3\left(z_1 z_3 + b_1 z_1 z_2 z_3^2 + b_2 z_2^2 z_3^2 + a_1 z_1^3 z_2 + a_2 z_1^2 z_2^2 + a_3 z_1 z_2^3\right), \\ z_3' = \left(z_3^2 - 1\right)\left(b_1 z_1 z_3^2 + b_2 z_2 z_3^2 + a_1 z_1^3 + a_2 z_1^2 z_2 + a_3 z_1 z_2^2\right). \end{cases} \tag{5}$$

We will say that a vector field has a global center on the sphere if every solution is periodic, except for a null-measure set containing critical points, homoclinic and heteroclinic connections.

Theorem

If system (4) *has a center at the origin, then it is a global center of the vector field* (5) *on the sphere.*

Periodic orbit on the sphere

Theorem

If system (5) has two cusps and no other critical points on the equator, then it always has a periodic solution in the sphere, symmetric with respect to the origin and its intersection with the equator consists of two (symmetric) regular points.

Proof.

If we denote the equator of the sphere as Q, the key point is proving that a solution starting in $Q^+ = \{(z_1, z_2, 0) | z_2 > 0\}$ intersects Q^- (or viceversa). If this was not true there would exist a nodal sector, but none exist. Hence, we have a map from Q^+ to Q^- (or viceversa).

Composing it with the symmetry with respect the center of the sphere, we have a map from Q^+ to Q^+ (resp. from Q^- to Q^-). This map extends to the closure, but reverts the order. Hence, there cannot be any fixed point in the border. Applying Brower's fixed point theorem there is a fixed point, that is a solution crossing Q at symmetric points. By the symmetry of the system, it is a periodic solution. □

Periodic orbit on the sphere

Theorem

If system (5) has two cusps and no other critical points on the equator, then it always has a periodic solution in the sphere, symmetric with respect to the origin and its intersection with the equator consists of two (symmetric) regular points.

Proof.

If we denote the equator of the sphere as Q, the key point is proving that a solution starting in $Q^+ = \{(z_1, z_2, 0) | z_2 > 0\}$ intersects Q^- (or viceversa). If this was not true there would exist a nodal sector, but none exist. Hence, we have a map from Q^+ to Q^- (or viceversa).

Composing it with the symmetry with respect the center of the sphere, we have a map from Q^+ to Q^+ (resp. from Q^- to Q^-). This map extends to the closure, but reverts the order. Hence, there cannot be any fixed point in the border. Applying Brower's fixed point theorem there is a fixed point, that is a solution crossing Q at symmetric points. By the symmetry of the system, it is a periodic solution. □

Periodic orbit on the sphere

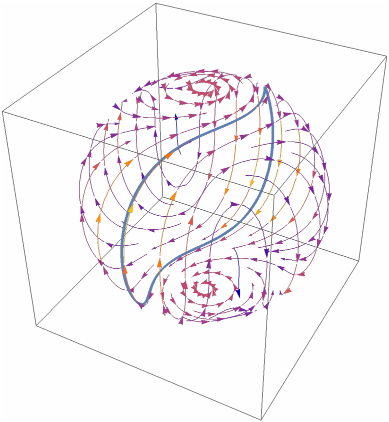
Theorem

If system (5) has two cusps and no other critical points on the equator, then it always has a periodic solution in the sphere, symmetric with respect to the origin and its intersection with the equator consists of two (symmetric) regular points.

Proof.

If we denote the equator of the sphere as Q, the key point is proving that a solution starting in $Q^+ = \{(z_1, z_2, 0) | z_2 > 0\}$ intersects Q^- (or viceversa). If this was not true there would exist a nodal sector, but none exist. Hence, we have a map from Q^+ to Q^- (or viceversa).
Composing it with the symmetry with respect the center of the sphere, we have a map from Q^+ to Q^+ (resp. from Q^- to Q^-). This map extends to the closure, but reverts the order. Hence, there cannot be any fixed point in the border. Applying Brower's fixed point theorem there is a fixed point, that is a solution crossing Q at symmetric points. By the symmetry of the system, it is a periodic solution. □

Periodic orbit on the sphere



Periodic orbit on the sphere (II)

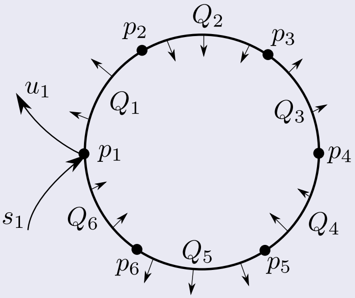
Theorem

If system (5) *has six cusps on the equator, then it either has an homoclinic or heteroclinic connection, or a periodic solution in the sphere. In this last case, the periodic solution is symmetric with respect to the origin and its intersection with the equator consists of two or six (symmetric) regular points.*

Proof of the Theorem about the periodic orbit on the sphere

Proof.

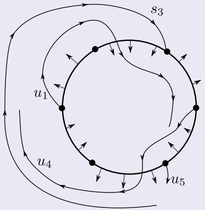
- p_i the infinite critical points in clockwise order. They are cusps.
- s_i, u_i the stable and unstable varieties of p_i, respectively,
- Q_i the sector of the equator Q between the critical point p_i and p_{i+1}, where $p_7 = p_1$.



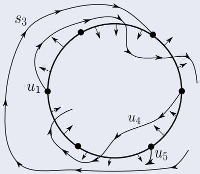
Proof of the Theorem about the periodic orbit on the sphere

Proof (cont).

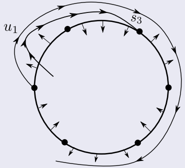
Claim 1: There exists $i \in \{1, \ldots, 6\}$ such that either u_i cuts the equator in the first turn in he sector symmetric to Q_i, or s_i cuts the equator in the first turn in the sector symmetric to Q_{i-1}.



Case 1. u_1 intersects Q_2, but then it does not intersect Q_3 in the first turn around the center.



Case 2. u_1 intersects Q_2 and then it intersects Q_3 in the first turn around the center.



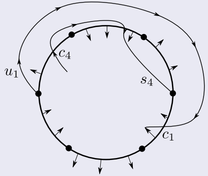
Case 3. u_1 does not intersect Q_2 in the first turn around the center.

Proof of the Theorem about the periodic orbit on the sphere

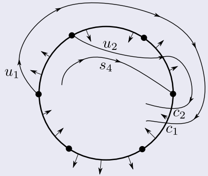
Proof (cont).

Claim 2: There exists a periodic solution crossing the equator. We may assume that u_1 intersects Q_4 in the first turn.

- If u_1 does not intersect the equator before Q_4 :



Case 1: if s_4 intersects Q_1 in a point c_4.



Case 2: if s_4 does not intersect Q_1 in a point. Then u_2 must intersect Q_4 in a point c_2.

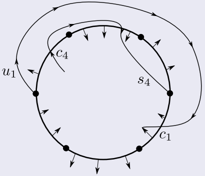
- If u_1 intersects Q at one point previous to Q_4, it must be at Q_2, but in order to intersect Q_4 afterwards, it must intersect Q_3 as well. We can argue as before.

Proof of the Theorem about the periodic orbit on the sphere

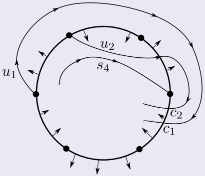
Proof (cont).

Claim 2: There exists a periodic solution crossing the equator. We may assume that u_1 intersects Q_4 in the first turn.

- If u_1 does not intersect the equator before Q_4 :



Case 1: if s_4 intersects Q_1 in a point c_4.



Case 2: if s_4 does not intersect Q_1 in a point. Then u_2 must intersect Q_4 in a point c_2.

- If u_1 intersects Q at one point previous to Q_4, it must be at Q_2, but in order to intersect Q_4 afterwards, it must intersect Q_3 as well. We can argue as before.

Limit cycles in the sphere

Proposition

There are systems inside family (5) having at least 3 periodic orbits in the sphere.

Proof.

If we compactify system $(\star)$, we get that there is at least one limit cycle in each one of both hemispheres. We prove that for this system $D = a_2^2 - 4a_1a_3 < 0$. Hence, only two cusps appear on the equator.
Applying the first theorem on periodic orbits, another periodic orbit exists in the sphere. If this periodic orbit is isolated, then it will be a limit cycle. □

Limit cycles in the sphere

Proposition

There are systems inside family (5) having at least 3 periodic orbits in the sphere.

Proof.

If we compactify system $(\star)$, we get that there is at least one limit cycle in each one of both hemispheres. We prove that for this system $D = a_2^2 - 4a_1a_3 < 0$. Hence, only two cusps appear on the equator. Applying the first theorem on periodic orbits, another periodic orbit exists in the sphere. If this periodic orbit is isolated, then it will be a limit cycle. □

Invariant straight lines

Proposition

The vector field (5) *has an invariant straight line if and only if*

$$4a_1a_3 - a_2^2 = 0, \quad 4a_2a_3b_1b_2^2 + 8a_3^2b_2^3 + a_2^3 = 0.$$

In this case, the invariant straight line is

$$-a_2b_2x + 2a_3b_2y + a_2 = 0.,$$

and it is a heteroclinic connection joining two degenerate infinite critical points.

Contents

Open questions

$$\begin{cases} x' = -y + x(b_1 x + b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2), \\ y' = \, x + y(b_1 x + b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2). \end{cases} \tag{4}$$

Open question 1

Is one the maximum number of limit cycles of system (4)?

$$\begin{cases} z_1' = z_3 \left(-z_2 z_3 + b_1 z_1^2 z_3^2 + b_2 z_1 z_2 z_3^2 + a_1 z_1^4 + a_2 z_1^3 z_2 + a_3 z_1^2 z_2^2\right), \\ z_2' = z_3 \left(z_1 z_3 + b_1 z_1 z_2 z_3^2 + b_2 z_2^2 z_3^2 + a_1 z_1^3 z_2 + a_2 z_1^2 z_2^2 + a_3 z_1 z_2^3\right), \\ z_3' = \left(z_3^2 - 1\right)\left(b_1 z_1 z_3^2 + b_2 z_2 z_3^2 + a_1 z_1^3 + a_2 z_1^2 z_2 + a_3 z_1 z_2^2\right). \end{cases} \tag{5}$$

Open question 2

If system (5) has an annulus or periodic orbits, is it a global center?

Open question 3

Which is the maximum number of limit cycles that system (5) can have?

Open questions

$$\begin{cases} x' = -y + x(b_1 x + b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2), \\ y' = \quad x + y(b_1 x + b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2). \end{cases} \quad (4)$$

Open question 1

Is one the maximum number of limit cycles of system (4)?

$$\begin{cases} z_1' = z_3 \left(-z_2 z_3 + b_1 z_1^2 z_3^2 + b_2 z_1 z_2 z_3^2 + a_1 z_1^4 + a_2 z_1^3 z_2 + a_3 z_1^2 z_2^2\right), \\ z_2' = z_3 \left(z_1 z_3 + b_1 z_1 z_2 z_3^2 + b_2 z_2^2 z_3^2 + a_1 z_1^3 z_2 + a_2 z_1^2 z_2^2 + a_3 z_1 z_2^3\right), \\ z_3' = \left(z_3^2 - 1\right) \left(b_1 z_1 z_3^2 + b_2 z_2 z_3^2 + a_1 z_1^3 + a_2 z_1^2 z_2 + a_3 z_1 z_2^2\right). \end{cases} \quad (5)$$

Open question 2

If system (5) has an annulus or periodic orbits, is it a global center?

Open question 3

Which is the maximum number of limit cycles that system (5) can have?

Open questions

$$
\begin{cases} x' = -y + x(b_1 x + b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2), \\ y' = \quad x + y(b_1 x + b_2 y + a_1 x^3 + a_2 x^2 y + a_3 x y^2). \end{cases} \tag{4}
$$

Open question 1

Is one the maximum number of limit cycles of system (4)?

$$
\begin{cases} z_1' = z_3 \left(-z_2 z_3 + b_1 z_1^2 z_3^2 + b_2 z_1 z_2 z_3^2 + a_1 z_1^4 + a_2 z_1^3 z_2 + a_3 z_1^2 z_2^2 \right), \\ z_2' = z_3 \left(z_1 z_3 + b_1 z_1 z_2 z_3^2 + b_2 z_2^2 z_3^2 + a_1 z_1^3 z_2 + a_2 z_1^2 z_2^2 + a_3 z_1 z_2^3 \right), \\ z_3' = \left(z_3^2 - 1 \right) \left(b_1 z_1 z_3^2 + b_2 z_2 z_3^2 + a_1 z_1^3 + a_2 z_1^2 z_2 + a_3 z_1 z_2^2 \right). \end{cases} \tag{5}
$$

Open question 2

If system (5) has an annulus or periodic orbits, is it a global center?

Open question 3

Which is the maximum number of limit cycles that system (5) can have?

More open questions

$$\begin{cases} z_1' = z_3\left(-z_2z_3 + b_1z_1^2z_3^2 + b_2z_1z_2z_3^2 + a_1z_1^4 + a_2z_1^3z_2 + a_3z_1^2z_2^2\right), \\ z_2' = z_3\left(z_1z_3 + b_1z_1z_2z_3^2 + b_2z_2^2z_3^2 + a_1z_1^3z_2 + a_2z_1^2z_2^2 + a_3z_1z_2^3\right), \\ z_3' = \left(z_3^2 - 1\right)\left(b_1z_1z_3^2 + b_2z_2z_3^2 + a_1z_1^3 + a_2z_1^2z_2 + a_3z_1z_2^2\right). \end{cases} \tag{5}$$

Open question 4

Does system (5) have always a periodic orbit that intersects the equator?

Open question 5

Do the systems inside family (5) having heteroclinic connections constitute a zero measure set in the set of parameters?

Open question 6

Can the periodic orbit of system (5) that crosses the equator be algebraic?

More open questions

$$\begin{cases} z_1' = z_3\left(-z_2 z_3 + b_1 z_1^2 z_3^2 + b_2 z_1 z_2 z_3^2 + a_1 z_1^4 + a_2 z_1^3 z_2 + a_3 z_1^2 z_2^2\right), \\ z_2' = z_3\left(z_1 z_3 + b_1 z_1 z_2 z_3^2 + b_2 z_2^2 z_3^2 + a_1 z_1^3 z_2 + a_2 z_1^2 z_2^2 + a_3 z_1 z_2^3\right), \\ z_3' = \left(z_3^2 - 1\right)\left(b_1 z_1 z_3^2 + b_2 z_2 z_3^2 + a_1 z_1^3 + a_2 z_1^2 z_2 + a_3 z_1 z_2^2\right). \end{cases} \tag{5}$$

Open question 4

Does system (5) have always a periodic orbit that intersects the equator?

Open question 5

Do the systems inside family (5) having heteroclinic connections constitute a zero measure set in the set of parameters?

Open question 6

Can the periodic orbit of system (5) that crosses the equator be algebraic?

More open questions

$$
\begin{cases}
z_1' = z_3 \left(-z_2 z_3 + b_1 z_1^2 z_3^2 + b_2 z_1 z_2 z_3^2 + a_1 z_1^4 + a_2 z_1^3 z_2 + a_3 z_1^2 z_2^2\right), \\
z_2' = z_3 \left(z_1 z_3 + b_1 z_1 z_2 z_3^2 + b_2 z_2^2 z_3^2 + a_1 z_1^3 z_2 + a_2 z_1^2 z_2^2 + a_3 z_1 z_2^3\right), \\
z_3' = \left(z_3^2 - 1\right) \left(b_1 z_1 z_3^2 + b_2 z_2 z_3^2 + a_1 z_1^3 + a_2 z_1^2 z_2 + a_3 z_1 z_2^2\right).
\end{cases} \tag{5}
$$

Open question 4

Does system (5) have always a periodic orbit that intersects the equator?

Open question 5

Do the systems inside family (5) having heteroclinic connections constitute a zero measure set in the set of parameters?

Open question 6

Can the periodic orbit of system (5) that crosses the equator be algebraic?

Bibliography

- A.Algaba, M. Reyes, *Computing center conditions for vector fields with constant angular speed.*
- A.Algaba, M.Reyes, A.Bravo, *Uniformly isochronous quintic planar vector fields.*
- R.Conti, *Uniformly Isochronous of Polynomial Systems in* R^2.
- F.Dias, L.F.Mello, *The center-focus problem and small amplitude limit cycles in rigid systems.*
- A.Gasull, H.Giacomini, *Effectiveness of the Bendixson?Dulac theorem.*
- A.Gasull, J.Torregrosa, *Exact number of limit cycles for a family of rigid systems.*
- A.Gasull, J.Torregrosa(2), *Some results on rigid systems.*

Bibliography

- L.V.Ivanov, E.P.Volokitin, *Uniformly isochronous polynomial centers.*
- J.Llibre, R.Rabanal, *Explicit focal basis for some planar rigid polynomial differential systems*
- A.Lins-Neto, *On the number of solutions of the equation* $dx/dt = \sum_{j=0}^{n} a_j(t)x^j, 0 \leq t \leq 1$, *for which* $x(0) = x(1)$.
- A.E.Rudenok, *Strong isochronism of a center. Periods of limit cycles of a Liénard's system.*

Rigid systems in the plane. Overview and new results

M.J. Álvarez (Universitat de les Illes Balears)