# Abelian Integrals and Limit Cycles

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# Formulation H16P

1. Projective classification of the ovals of a real plane algebraic curve:

$$\left\{(x,y)\in \mathbf{R}^2 : H(x,y)=0\right\},$$

where $H$ is a polynomial of degree $n$

2. Determine the maximum number $\mathcal{H}(n)$ of limit cycles of $X \leftrightarrow P_n dy - Q_n dx$,

$$X \leftrightarrow \begin{cases} \dot{x} = P_n(x,y) = \sum_{0\le i+j\le n} a_{ij}x^i y^j \\ \dot{y} = Q_n(x,y) = \sum_{0\le i+j\le n} b_{ij}x^i y^j \end{cases}, a_{ij}, b_{ij} \in \mathbb{R}, x, y \in \mathbb{R}$$

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# Analogy

- Method of continuous variation of the coefficients
- Algebraic Ovals - Theorem of Harnack
- Limit cycles (transcendental) - Finiteness of the Hilbert numbers $\mathcal{H}(n)$ far from complete

# Some known results

- $\mathcal{H}(2) \geq 4$ [1979, Shi, Chen and Wang]
- $\mathcal{H}(3) \geq 12$ [2005, Yu and Han]
- $\mathcal{H}(4) \geq 22$ [2005, Christopher]
- $\mathcal{H}(n) \geq kn^2 \ln n$ for some constant $k$ [1995, Christopher and Lloyd]
- $\mathcal{H}(n) \geq \frac{1}{4}(n+1)^2 \left(1.442695 \ln(n+1) - \frac{1}{6}\right) + n - \frac{2}{3}$ [2003, J. Li]

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- Individual finiteness
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  - 1990s Ilyashenko and Ecalle
- Uniform finiteness
  - By compactification of phase and parameter space
  - Roussarie reduction to prove local finite cyclicity of limit periodic sets

# Setting and notations

- $X_H \leftrightarrow \begin{cases} \dot{x} = \dfrac{\partial H}{\partial y}(x,y) \\ \dot{y} = -\dfrac{\partial H}{\partial x}(x,y) \end{cases}$, where $H$ is polynomial of degree $n+1$

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- period annulus
- $\gamma(h) \equiv \{(x, y) \in \mathbf{R}^2 : H(x, y) = h\}$

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# Formulation of weak H16P for limit cycles

- $X = X_H +$ 'polynomial perturbation'
- Weakened Hilbert's 16th Problem
- Tangential Hilbert's 16th Problem
- Infinitesimal Hilbert's 16th Problem

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1. Determine $LC\,(n, H) = \sup\{$number of limit cycles of $X_\lambda$ that bifurcate from the period annulus of $X_H\}$, where the sup is taken over all polynomial vector fields $X_\lambda$ of degree $n$ for which $X_{\lambda_0} = X_H$.

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2. Determine $LC(n) = sup\{LC(n, H) : H$ generic polynomial of degree $n+1\}$

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- Abelian integral is the integral of a rational 1-form along an algebraic oval

# Formulation of weak H16P for zeroes of associated Abelian integral

1. Determine
$Z(n, H) = \sup \{\text{number of zeroes of } I(h) \text{ where } h \in [0, h_0]\}$,
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- Conjecture: $Z(n) = \frac{n(n+1)}{2} - 1$

# Limit cycles and Abelian integral

- Displacement map:

$$\delta(h,\varepsilon) = P(h,\varepsilon) - h = \int_{\gamma_\varepsilon(h)} \mathrm{d}H$$

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- For $h$ in the interior of a period annulus:

$$\delta(h,\varepsilon) = \varepsilon\left[I(h) + \varepsilon\varphi(h,\varepsilon)\right]$$
$$\varphi(h,\varepsilon) = O(\varepsilon), \varepsilon \to 0$$

# Weak Hilbert's 16th Problem for Abelian integrals

## Theorem (Pontryagin)

*Suppose that $I(h)$ is not identically zero for $h \in (a, b)$, then the following statements hold:*

- *If $X_\varepsilon$ has a limit cycle bifurcating from $\gamma_{h^*}$, then $I(h^*) = 0$*

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- *If there exists an $h^* \in (a, b)$ such that $I(h^*) = 0$ and $I'(h^*) \neq 0$, then $X_\varepsilon$ has a unique limit cycle bifurcating from $\gamma(h^*)$; moreover this limit cycle is hyperbolic.*

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- *If there exists an $h^* \in (a, b)$ such that $I(h^*) = I'(h^*) = \ldots = I^{(k-1)}(h^*) = 0$ and $I^{(k)}(h^*) \neq 0$, then $X_\varepsilon$ has at most $k$ limit cycles bifurcating from $\gamma(h^*)$, taking into account the multiplicity of the limit cycles.*

# Example

- Van der Pol equation $x'' + \varepsilon\left(x^2 - 1\right)x' + x = 0$,

$$x' = y, y' = -x + \varepsilon\left(1 - x^2\right)y$$

- For $\varepsilon = 0$ : Hamiltonian system with $\gamma(h) = \left\{x^2 + y^2 = h, h > 0\right\}$

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- $I (h) = -\int_{\gamma(h)} \left(1 - x^2\right) ydx = \int_0^{2\pi} \left(1 - h\cos^2\theta\right) \sin^2\theta \mathrm{d}\theta =$ $\pi h \left(\frac{h}{4} - 1\right)$

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- $h = 0$ corresponds to the singularity of $X_H$
- $h = 4$ corresponds to the periodic orbit $x^2 + y^2 = 4$; unique and hyperbolic

# Melnikov functions

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- If $\delta$ and $M_k$ analytic, then $M_k \equiv 0, \forall k \implies X_\varepsilon$ integrable vector field

# Algorithm to compute Melnikov functions

## Definition

$H$ satisfies the condition $(*)$ if for any analytic 1-form $\omega$ holds the following:

$$\int_{\gamma(h)} \omega \equiv 0, h \in \sigma$$

if and only if $\omega = \mathrm{d}R + g\mathrm{d}H$, for some analytic functions $g, R$

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## Lemma

*If $H$ satisfies the condition $(*) : M_j \equiv 0, \forall 1 \leq j \leq k-1$, then there exist $q_1, \ldots, q_k, R_1, \ldots, R_k$ such that $\omega = q_1 dH + dR_1, q_1 = q_2 dH + dR_2, \ldots, q_{k-1}\omega = q_k dH + dR_k$ and*

$$M_k(h) = \int_{\gamma(h)} q_k \omega$$

*where $\omega_i$ is defined by $\omega_\varepsilon = dH + \varepsilon\omega_1 + \varepsilon^2\omega_2 + \ldots + \varepsilon^k w_k + o\left(\varepsilon^k\right), \varepsilon \to 0$.*

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- [Gavrilov, 1998] $H$ semi-weighted Morse polynomial, family $(\gamma(h))$ surrounds only 1 critical point of $H$

# Study AI related to Harmonic oscillator

Elliptic Hamiltonian $H(x, y) = y^2/2 + P_2(x, y)$ [Iliev]

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$$\int_{\gamma(h^2)} \omega = h^2 Q_{n-1}(h),$$

*for a polynomial $Q_{n-1}(h)$ of degree $n-1$ with $Q_{n-1}(h) = Q_{n-1}(-h)$*

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**Corollary**

*$I(h)$ has at most $(n-1)/2$ zeroes except the trivial zero $h = 0$*

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- Define $I_j(h) = \int_{\gamma(h)} x^k ydx$
- $0 \equiv dH = \left(1 - x^2\right) dx + ydy \implies \left(1 - x^2\right) ydx + y^2 dy \equiv 0 \implies I_0(h) \equiv I_2(h)$
- $(2k+9) I_{k+2}(h) - 3(2k+3) I_k(h) + 6khI_{k-1}(h) = 0$, where $k \geq 1$

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- $I(h)$ can be expressed as linear combination of the $n = n_0 + n_1 + 2$ independent functions

$$I_0(h), hI_0(h), h^2 I_0(h), \ldots, h^{n_0} I_0(h),$$
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- Any non-trivial $I_h$ has at most $n$ zeroes; moreover there exists a 1-form $\omega$ such that $I(h)$ has exactly $(n-1)$ zeroes

# Chebychev systems

## Definition

An $n$-tuple of smooth functions $J_0, J_1, \ldots, J_{n-1}$ defined on a closed interval $(h_0, h_1)$ is said to be a Chebychev system if every non-linear combination of the $n$ functions has at most $n-1$ zeroes in $(h_0, h_1)$ counting their multiplicity.

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## Lemma

*The bifurcation diagram of zeroes of a linear combination*

$$\sum_{i=0}^{n-1} \alpha_i J_i$$

*with respect to $(\alpha_0, \ldots, \alpha_{n-1})$ is topologically equivalent to the one of a polynomial of degree $n-1$.*

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- Petrov used the argument principle on the complexification of $I$ to prove Chebychev property

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  - heteroclinic saddle loop: more limit cycles than zeroes of the Abelian integral
    - ★ No 1-1 transfer zeroes of AI and limit cycles
    - ★ Some transfer is possible [C, Dumortier, Roussarie and Luca, 2005, 2007, 2009; Gavrilov, 2010]

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- unbounded period annulus: phenomenon of limit cycles that bifurcate from infinity
- Conclusion 1-1 transfer compact annulus for which the boundary exists of singular point, periodic orbit or saddle loop

# Limit cycle bifurcating from infinity

- Unfolding $X_\varepsilon$ of the Harmonic oscillator [Iliev]

$$x' = y + \varepsilon y^2 + \varepsilon^3 \left(-ax + (x + 1/3y)^3\right)$$
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- $M_3(h) = 0$ and $\frac{dM_3}{dh}(18a/5) = \pi a > 0$
- In original coordinates: for all $\varepsilon > 0$ there exists a unique limit cycle of $X_\varepsilon$ approaching the circle

$$(x - 1/(3\varepsilon))^2 + (y + 1/\varepsilon)^2 = 18a/5$$

# Upper bound for limit cycles

## Theorem (Upper bound - Dumortier, Roussarie)

*Under extra genericity conditions* **(AI)** *and* **(C)**

$$Cycl\left(X_\lambda, (\Gamma, (\nu_0, 0))\right) \le 2k - 1 + \frac{k(k-1)}{2} \text{ if } codim I_\nu = 2k - 1, \text{ and}$$
$$\le 2k + \frac{k(k-1)}{2} \text{ if } codim I_\nu = 2k.$$

| $I_\nu$ | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | $13 \cdots$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| $X_{(\nu,\varepsilon)}$ | 1 | 2 | 4 | 5 | 8 | 9 | 13 | 14 | 19 | 20 | 26 | 27 | $34 \cdots$ |

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# Generic codimension 4 example

- Hamiltonian $H(x,y) = y\left(x^2 + \frac{1}{12}y^2 - 1\right)$
- $X_{(\nu,\varepsilon)} \leftrightarrow$

$$\begin{cases} \dot{x} &= 1 - \frac{1}{4}y^2 - x^2 \\ & \quad +\varepsilon\left(\nu_3 xy + \nu_4 xy^2 + y\left(x^2 + \frac{1}{12}y^2 - 1\right)\left(x - \frac{\sqrt{3}\pi}{8}xy\right)\right) \\ \dot{y} &= 2xy + \varepsilon y\left(\nu_1 + \nu_2 x\right) \end{cases}$$

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# Genericity conditions **(AI)**- Abelian integral

- Abelian integral

$$I(h,\nu) = p(\nu) + q(\nu)\,h\log h + r(\nu)\,h + s(\nu)\,h^2\log h + O\left(h^2\right), h \downarrow 0,$$

where

$$\begin{aligned} p(\nu) &= -\sqrt{3}\pi\nu_1 + 8\nu_3 - 3\sqrt{3}\pi\nu_4 \\ q(\nu) &= -\nu_1 \\ r(\nu) &= a_1\nu_1 - \sqrt{3}\pi\nu_3 + 12\nu_4 \\ s(\nu) &= a_4\nu_1 + b_4\nu_3 + c_3\nu_4 - 1 \end{aligned}$$

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- $\alpha_1(\nu) = \frac{1}{2}(\nu_1 - \nu_2)$.
- **(AI)** $\nu \mapsto (p(\nu), q(\nu), r(\nu), \alpha_1(\nu, 0))$ is a local submersion at $\nu = \nu_0$

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# Codimension 4 extra genericity condition **(C)**

$\Delta = D_2 \circ R_2 - R_1 \circ D_1 = \varepsilon I_\nu + \text{"}O\left(\varepsilon^2\right)\text{"}, \varepsilon \to 0$
Expressed in apropriate normal form coordinates locally:

- $(u, v)$ near $s_1$
- $(z, w)$ near $s_2$

$R_1(v) = v + \varepsilon\left(-\beta_1(\nu, \varepsilon) + \gamma_1 v + \eta_1(\nu, \varepsilon) v^2 + O\left(v^3\right)\right), v \downarrow 0.$
$R_2(u) = u + \varepsilon\left(\eta_2(\nu, \varepsilon) u^2 + O\left(u^3\right)\right), u \downarrow 0$
**(C)**: $\eta_2(0) \neq 2\eta_1(0)$

# Limit cycle bifurcating from 2-saddle cycle

## Theorem (Caubergh, Dumortier, Roussarie)

- *$C^\infty$ Unfolding leaving 1 connection unbroken*
- *$codim I_\nu = 2k - 1$ and extra genericity condition*

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$$X_{(\nu,\varepsilon)} \leftrightarrow \begin{cases} \dot{z} & = -z \\ \dot{w} & = w\,(1+\varepsilon\alpha) \end{cases}$$

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$\Rightarrow \exists\,(k-2)$ *alien limit cycles.*