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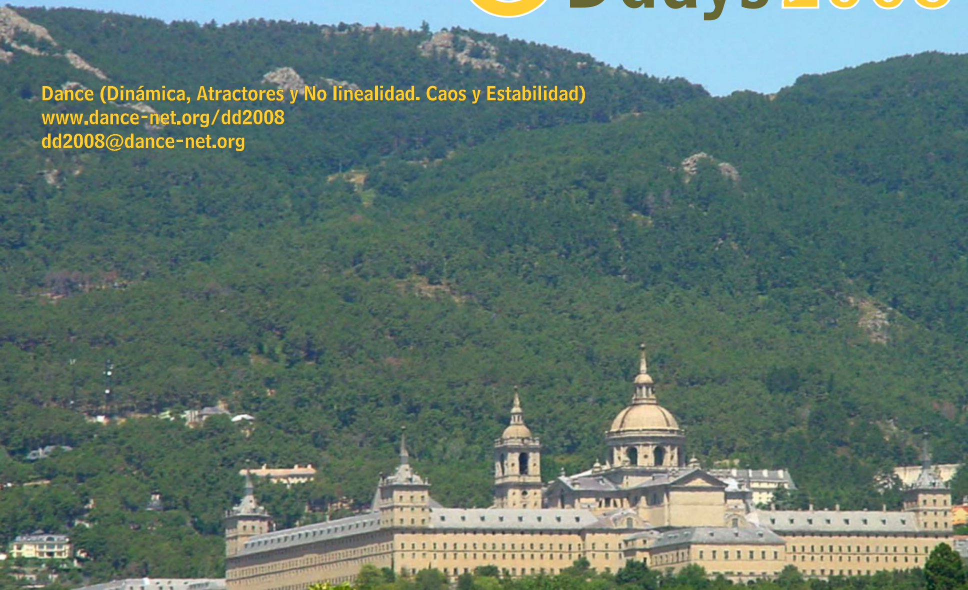

The role of hyperbolic invariant objects: From Arnold diffusion to biological clocks

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October, 2008

Outline

1. Arnold diffusion for a priori unstable Hamiltonian Systems
   - Arnold diffusion
   - Main result
   - Sketch of the Proof

2. Fast numerical algorithms to compute invariant tori in Hamil. Systems

3. A computational and geometric approach to PRC and PRS

Part I: Arnold diffusion for apriori unstable Hamiltonian Systems

Arnold diffusion

- Nearly-integrable Hamiltonian systems of n-degrees of freedom

$$H(I,\varphi) = H_0(I) + \epsilon H_1(I,\varphi)$$

where $(I,\varphi) \in \mathbb{R}^n \times \mathbb{T}^n$.

- For $\epsilon = 0$ all the trajectories lie on an invariant tori $I = ct$. All trajectories are stable.
- KAM theorem. Under a suitable non-degeneracy condition the n-dimensional invariant tori $I = ct$ with Diophantine frequency $\omega(I)$ survive, with some deformation, for ϵ small enough. Provides stability for $n \leq 2$.
- Question: What happens for the trajectories which do not lie on the invariant tori, for $n > 2$? Do there exist unstable orbits, that is, orbits whose action variable (slow variable) experiences a drift of order 1?

Main contributions in Arnold diffusion

- Arnold'64 (Example of a Hamiltonian of 2+1/2 degrees of freedom with 2 parameters).
- A priori unstable case (the unperturbed Hamiltonian H_0 presents hyperbolicity: integrable pendulum) without gaps (non-generic perturbation H_1): Chierchia and Gallavotti '94 '98, Berti, Biasco, Bolle '02 '03 (with time estimates).
- A priori unstable case overcoming the large gap problem: Cheng and Yan '04 (variational methods), Treschev '04 (separatrix map), Delshams, de la Llave and Seara '03 '06 (Geometric methods), de la Llave and Gidea '06 (Topological methods).

Goal: Generalize the result in [DLS06] for generic perturbations.

[DLS06] A. Delshams, R. de la Llave and T.M. Seara. A geometric mechanism for diffusion in Hamiltonian systems overcoming the large gap problem: heuristics and rigorous verification on a model. *Mem. Amer. Math. Soc.*, 179 (844), 2006.

Instability for a priori unstable Hamiltonian systems

We consider a 2π-periodic in time perturbation of a pendulum and a rotor described by the non-autonomous Hamiltonian of 2+1/2-dof,

$$\begin{aligned} H_\epsilon(p,q,I,\varphi,t) &= H_0(p,q,I) + \epsilon h(p,q,I,\varphi,t;\epsilon) \\ &= P_\pm(p,q) + \frac{1}{2}I^2 + \epsilon h(p,q,I,\varphi,t;\epsilon) \end{aligned} \tag{1}$$

where $(p,q,I,\varphi,t) \in (\mathbb{R}\times\mathbb{T})^2 \times \mathbb{T}$ and

$$P_\pm(p,q) = \pm\left(\frac{1}{2}p^2 + V(q)\right) \tag{2}$$

and $V(q)$ is a 2π-periodic function. We will refer to $P_\pm(p,q)$ as the *pendulum*.

Main result

Theorem

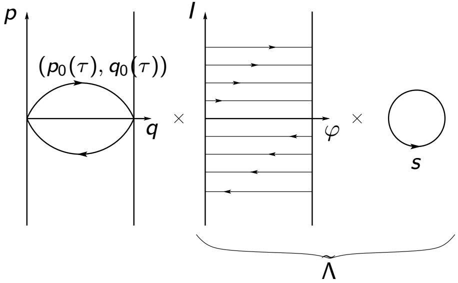
Consider the Hamiltonian (1) and that V and h are uniformly $\mathcal{C}^{r+2}$ for $r \geq r_0$, sufficiently large. Assume,

- **H1** *The potential $V : \mathbb{T} \to \mathbb{R}$ has a unique global maximum at $q = 0$ which is non-degenerate. Denote by $(q_0(t), p_0(t))$ an orbit of the pendulum $P_\pm(p, q)$ homoclinic to $(0, 0)$.*
- **H2** *The Melnikov potential, associated to h (and to the homoclinic orbit (p_0, q_0)): satisfies concrete non-degeneracy conditions.*
- **H3** *The perturbation term h satisfies concrete non-degeneracy conditions.*

Then, there is $\epsilon^* > 0$ such that for $0 < |\epsilon| < \epsilon^*$, and for any interval $[I_-^*, I_+^*] \in (I_-, I_+)$, there exists a trajectory $\widetilde{x}(t)$ of the system (1) such that for some $T > 0$,

$$I(\widetilde{x}(0)) \leq I_-^*; \qquad I(\widetilde{x}(T)) \geq I_+^*.$$

$\epsilon = 0$



- Normally hyperbolic invariant manifold (3D)

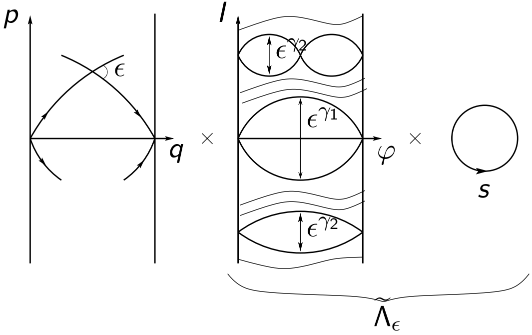
$$\tilde{\Lambda} = \{(0, 0, I, \varphi, s) : (I, \varphi, s) \in \mathbb{R} \times \mathbb{T}^2\}$$

- Invariant manifolds (4D):

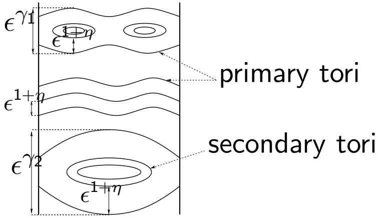
$$W^s\widetilde{\Lambda} = W^u\widetilde{\Lambda} = \{(p_0(\tau), q_0(\tau), I, \varphi, s) : \tau \in \mathbb{R}, I \in [I_-, I_+], (\varphi, s) \in \mathbb{T}^2\}$$

where $(p_0(t), q_0(t))$ is an orbit of $P_\pm(p, q)$ homoclinic to $(0, 0)$.

$0 < \epsilon \ll 1$



- On $[I_-, I_+]$, $\widetilde{\Lambda}$ persists to $\widetilde{\Lambda}_\epsilon$
- $W^s\widetilde{\Lambda}_\epsilon$ and $W^u\widetilde{\Lambda}_\epsilon$ are ϵ-close to the unperturbed ones.
- Using hypothesis **H2'**, $W^s\widetilde{\Lambda}_\epsilon \pitchfork W^u\widetilde{\Lambda}_\epsilon$ along Γ_ϵ.



- Combine the inner and the outer dynamics to construct a transition chain along $\widetilde{\Lambda}_\epsilon$: sequence of whiskered tori with heteroclinic intersections, i.e. $\{\mathcal{T}_{I_i}\}_{i=1}^N$, such that $\mathcal{W}^u(\mathcal{T}_{I_i}) \pitchfork \mathcal{W}^s(\mathcal{T}_{I_{i+1}})$ and $|I_N - I_1| = \mathcal{O}(1)$.
- Use that

$$S_\epsilon(\tau_i) \pitchfork_{\widetilde{\Lambda}_\epsilon} \tau_{i+1} \Rightarrow W^u_{\tau_i} \pitchfork W^s_{\tau_{i+1}}$$

(conditions **H2"**, **H3"** and **H3"'**)

- There is an orbit $\widetilde{x}(t)$ that shadows the transition chain.

Part II: Fast numerical algorithms for the computation of invariant tori in Hamiltonian systems

(in collaboration with Rafael de la Llave and Yannick Sire)

Computation of Invariant tori

- Invariant torus of dimension ℓ = quasi-periodic solution with ℓ independent frequencies (primary and secondary, maximal and whiskered).
- Importance. Together with their connections organize the long term behavior of the system (celestial mechanics, chemistry, ...)
- Numerical computation. Contributions of people in the Dynamical Systems group of Barcelona (de la Llave, Gómez, Haro, Jorba, Mondelo, Simó, Villanueva, ...).
- Goal: Develop numerical algorithms following the theoretical results of KAM Theorem without Action-Angle variables ([dlLGJV05],[FLS08]) and implement them numerically.

[dlLGJV05] R. de la Llave, A. González, A. Jorba and J. Villanueva. KAM theory without action-angle variables, *Nonlinearity*,18(2):855–895,2005.
[FLS08] E. Fontich, R. de la Llave and Y. Sire. Construction of invariant whiskered tori by a parametrization method. Part I: Maps and flows in finite dimensions. Preprint, 2008.

The invariance equation

- Consider a map F exact symplectic defined on $(U \subset \mathbb{R}^d) \times \mathbb{T}^d$.
- Assume that $\omega \in \mathbb{R}^\ell$ is fixed and Diophantine, i.e. for some $\nu, \tau > 0$,

$$|\omega \cdot k - n|^{-1} \leq \nu |k|^\tau \ \forall\ k \in \mathbb{Z}^\ell - \{0\},\ \ n \in \mathbb{Z}$$

- We seek for an embedding $K : \mathbb{T}^\ell \to \mathbb{R}^d \times \mathbb{T}^d$ that satisfies the invariance equation

$$F \circ K - K \circ T_\omega = 0$$

  where $T_\omega(\theta) = \theta + \omega$.
- The dynamics of F restricted on the invariant torus (range of K) is conjugated to a rigid rotation of frequency ω.
- Develop a Newton method to compute K.

Some remarks on the algorithms

- Algorithms are efficient in the following sense: If we discretize K using N Fourier coefficients, the algorithm requires storage of $\mathcal{O}(N)$ and the Newton step takes $\mathcal{O}(N \log N)$ operations using FFT.
- The method does not require the system to be written in Action-Angle variables (it can deal in a unified way with both primary and secondary KAM tori).
- The system is not required to be close to the integrable case.

KAM tori for the Standard Map

2D exact symplectic map defined on the cylinder $\mathbb{R} \times \mathbb{T}$.

$$\bar{p} = p + \varepsilon/(2\pi)\sin(2\pi q)$$
$$\bar{q} = q + \bar{p} \qquad (\mathrm{mod}\ 1)$$

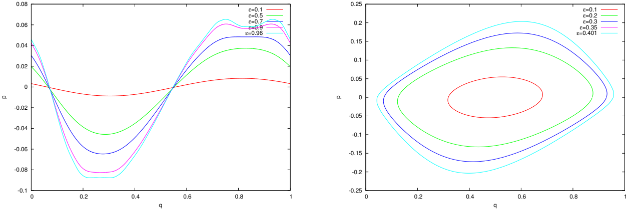
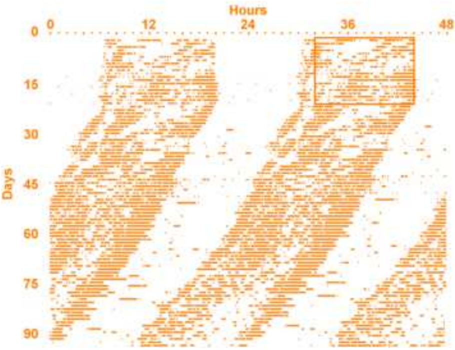


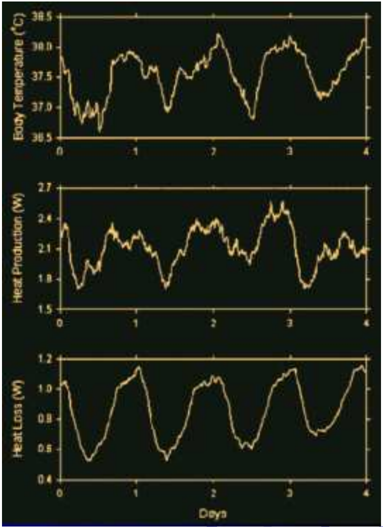
Figure: (Left) Primary tori of frequency $\omega_g = (\sqrt{5} - 1)/2$ (golden mean) for values of $\epsilon = 0.1, 0.5, 0.7, 0.9, 0.96$. They are shifted to have 0 offset. We used $N = 2^{11}$ Fourier modes. It takes 0.03 sec to perform one step of the continuation method. (Right) Secondary tori of frequency $3/40\omega_g$ for values of $\epsilon = 0.1, 0.2, 0.3, 0.35, 0.401$. We used $N = 2^9$ Fourier modes. It takes 0.01 sec to perform one step of the continuation method.

Part III: A computational and geometric approach to Phase Resetting Curves and Surfaces

Biological motivation: circadian rhythms

Biological clocks ≈ Presence of limit cycles/oscillators





Consider an autonomous system

$$\dot{x} = X(x), \quad x \in \mathbb{R}^d, d \geq 2$$

with a periodic orbit γ of period T

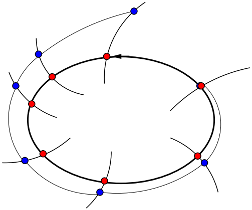
Definition

A point $q \in \Omega \subset \mathbb{R}^d$, Ω open domain containing the limit cycle γ, is in asymptotic phase with a point $p \in \gamma$ if

$$\lim_{t\to\infty} |\Phi_t(q) - \Phi_t(p)| = 0,$$

where Φ_t is the flow associated to the vector field X.

The set of points having the same asymptotic phase is called isochron. γ is isochronous if every point in Ω is in phase with a point on γ.



[Guck75] If γ is a stable limit cycle, then γ is isochronous. The isocrhons are the leaves of the stable manifold $W^s_{\gamma(\theta)}$.

Generalization of the phase in a neighborhood of the limit cycle

In a neighborhood Ω of the limit cycle γ there exists a unique scalar function

$$\begin{aligned} \vartheta : \quad \Omega \subset \mathbb{R}^d \quad &\rightarrow \mathbb{T} = [0,1) \\ x \quad &\mapsto \vartheta(x) \end{aligned}$$

such that

$$\lim_{t \to \infty} |\Phi_t(x) - \gamma(\vartheta(x) + t/T)| = 0.$$

The value $\vartheta(x)$ is the asymptotic phase of x. The isocrhons are the level sets of the function $\vartheta(x)$.

Phase Resetting Curves

Consider

$$\dot{x} = X(x) + \epsilon\, \delta(t(1 - \theta_s))$$

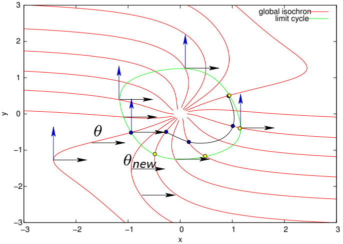
where $\epsilon = (\epsilon_1, \ldots, \epsilon_d)$, then

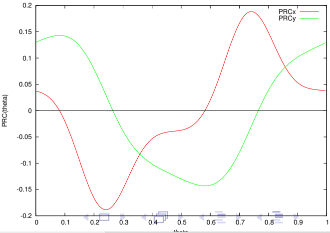
$$PRC(\theta) = \theta_{new} - \theta.$$

For weak perturbations, $|\epsilon| \ll 1$, the infinitesimal PRC

$$PRC(\vartheta(x)) = \epsilon \cdot \nabla\vartheta(x).$$

PRC: $x \in \gamma$.
PRS: Generalization for $x \in \Omega$.
Biological relevance of PRS





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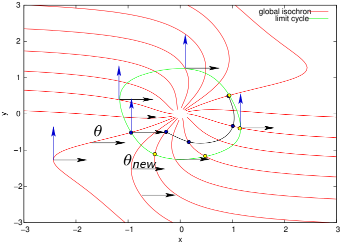

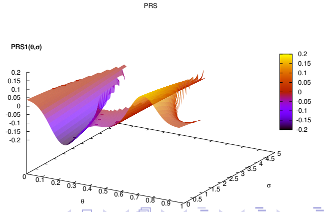

Conclusions

- We proved the existence of diffusing orbits for a priori unstable Hamiltonian systems with a generic perturbation h assuming that it is regular enough.
- We developed fast numerical algorithms to compute invariant to tori (primary and secondary, maximal and hyperbolic).
- We implemented them and we applied them to compute primary and secondary maximal tori of the standard map and primary maximal and whiskered tori of the Froeshclé map.
- We extend the Phase Resetting Curves to a neighborhood of the limit cycle, obtaining what we call the Phase Resetting Surface and we computed them numerically.