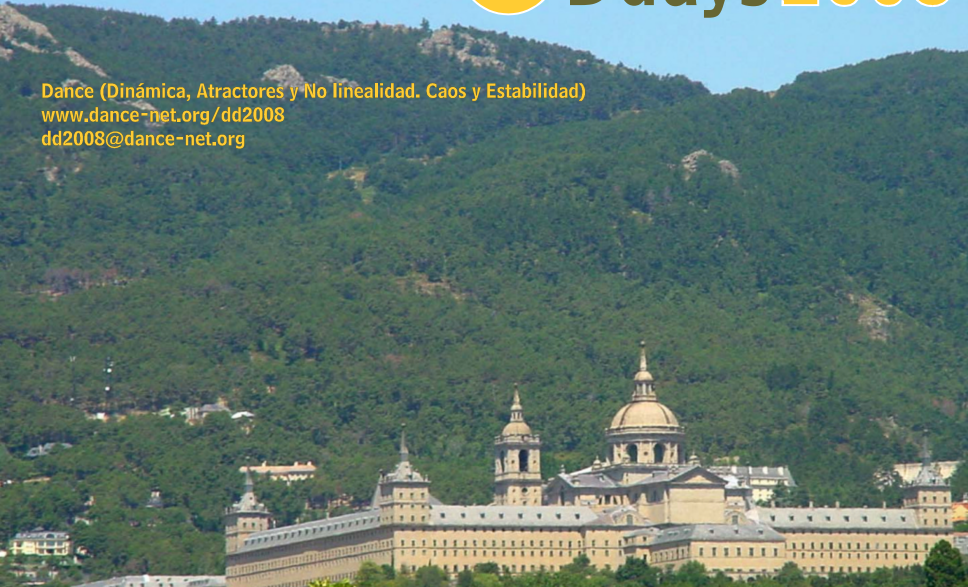
22, 23 y 24 de Octubre de 2008. El Escorial (Madrid)

Cuarta reunión de la red temática Dance

# Ddays 2008

Dance (Dinámica, Atractores y No linealidad. Caos y Estabilidad)
www.dance-net.org/dd2008
dd2008@dance-net.org

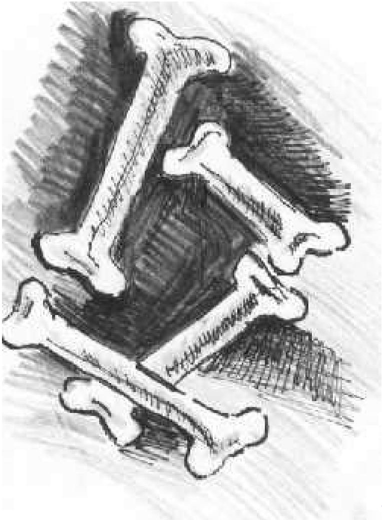


# Numerical Computation of Families of Periodic Orbits. Orbital dynamics around closed potential forms

E. Tresaco

GME. Universidad de Zaragoza

# HUESOS



# Table of contents

# Framework

*Study of orbital dynamics based on a Hamiltonian formulation of rather simple models, to illustrate the relevant structures in phase space*

- Closed form of the Potential function
- Hamiltonian formulation
- Autonomous problem

# Computation tools

- Poincaré surface of section
- Continuation of families of periodic orbits
  - Poincaré Map computation
  - Local Continuation of the Family: Deprit-Henrard algorithm
- Lineal stability computation

# Computation tools

- Poincaré surface of section
- Continuation of families of periodic orbits
  - Poincaré Map computation
  - Local Continuation of the Family: Deprit-Henrard algorithm
- Lineal stability computation

# Computation tools

- Poincaré surface of section
- Continuation of families of periodic orbits
  - Poincaré Map computation
  - Local Continuation of the Family: Deprit-Henrard algorithm
- Lineal stability computation

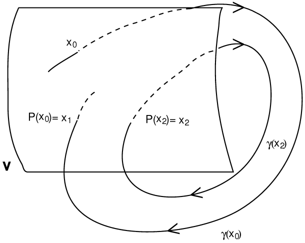
# Computation tools

- Poincaré surface of section
- Continuation of families of periodic orbits
  - Poincaré Map computation
  - Local Continuation of the Family: Deprit-Henrard algorithm
- Lineal stability computation

# Computation tools

- Poincaré surface of section
- Continuation of families of periodic orbits
  - Poincaré Map computation
  - Local Continuation of the Family: Deprit-Henrard algorithm
- Lineal stability computation

# Exploring an Energy level: Poincaré surface of section




$$\mathcal{H}(x,y,X,Y) = \frac{1}{2}(y_1^2+y_2^2)+U(x_1,x_2,y_1,y_2)$$

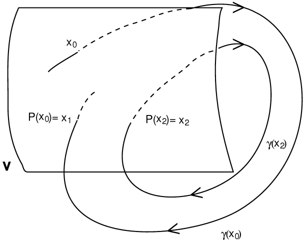
$$\Downarrow$$

$$y_2 = y_2(x_1,x_2,y_1,J)$$

$$\Downarrow$$

$$(x_1,y_1)$$

# Exploring an Energy level: Poincaré surface of section




$$J = \frac{1}{2}(y_1^2 + y_2^2) + U(x_1, x_2, y_1, y_2)$$

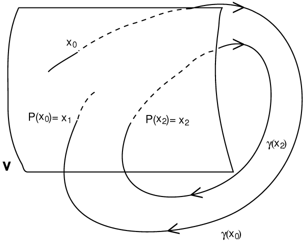
$$\Downarrow$$

$$y_2 = y_2(x_1, x_2, y_1, J)$$

$$\Downarrow$$

$$(x_1, y_1)$$

# Exploring an Energy level: Poincaré surface of section




$$J = \frac{1}{2}(y_1^2 + y_2^2) + U(x_1, x_2, y_1, y_2)$$

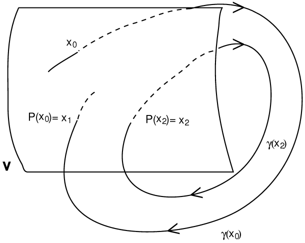
$$\Downarrow$$

$$y_2 = y_2(x_1, x_2, y_1, J)$$

$$\Downarrow$$

$$(x_1, y_1)$$

# Exploring an Energy level: Poincaré surface of section




$$J = \frac{1}{2}(y_1^2 + y_2^2) + U(x_1, x_2, y_1, y_2)$$

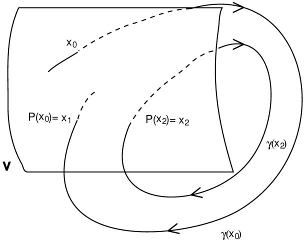
$$\Downarrow$$

$$y_2 = y_2(x_1, x_2, y_1, J)$$

$$\Downarrow$$

$$(x_1, y_1)$$

# Exploring an Energy level: Poincaré surface of section




$$J = \frac{1}{2}(y_1^2 + y_2^2) + U(x_1, x_2, y_1, y_2)$$

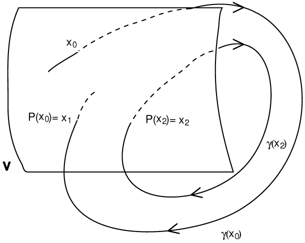
$$\Downarrow$$

$$y_2 = y_2(x_1, x_2 = 0, y_1, J)$$

$$\Downarrow$$

$$(x_1, y_1)$$

# Exploring an Energy level: Poincaré surface of section




$$J = \frac{1}{2}(y_1^2 + y_2^2) + U(x_1, x_2, y_1, y_2)$$

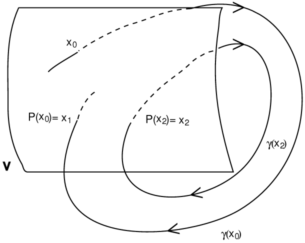
$$\Downarrow$$

$$y_2 = y_2(x_1, x_2 = 0, y_1, J)$$

$$\Downarrow$$

$$(x_1, y_1)$$

# Exploring an Energy level: Poincaré surface of section




$$J = \frac{1}{2}(y_1^2 + y_2^2) + U(x_1, x_2, y_1, y_2)$$

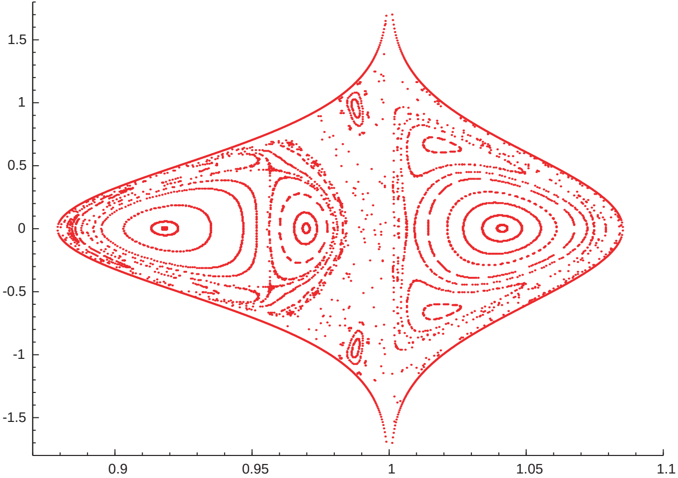
$$\Downarrow$$

$$y_2 = y_2(x_1, x_2 = 0, y_1, J)$$

$$\Downarrow$$

$$(x_1, y_1)$$

# Exploring an Energy level: Poincaré surface of section



# Poincaré Map computation

*Successive trajectory intersections with a Poincaré section define a map from one (transversal) crossing of the surface to the next, the so called Poincaré return map.*

$$J = \frac{1}{2}(y_1^2 + y_2^2) + U(x_1, x_2) \qquad \rightsquigarrow \qquad \Sigma_J^{(i)} := x_2 = 0$$

$$PM = P_2 \circ \tilde{P} \circ \Pi$$

$$\begin{array}{ccccccc} & P_2 & & \tilde{P} & & \Pi & \\ \Sigma_J^{(2)} & \rightarrow & \mathbb{R}^4 & \rightarrow & \mathbb{R}^4 & \rightarrow & \Sigma_J^{(2)} \\ \begin{Bmatrix} x_1 \\ y_1 \end{Bmatrix} & \mapsto & \begin{Bmatrix} x_1 \\ 0 \\ y_1 \\ y_2(J,z) \end{Bmatrix} & \mapsto & \begin{Bmatrix} \tilde{P}_1(\tilde{z}) \\ \tilde{P}_2(\tilde{z}) \\ \tilde{P}_3(\tilde{z}) \\ \tilde{P}_4(\tilde{z}) \end{Bmatrix} & \mapsto & \begin{Bmatrix} \tilde{P}_1(\tilde{z}) \\ \tilde{P}_3(\tilde{z} \end{Bmatrix} \end{array}$$

# Poincaré Map computation

*A periodic orbit $\vec{z}_0$ is a fixed point of the application $PM$: find zeros of* $\vec{F} = \vec{PM} - \vec{Id}$

Two reductions:

- Transversal condition: $\dot{y}_2 \neq 0$ whenever $x_2 = 0$
- Conservation of the Jacobi Integral

$\Downarrow$

Two-dimensional map from the Poincaré surface-of-section

*These reductions remove the unity eigenvalues and allow the reduced map to be used to iteratively solve for the fixed points of the map that correspond to the closed periodic orbits.*

formulation

# Poincaré Map computation

*A periodic orbit $\vec{z}_0$ is a fixed point of the application $PM$: find zeros of* $\vec{F} = \vec{PM} - \vec{Id}$

Two reductions:

- Transversal condition: $\dot{y}_2 \neq 0$ whenever $x_2 = 0$
- Conservation of the Jacobi integral

$\Downarrow$

Two-dimensional map from the Poincaré surface to itself

*These reductions remove the unity eigenvalues and allow the reduced map to be used to iteratively solve for the fixed points of the map that correspond to the closed periodic orbits.*

formulation

# Poincaré Map computation

*A periodic orbit $\vec{z}_0$ is a fixed point of the application $PM$: find zeros of $\vec{F} = \vec{PM} - \vec{Id}$*

Two reductions:

- Transversal condition: $\dot{y}_2 \neq 0$ whenever $x_2 = 0$
- Conservation of the Jacobi integral

$\Downarrow$

Two-dimensional map from the Poincaré surface to itself

*These reductions remove the unity eigenvalues and allow the reduced map to be used to iteratively solve for the fixed points of the map that correspond to the closed periodic orbits.*

formulation

# Poincaré Map computation

*A periodic orbit $\vec{z}_0$ is a fixed point of the application $PM$: find zeros of* $\vec{F} = \vec{PM} - \vec{Id}$

Two reductions:

- Transversal condition: $\dot{y}_2 \neq 0$ whenever $x_2 = 0$
- Conservation of the Jacobi integral

$\Downarrow$

Two-dimensional map from the Poincaré surface to itself

*These reductions remove the unity eigenvalues and allow the reduced map to be used to iteratively solve for the fixed points of the map that correspond to the closed periodic orbits.*

formulation

# Poincaré Map computation

*A periodic orbit* $\vec{z}_0$ *is a fixed point of the application* $PM$*: find zeros of* $\vec{F} = \vec{PM} - \vec{Id}$

Two reductions:

- Transversal condition: $\dot{y}_2 \neq 0$ whenever $x_2 = 0$
- Conservation of the Jacobi integral

$$\Downarrow$$

Two-dimensional map from the Poincaré surface to itself

*These reductions remove the unity eigenvalues and allow the reduced map to be used to iteratively solve for the fixed points of the map that correspond to the closed periodic orbits.*

formulation

# Poincaré Map computation

*A periodic orbit $\vec{z}_0$ is a fixed point of the application $PM$: find zeros of* $\vec{F} = \vec{PM} - \vec{Id}$

Two reductions:

- Transversal condition: $\dot{y}_2 \neq 0$ whenever $x_2 = 0$
- Conservation of the Jacobi integral

$$\Downarrow$$

Two-dimensional map from the Poincaré surface to itself

*These reductions remove the unity eigenvalues and allow the reduced map to be used to iteratively solve for the fixed points of the map that correspond to the closed periodic orbits.*

formulation

# Poincaré Map computation

*A periodic orbit* $\vec{z}_0$ *is a fixed point of the application* $PM$*: find zeros of* $\vec{F} = \vec{PM} - \vec{Id}$

Two reductions:

- Transversal condition: $\dot{y}_2 \neq 0$ whenever $x_2 = 0$
- Conservation of the Jacobi integral

$$\Downarrow$$

Two-dimensional map from the Poincaré surface to itself

*These reductions remove the unity eigenvalues and allow the reduced map to be used to iteratively solve for the fixed points of the map that correspond to the closed periodic orbits.*

formulation

# Poincaré Map computation

- *This reduction procedure removes the unity eigenvalues and allow the reduced map $\vec{D}PM$ to be used to iteratively solve for the fixed points of the map that correspond to the closed periodic orbits.*
- *Analyzing the eigenvalues of this matrix also provides details on the stability of the periodic orbits.*
- *Computation of a new neighboring orbit with a slightly Jacobi constant:*

$$\delta\vec{x} = (I - DPM)^{-1}\frac{\partial\vec{x}_0}{\partial J}\delta J$$

- *This method performs the continuation of families of periodic orbits, jointed with the determination of their linear stability through the analysis of the reduced monodromy matrix.*

# Henrard-Deprit Algorithm

*Given a periodic solution $\vec{x}_0$, in an small interval around $\sigma_0$ there exists a family of periodic orbits $O(\sigma)$*

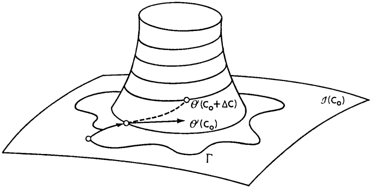
$$\vec{X} = \vec{x}_0 + \Sigma_{k\geq 1} \frac{\Delta\sigma^k}{k!} \frac{\partial^k \vec{x}}{\partial \sigma^k}$$

$$\vec{T} = T_0 + \Sigma_{k\geq 1} \frac{\Delta\sigma^k}{k!} \frac{d^k T}{d\sigma^k}$$

*A Natural Family of Periodic Orbit*

# Henrard-Deprit Algorithm

*It defines a cylinder upon which, given an initial point on each orbit, $t$ and $\sigma$ constitute a system of analytic coordinates.*

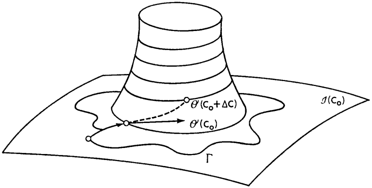



*The idea is continuing the cylinder from the initial orbit estimating only the first few coefficients, thereby determining approximately the deformation path of the initial conditions on the cylinder $O(\sigma)$.*

- Predictor-Corrector scheme

# Henrard-Deprit Algorithm

*It defines a cylinder upon which, given an initial point on each orbit, $t$ and $\sigma$ constitute a system of analytic coordinates.*




*The idea is continuing the cylinder from the initial orbit estimating only the first few coefficients, thereby determining approximately the deformation path of the initial conditions on the cylinder $O(\sigma)$.*

- Predictor-Corrector scheme

# Henrard-Deprit Algorithm

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta}_0,\sigma_0) = \vec{x}(t+T_0,\vec{\zeta}_0,\sigma_0)$$

$$\vec{\zeta}_1 = \vec{\zeta}_0 + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$x(T_0+\Delta T, \vec{\zeta}_0 + \Delta\vec{\zeta}, \sigma_0 + \Delta\sigma) - (\vec{\zeta}_0 + \Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial\vec{x}}{\partial\sigma} \quad \mapsto \textbf{Predictor}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta}_1 + \vec{F}(\vec{x},\sigma_1)\Delta T_1 = -(\vec{x} - \vec{\zeta}_1) \quad \mapsto \textbf{Corrector}$$

formulation

# Henrard-Deprit Algorithm

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta}_0,\sigma_0) = \vec{x}(t+T_0,\vec{\zeta}_0,\sigma_0)$$

$$\vec{\zeta}_1 = \vec{\zeta}_0 + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$x(T_0+\Delta T,\vec{\zeta}_0+\Delta\vec{\zeta},\sigma_0+\Delta\sigma) - (\vec{\zeta}_0+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial\vec{x}}{\partial\sigma} \quad \mapsto \textbf{Predictor}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta}_1 + \vec{F}(\vec{x},\sigma_1)\Delta T_1 = -(\vec{x} - \vec{\zeta}_1) \quad \mapsto \textbf{Corrector}$$

formulation

# Henrard-Deprit Algorithm

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta}_0,\sigma_0) = \vec{x}(t+T_0,\vec{\zeta}_0,\sigma_0)$$

$$\vec{\zeta}_1 = \vec{\zeta}_0 + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$x(T_0+\Delta T,\vec{\zeta}_0+\Delta\vec{\zeta},\sigma_0+\Delta\sigma) - (\vec{\zeta}_0+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial\vec{x}}{\partial\sigma} \quad \mapsto \textbf{Predictor}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta}_1 + \vec{F}(\vec{x},\sigma_1)\Delta T_1 = -(\vec{x}-\vec{\zeta}_1) \quad \mapsto \textbf{Corrector}$$

formulation

# Henrard-Deprit Algorithm

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta_0},\sigma_0) = \vec{x}(t+T_0,\vec{\zeta_0},\sigma_0)$$

$$\vec{\zeta_1} = \vec{\zeta_0} + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$x(T_0+\Delta T, \vec{\zeta_0}+\Delta\vec{\zeta}, \sigma_0+\Delta\sigma) - (\vec{\zeta_0}+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial\vec{x}}{\partial\sigma}$$ $\mapsto$ **Predictor**

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta_1} + \vec{F}(\vec{x},\sigma_1)\Delta T_1 = -(\vec{x} - \vec{\zeta_1})$$ $\mapsto$ **Corrector**

formulation

# Henrard-Deprit Algorithm

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta_0},\sigma_0) = \vec{x}(t+T_0,\vec{\zeta_0},\sigma_0)$$

$$\vec{\zeta_1} = \vec{\zeta_0} + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$x(T_0+\Delta T,\vec{\zeta_0}+\Delta\vec{\zeta},\sigma_0+\Delta\sigma) - (\vec{\zeta_0}+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial\vec{x}}{\partial\sigma}$$ $\mapsto$ **Predictor**

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta_1} + \vec{F}(\vec{x},\sigma_1)\Delta T_1 = -(\vec{x} - \vec{\zeta_1})$$ $\mapsto$ **Corrector**

formulation

# Henrard-Deprit Algorithm

Intrinsic formulation:

$$(\vec{t}, \vec{n}, \vec{b})$$

$$\Downarrow$$

*variational equations turn out to be separable*

$$\ddot{q} = Q_1 q + Q_2 \dot{r} + Q_3 r + (Q_4 U_\sigma + Q_5)\delta\sigma$$

$$\ddot{r} = R_1 r + R_2 \dot{q} + R_3 q + (R_4 U_\sigma + R_5)\delta\sigma$$

$$\frac{d}{dt}\left(\frac{p}{V}\right) = \frac{1}{V}\left[(2N + \vec{B}\vec{b})q - (\vec{B}\vec{n})r + \frac{U_\sigma}{V}\right]$$

# Henrard-Deprit Algorithm

Intrinsic formulation:

$$(\vec{t}, \vec{n}, \vec{b})$$

$$\Downarrow$$

*variational equations turn out to be separable*

$$\ddot{q} = Q_1 q + Q_2 \dot{r} + Q_3 r + (Q_4 U_\sigma + Q_5)\delta\sigma$$

$$\ddot{r} = R_1 r + R_2 \dot{q} + R_3 q + (R_4 U_\sigma + R_5)\delta\sigma$$

$$\frac{d}{dt}\left(\frac{p}{V}\right) = \frac{1}{V}\left[(2N + \vec{B}\vec{b})q - (\vec{B}\vec{n})r + \frac{U_\sigma}{V}\right]$$

# Henrard-Deprit Algorithm

Intrinsic formulation:

$$(\vec{t}, \vec{n}, \vec{b})$$

$$\Downarrow$$

*variational equations turn out to be separable*

$$\ddot{q} = Q_1 q + Q_2 \dot{r} + Q_3 r + (Q_4 U_\sigma + Q_5)\delta\sigma$$

$$\ddot{r} = R_1 r + R_2 \dot{q} + R_3 q + (R_4 U_\sigma + R_5)\delta\sigma$$

$$\frac{d}{dt}(\frac{p}{V}) = \frac{1}{V}\left[(2N + \vec{B}\vec{b})q - (\vec{B}\vec{n})r + \frac{U_\sigma}{V}\right]$$

# Henrard-Deprit Algorithm

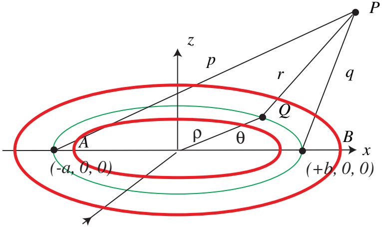
- *Continuing periodic orbits is reduced to find the displacements to the orbit i.e. the solutions of the variational equations*
- *A side effect is again the computation of the linear stability with no additional effort.*
- *Intrinsic formulation results in the separation of the tangent displacement, which retain the secular part of the variations. Nontrivial eigenvalues are computed from the normal and binormal variations, reducing the dimension of the STM and eliminating the trivial exponents.*
- *The critical case of any of the nontrivial eigenvalues having modulus 1 is a singularity of the algorithm. These are also the cases of no convergence of the Newton method in the Poincaré map computation.*

# Examples

## Annular Ring

# Periodic orbits around an Annular Ring

*Study the dynamics of orbits under the attraction of a homogeneous annulus disk on a fixed plane.*



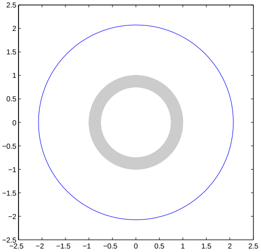

$$U(x,y,z;a,b) = \frac{\mu}{\pi(a^2-b^2)}\left(U(x,y,z;a) - U(x,y,z;b)\right)$$

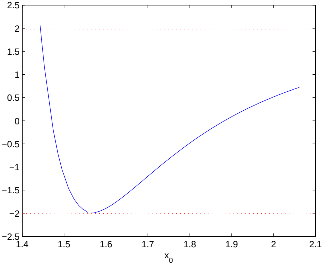
$$U(x,y,z;a) = 2(-pE(k) - \tfrac{a^2-r^2}{p}K(k) + |z|\left(\tfrac{\pi}{2} + \tfrac{\pi}{2}\,\mathrm{sign}(a-r)\right) -$$

$$z\,\mathrm{sign}(a-r)\left(E(k)F(\phi,k') + K(k)E(\phi,k') - K(k)F(\phi,k')\right))$$

equilibria

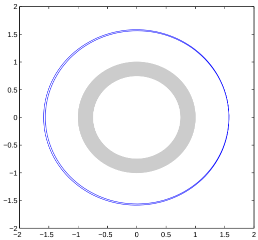
# Annular Ring: Dynamics on the equatorial plane

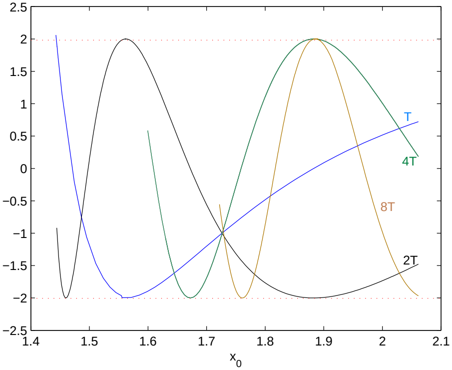



- There exists a family of trivial circular periodic orbits outside the ring.
- In order to compute the family we have chosen as continuation parameter the orbital energy.
- We plot the evolution of the stability index $k$ along the family

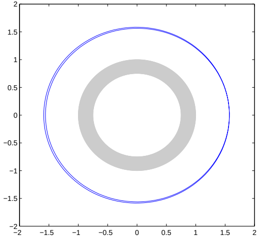
# Annular Ring: Dynamics on the equatorial plane

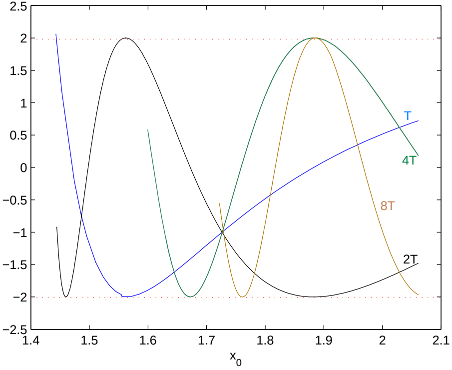



$$k_m = 2\cos(m \arccos(\frac{k}{2})), \qquad |k| \leq 2$$

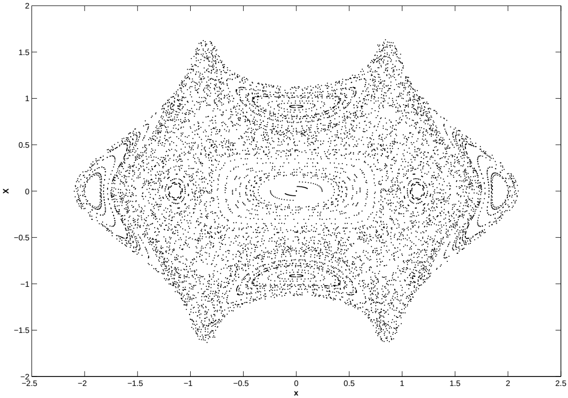
# Annular Ring: Dynamics on the equatorial plane




$$k_m = 2\cos(m \arccos(\frac{k}{2})), \qquad |k| \leq 2$$

# Annular Ring: Polar case



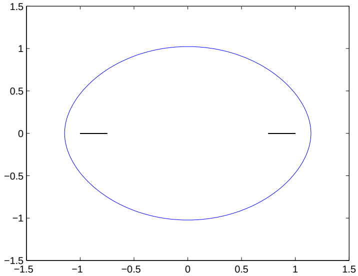

Poincaré section for energy $E = -0.5$, and $a = 1$, $b = 0.75$

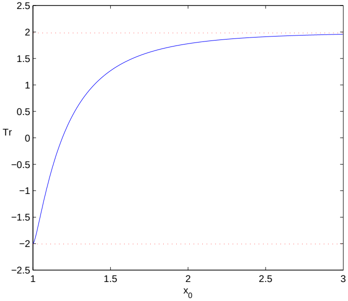
# Annular Ring: Polar case

| | $x$ | $z$ | $\dot{x}$ | $\dot{z}$ | $T$ | $tr$ |
|---|---|---|---|---|---|---|
| 1 | 1.14498558 | 0 | 0 | 1.08570023 | 8.2283 | -0.3716 |
| 2 | 1.93191413 | 0.00005241 | -0.00005393 | 0.31172844 | 13.3033 | 1.9627 |
| 3 | 1.73809965 | -0.00000046 | 0.00000097 | 0.19712184 | 10.35376 | 1.5428 |
| 4 | 0.61431996 | 0.74536418 | 0.81006851 | 0.78959165 | 50.2524 | -0.0072 |
| 5 | 0.38899022 | 0.00000771 | 0.00000213 | 1.39390805 | 39.9107 | 2.4758 |
| 6 | -0.60617609 | -0.00231883 | 0.62563965 | 1.19156741 | 46.8767 | 1.7487 |
| 7 | 0.45581169 | -0.00030177 | -0.00015502 | 0.98791160 | 7.6428 | -1.4308 |
| 8 | 0.70912248 | 0.00000166 | 0.00000332 | 1.18117588 | 12.9022 | 1.7661 |
| 9 | -0.00001481 | 0.00006441 | -0.21096655 | 0.91717358 | 24.7412 | -0.9258 |
| 10 | -0.00281186 | -1.63333399 | 0.13129963 | -0.00547436 | 25.5104 | 1.7511 |
| 11 | 0.44689964 | -0.00069537 | -0.046256793 | 1.03202274 | 44.0945 | 1.7951 |

# Annular Ring: Polar case

This family consists of nearly circular orbits around the annulus and perpendicular to its plane.

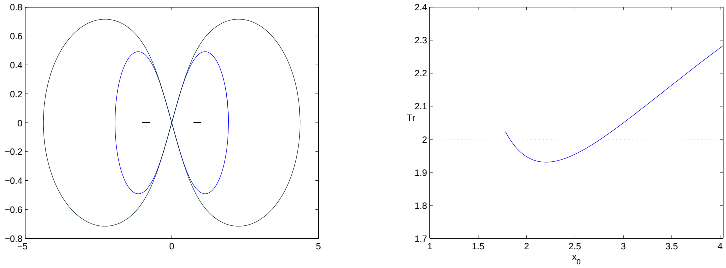



It begins with orbits of infinitesimal radius and ends having a collision on the annulus.

# Annular Ring: Polar case

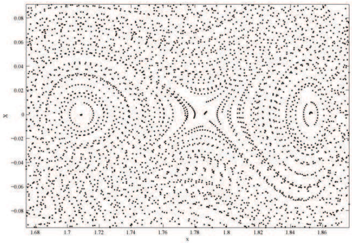
This 8-shape family consists of 2-arc symmetric periodic orbits around the annulus,and centered in the origin.




As long as the orbits goes larger the family enters in unstable motion, while when the orbit decreases the family stays in a stable region until it crosses the boundary value, leading to bifurcation with new families of periodic orbits.

# Annular Ring: Polar case

Poincaré section corresponding to the bifurcation point $x_0 \simeq 1.828$

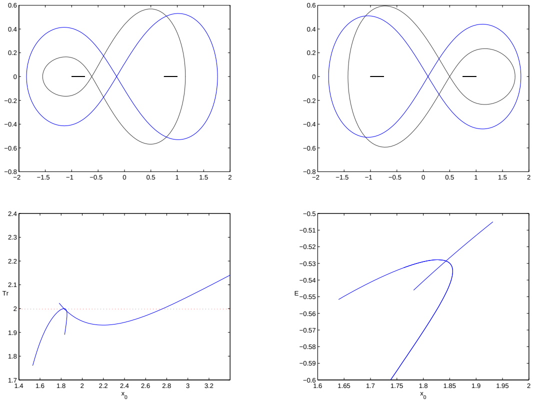


- Pitchfork bifurcation

The original family of stable orbits passes to be unstable, and appear two new families of stable orbits.

# Annular Ring: Polar case

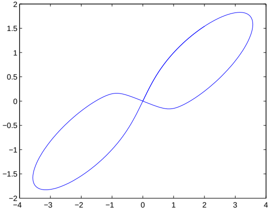
These bifurcated families are asymmetric arcs orbits displaced from the center of the annulus.




As energy decreases, orbits goes more asymmetric until the family ends with a collision on the annulus.

# Annular Ring: Polar case

The second bifurcation point at $x_0 \simeq 2.75$ where the original family again pass through the critical value $tr = 2$ and enters in a unstable region, also leads in new family.

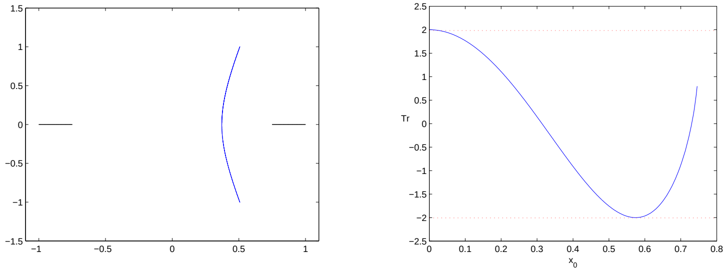



$$Type3 : \begin{pmatrix} 0.9983 & 0.0006 \\ -3.0810 & 0.9996 \end{pmatrix}$$

There is a bifurcation with with a non-symmetric family of the same period.

# Annular Ring: Polar case

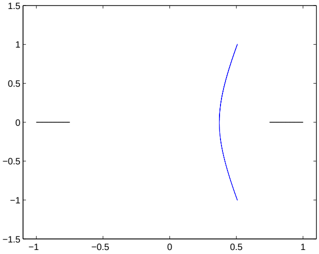
This family consists of single-line open orbits. It is a stable family that originates out a bifurcation with vertical oscillator, and ends with a collision orbit with the annulus.

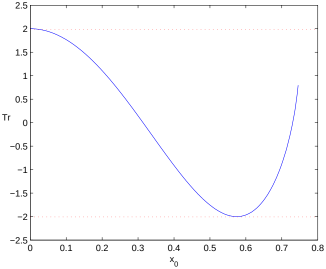



- Bifurcation with a doubling-period family

# Annular Ring: Polar case

This family consists of single-line open orbits. It is a stable family that originates out a bifurcation with vertical oscillator, and ends with a collision orbit with the annulus.

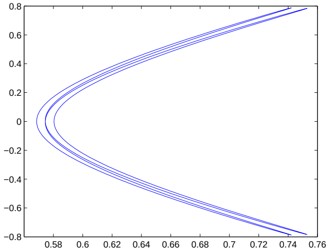



- Bifurcation with a doubling-period family

# Annular Ring: Polar case

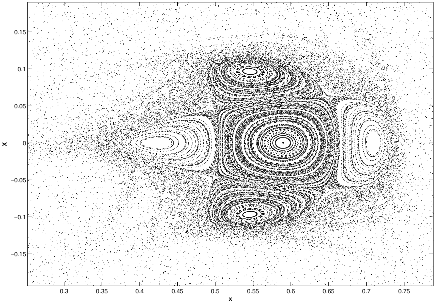
The doubling period family at the bifurcation point with the singular family, have all unit eigenvalues and thus a stability index equal to $+2$.




This point may lead in the bifurcation with new families.

# Annular Ring: Polar case

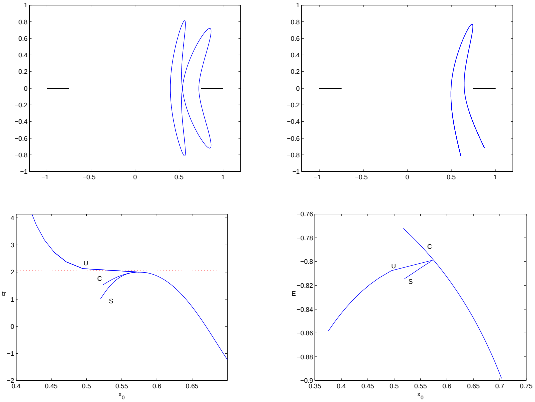
Poincaré section at this bifurcation point:



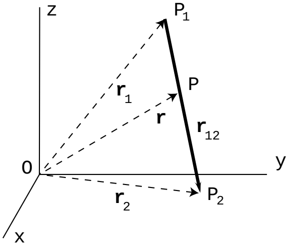

Central point correspond to the single arc orbit, surrounding by four isles, these correspond to a new bifurcated stable family,while the other four hyperbolic points are related to a new unstable family.

# Annular Ring: Polar case

Stable and unstable orbits respectively bifurcated out of the doubling-period. Both families end with a collision orbit with the annular ring



# Finite straight segment: Asteroids




$$\begin{aligned} \ddot{x} - 2\dot{y} &= W_x \\ \ddot{y} + 2\dot{x} &= W_y \\ \ddot{z} &= W_z \end{aligned}$$

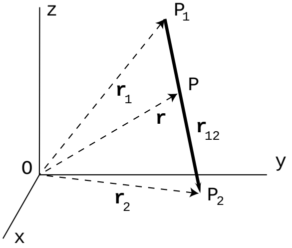
$$W = \frac{x^2 + y^2}{2} - \frac{\mu}{2\ell} \log\left(\frac{r_1 + r_2 + 2\ell}{r_1 + r_2 - 2\ell}\right)$$

$$\left( \; x = a, \quad y = 0, \quad z = 0, \quad x' = 0, \quad y' = an, \quad z' = 0 \; \right)$$

$$\left( \; x = a, \quad y = 0, \quad z = 0, \quad \dot{x} = x' + \omega y, \quad \dot{y} = y' - \omega x, \quad \dot{z} = 0 \; \right)$$

$$P_r = 2\pi/\tilde{n}, \quad \tilde{n} = n - \omega$$

# Finite straight segment: Asteroids




$$\begin{aligned} \ddot{x} - 2\dot{y} &= W_x \\ \ddot{y} + 2\dot{x} &= W_y \\ \ddot{z} &= W_z \end{aligned}$$

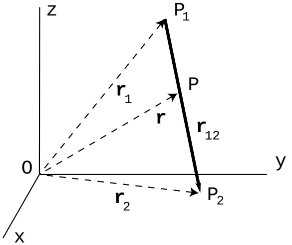
$$W = \frac{x^2 + y^2}{2} - \frac{\mu}{2\ell} \log\left(\frac{r_1 + r_2 + 2\ell}{r_1 + r_2 - 2\ell}\right)$$

$$\left(\ x = a, \quad y = 0, \quad z = 0, \quad x' = 0, \quad y' = an, \quad z' = 0\ \right)$$

$$\left(\ x = a, \quad y = 0, \quad z = 0, \quad \dot{x} = x' + \omega y, \quad \dot{y} = y' - \omega x, \quad \dot{z} = 0\ \right)$$

$$P_r = 2\pi/\tilde{n}, \quad \tilde{n} = n - \omega$$

# Finite straight segment: Asteroids




$$\begin{aligned} \ddot{x} - 2\dot{y} &= W_x \\ \ddot{y} + 2\dot{x} &= W_y \\ \ddot{z} &= W_z \end{aligned}$$

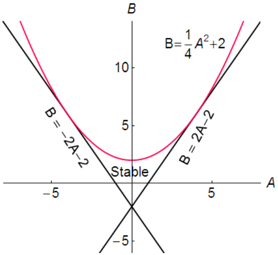
$$W = \frac{x^2 + y^2}{2} - \frac{\mu}{2\ell} \log\left(\frac{r_1 + r_2 + 2\ell}{r_1 + r_2 - 2\ell}\right)$$

$$\left(\ x = a, \quad y = 0, \quad z = 0, \quad x' = 0, \quad y' = an, \quad z' = 0\ \right)$$

$$\left(\ x = a, \quad y = 0, \quad z = 0, \quad \dot{x} = x' + \omega y, \quad \dot{y} = y' - \omega x, \quad \dot{z} = 0\ \right)$$

$$P_r = 2\pi/\tilde{n}, \quad \tilde{n} = n - \omega$$

# Finite straight segment: Asteroids



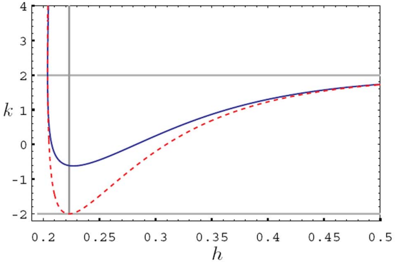

$$x^4 - Ax^3 + Bx^2 + Ax + 1 = 0$$

$$\begin{matrix} k_1 \\ k_2 \end{matrix} = \frac{A \pm \sqrt{A^2 - 4B + 8}}{2}$$

$$|k_1| < 2, |k_2| < 2 \quad \mapsto$$ Stability

# Finite straight segment: Asteroids

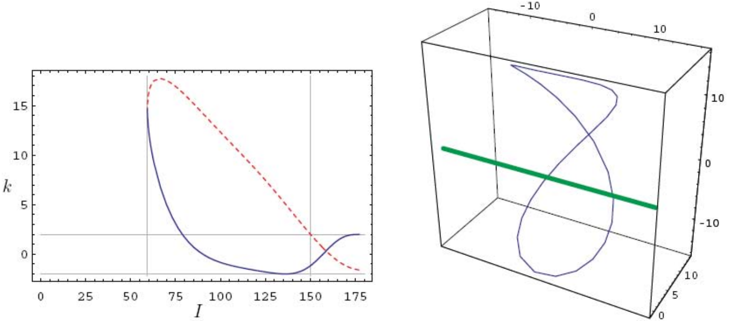
*Ellipses family which eccentricity increases as the orbits approach the segment*




*3-D p.o. are members of bifurcated families of the planar equatorial family. These bifurcations appear when there are commensurability between the two frequencies involved*

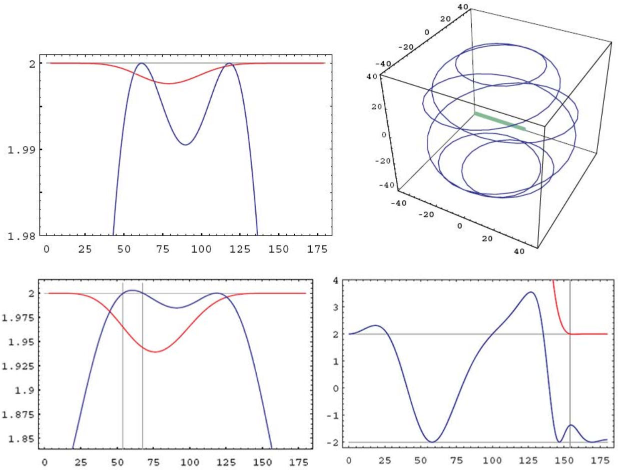
- *D:N resonant orbit*

# Finite straight segment: Asteroids




*3-D eight-shaped symmetric family, bifurcated at the 1:1 resonance and termination at the isosceles equilibrium*

# Finite straight segment: Asteroids




*Resonances that give orbits closer to the origin introduce larger intervals of mild instability*

# Thank you for your attention

# Bibliography:


- *Exploration Numérique du problème restreint, Hénon. Annales d'astrophysique 28, 992 (1965)*
- *The stability of the equilibrium position of a Hamiltonian system of ordinary differential equations in the general elliptic case, Arnold. Soviet Math. Dokl. 247 (1961)*
- *Numerical integration of periodic orbits in the main problem of the artificial satellite theory, Broucke. Celestial Mechanics and Dynamical Astronomy 58, 99 (1994)*
- *On the numerical continuation of periodic orbits, Lara and Peláez. AA 389, 692 (2002)*
- *Satellite Dynamics about Asteroids: Computing Poincaré Maps for the General Case, Scheeres. Hamiltonian Systems with Three or More Degrees of Freedom, 533, 554 (1999)*
- *A simple model to determine chaotic motions around asteroids, Elipe and Lara. RevMexAA (Serie de Conferencias), 25, 3 (2006)*

# Poincaré Map computation

*Find zeros of* $F = PM - Id$

$$\vec{D}F(z) = \vec{D}PM(z) - Id$$

$$\vec{D}PM(z) = M_2 \vec{D}\tilde{P}(\tilde{z}) \vec{D}P_2(z)$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \vec{D}P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ \partial_x y^t_2 & \partial_y y^t_1 \end{pmatrix}$$

where $\partial_x y_2 = \frac{\partial y_2}{\partial x_1}, \quad \partial_y y_2 = \frac{\partial y_2}{\partial y_1}$

▸ 3D formula

# Poincaré Map computation

*Find zeros of* $F = PM - Id$

$$\vec{D}F(z) = \vec{D}PM(z) - Id$$

$$\vec{D}PM(z) = M_2 \vec{D}\tilde{P}(\tilde{z}) \vec{D}P_2(z)$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \vec{D}P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ \partial_x y_2^t & \partial_y y_1^t \end{pmatrix}$$

where $\quad \partial_x y_2 = \frac{\partial y_2}{\partial x_1}, \quad \partial_y y_2 = \frac{\partial y_2}{\partial y_1}$

3D formula

# Poincaré Map computation

*Find zeros of* $F = PM - Id$

$$\vec{D}F(z) = \vec{D}PM(z) - Id$$

$$\vec{D}PM(z) = M_2 \vec{D}\tilde{P}(\tilde{z}) \vec{D}P_2(z)$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \vec{D}P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ \partial_x y_2^t & \partial_y y_1^t \end{pmatrix}$$

where $\partial_x y_2 = \frac{\partial y_2}{\partial x_1}, \quad \partial_y y_2 = \frac{\partial y_2}{\partial y_1}$

3D formula

# Poincaré Map computation

*Find zeros of* $F = PM - Id$

$$\vec{D}F(z) = \vec{D}PM(z) - Id$$

$$\vec{D}PM(z) = M_2 \vec{D}\tilde{P}(\tilde{z}) \vec{D}P_2(z)$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \vec{D}P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ \partial_x y_2^t & \partial_y y_1^t \end{pmatrix}$$

where $\partial_x y_2 = \frac{\partial y_2}{\partial x_1}, \quad \partial_y y_2 = \frac{\partial y_2}{\partial y_1}$

3D formula

# Poincaré Map computation

*Find zeros of* $F = PM - Id$

$$\vec{D}F(z) = \vec{D}PM(z) - Id$$

$$\vec{D}PM(z) = M_2 \vec{D}\tilde{P}(\tilde{z}) \vec{D}P_2(z)$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \vec{D}P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ \partial_x y_2^t & \partial_y y_1^t \end{pmatrix}$$

where $\partial_x y_2 = \frac{\partial y_2}{\partial x_1}, \quad \partial_y y_2 = \frac{\partial y_2}{\partial y_1}$

3D formula

# Poincaré Map computation

*Find zeros of* $F = PM - Id$

$$\vec{D}F(z) = \vec{D}PM(z) - Id$$

$$\vec{D}PM(z) = M_2 \vec{D}\tilde{P}(\tilde{z}) \vec{D}P_2(z)$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad \vec{D}P_2 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ \partial_x y_2^t & \partial_y y_1^t \end{pmatrix}$$

where $\partial_x y_2 = \frac{\partial y_2}{\partial x_1}, \quad \partial_y y_2 = \frac{\partial y_2}{\partial y_1}$

3D formula

# Poincaré Map computation

$$\vec{D}PM(z) = M_2\vec{D}\tilde{P}(\tilde{z})\vec{D}P_2(z)$$

$$where \quad \tilde{P}(\tilde{z}) = \varphi_{T(\tilde{z})}(\tilde{z}) \quad and \quad \dot{\varphi}_{T(\tilde{z})}(\tilde{z}) = f(\varphi_{T(\tilde{z})}(\tilde{z}))$$

$$\vec{D}\tilde{P}(\tilde{z}) = \dot{\varphi}_{T(\tilde{z})}(\tilde{z})\vec{D}T(\tilde{z}) + \vec{D}\varphi_{T(\tilde{z})}(\tilde{z})$$

$$\vec{D}T(\tilde{z}) = \frac{(\vec{D}\varphi_{T(\tilde{z})}(\tilde{z}))_2}{(\vec{f}(PM\tilde{z}))_2}$$

$$\vec{D}PM$$

# Poincaré Map computation

$$\vec{D}PM(z) = M_2 \vec{D}\tilde{P}(\tilde{z}) \vec{D}P_2(z)$$

$$where \quad \tilde{P}(\tilde{z}) = \varphi_{T(\tilde{z})}(\tilde{z}) \quad and \quad \dot{\varphi}_{T(\tilde{z})}(\tilde{z}) = f(\varphi_{T(\tilde{z})}(\tilde{z}))$$

$$\vec{D}\tilde{P}(\tilde{z}) = \dot{\varphi}_{T(\tilde{z})}(\tilde{z}) \vec{D}T(\tilde{z}) + \vec{D}\varphi_{T(\tilde{z})}(\tilde{z})$$

$$\vec{D}T(\tilde{z}) = \frac{(\vec{D}\varphi_{T(\tilde{z})}(\tilde{z}))_2}{(\vec{f}(PM\tilde{z}))_2}$$

$$\vec{D}PM$$

# Poincaré Map computation

$$\vec{D}PM(z) = M_2 \vec{D}\tilde{P}(\tilde{z}) \vec{D}P_2(z)$$

$$where \quad \tilde{P}(\tilde{z}) = \varphi_{T(\tilde{z})}(\tilde{z}) \quad and \quad \dot{\varphi}_{T(\tilde{z})}(\tilde{z}) = f(\varphi_{T(\tilde{z})}(\tilde{z}))$$

$$\vec{D}\tilde{P}(\tilde{z}) = \dot{\varphi}_{T(\tilde{z})}(\tilde{z})\vec{D}T(\tilde{z}) + \vec{D}\varphi_{T(\tilde{z})}(\tilde{z})$$

$$\vec{D}T(\tilde{z}) = \frac{(\vec{D}\varphi_{T(\tilde{z})}(\tilde{z}))_2}{(\vec{f}(PM\tilde{z}))_2}$$

$$\vec{D}PM$$

# Poincaré Map computation

$$\vec{D}PM(z) = M_2\vec{D}\tilde{P}(\tilde{z})\vec{D}P_2(z)$$

$$where \quad \tilde{P}(\tilde{z}) = \varphi_{T(\tilde{z})}(\tilde{z}) \quad and \quad \dot{\varphi}_{T(\tilde{z})}(\tilde{z}) = f(\varphi_{T(\tilde{z})}(\tilde{z}))$$

$$\vec{D}\tilde{P}(\tilde{z}) = \dot{\varphi}_{T(\tilde{z})}(\tilde{z})\vec{D}T(\tilde{z}) + \vec{D}\varphi_{T(\tilde{z})}(\tilde{z})$$

$$\vec{D}T(\tilde{z}) = \frac{(\vec{D}\varphi_{T(\tilde{z})}(\tilde{z}))_2}{(\vec{f}(PM\tilde{z}))_2}$$

$$\vec{D}PM$$

# Poincaré Map computation

$$\vec{D}PM(z) = M_2 \vec{D}\tilde{P}(\tilde{z})\vec{D}P_2(z)$$

$$where \quad \tilde{P}(\tilde{z}) = \varphi_{T(\tilde{z})}(\tilde{z}) \quad and \quad \dot{\varphi}_{T(\tilde{z})}(\tilde{z}) = f(\varphi_{T(\tilde{z})}(\tilde{z}))$$

$$\vec{D}\tilde{P}(\tilde{z}) = \dot{\varphi}_{T(\tilde{z})}(\tilde{z})\vec{D}T(\tilde{z}) + \vec{D}\varphi_{T(\tilde{z})}(\tilde{z})$$

$$\vec{D}T(\tilde{z}) = \frac{(\vec{D}\varphi_{T(\tilde{z})}(\tilde{z}))_2}{(\vec{f}(PM\tilde{\vec{z}}))_2}$$

$$\vec{D}PM$$

# Poincaré Map computation

$$\vec{D}PM(z) = M_2\vec{D}\tilde{P}(\tilde{z})\vec{D}P_2(z)$$

$$where \quad \tilde{P}(\tilde{z}) = \varphi_{T(\tilde{z})}(\tilde{z}) \quad and \quad \dot{\varphi}_{T(\tilde{z})}(\tilde{z}) = f(\varphi_{T(\tilde{z})}(\tilde{z}))$$

$$\vec{D}\tilde{P}(\tilde{z}) = \dot{\varphi}_{T(\tilde{z})}(\tilde{z})\vec{D}T(\tilde{z}) + \vec{D}\varphi_{T(\tilde{z})}(\tilde{z})$$

$$\vec{D}T(\tilde{z}) = \frac{(\vec{D}\varphi_{T(\tilde{z})}(\tilde{z}))_2}{(\vec{f}(PM\tilde{\vec{z}}))_2}$$

$$\vec{D}PM$$

# Henrard-Deprit Algorithm I

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta}_0,\sigma_0) = \vec{x}(t+T_0,\vec{\zeta}_0,\sigma_0)$$

$$\vec{\zeta}_1 = \vec{\zeta}_0 + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$\vec{0} = x(T_1,\vec{\zeta}_1,\sigma_1) -$$

$$x(T_0+\Delta T, \vec{\zeta}_0+\Delta\vec{\zeta}, \sigma_0+\Delta\sigma) - (\vec{\zeta}_0+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta} + \vec{F}(\vec{x},\sigma_0)\Delta T + \frac{\partial \vec{x}}{\partial \sigma}\Delta\sigma = -(\vec{x} - \vec{\zeta}_0)$$

$$(\nabla_\zeta \vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial \vec{x}}{\partial \sigma} \quad \mapsto$$ **Predictor**

# Henrard-Deprit Algorithm I

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta}_0,\sigma_0) = \vec{x}(t+T_0,\vec{\zeta}_0,\sigma_0)$$

$$\vec{\zeta}_1 = \vec{\zeta}_0 + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$\vec{0} = x(T_1,\vec{\zeta}_1,\sigma_1) -$$

$$x(T_0+\Delta T, \vec{\zeta}_0+\Delta\vec{\zeta}, \sigma_0+\Delta\sigma) - (\vec{\zeta}_0+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta} + \vec{F}(\vec{x},\sigma_0)\Delta T + \frac{\partial \vec{x}}{\partial \sigma}\Delta\sigma = -(\vec{x} - \vec{\zeta}_0)$$

$$(\nabla_\zeta \vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial \vec{x}}{\partial \sigma} \quad \mapsto$$ **Predictor**

# Henrard-Deprit Algorithm I

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta_0},\sigma_0) = \vec{x}(t+T_0,\vec{\zeta_0},\sigma_0)$$

$$\vec{\zeta_1} = \vec{\zeta_0} + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$\vec{0} = x(T_1,\vec{\zeta_1},\sigma_1) - x(0,\vec{\zeta_1},\sigma_1)$$

$$x(T_0+\Delta T,\vec{\zeta_0}+\Delta\vec{\zeta},\sigma_0+\Delta\sigma) - (\vec{\zeta_0}+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_{\zeta}\vec{x} - \vec{I})\Delta\vec{\zeta} + \vec{F}(\vec{x},\sigma_0)\Delta T + \frac{\partial\vec{x}}{\partial\sigma}\Delta\sigma = -(\vec{x} - \vec{\zeta_0})$$

$$(\nabla_{\zeta}\vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial\vec{x}}{\partial\sigma} \qquad \mapsto \textbf{Predictor}$$

# Henrard-Deprit Algorithm I

$$\dot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta}_0,\sigma_0) = \vec{x}(t+T_0,\vec{\zeta}_0,\sigma_0)$$

$$\vec{\zeta}_1 = \vec{\zeta}_0 + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$\vec{0} = x(T_1,\vec{\zeta}_1,\sigma_1) - \vec{\zeta}_1$$

$$x(T_0+\Delta T,\vec{\zeta}_0+\Delta\vec{\zeta},\sigma_0+\Delta\sigma) - (\vec{\zeta}_0+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta\vec{x} - \vec{I})\Delta\vec{\zeta} + \vec{F}(\vec{x},\sigma_0)\Delta T + \frac{\partial\vec{x}}{\partial\sigma}\Delta\sigma = -(\vec{x}-\vec{\zeta}_0)$$

$$(\nabla_\zeta\vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial\vec{x}}{\partial\sigma} \qquad \mapsto \textbf{Predictor}$$

# Henrard-Deprit Algorithm I

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta_0},\sigma_0) = \vec{x}(t+T_0,\vec{\zeta_0},\sigma_0)$$

$$\vec{\zeta_1} = \vec{\zeta_0} + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$\vec{0} = x(T_1,\vec{\zeta_1},\sigma_1) - \vec{\zeta_1}$$

$$x(T_0+\Delta T,\vec{\zeta_0}+\Delta\vec{\zeta},\sigma_0+\Delta\sigma) - (\vec{\zeta_0}+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta\vec{x} - \vec{I})\Delta\vec{\zeta} + \vec{F}(\vec{x},\sigma_0)\Delta T + \frac{\partial\vec{x}}{\partial\sigma}\Delta\sigma = -(\vec{x} - \vec{\zeta_0})$$

$$(\nabla_\zeta\vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial\vec{x}}{\partial\sigma} \qquad \mapsto \textbf{Predictor}$$

# Henrard-Deprit Algorithm I

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta}_0,\sigma_0) = \vec{x}(t+T_0,\vec{\zeta}_0,\sigma_0)$$

$$\vec{\zeta}_1 = \vec{\zeta}_0 + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$\vec{0} = x(T_1,\vec{\zeta}_1,\sigma_1) - \vec{\zeta}_1$$

$$x(T_0+\Delta T,\vec{\zeta}_0+\Delta\vec{\zeta},\sigma_0+\Delta\sigma) - (\vec{\zeta}_0+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta\vec{x} - \vec{I})\Delta\vec{\zeta} + \vec{F}(\vec{x},\sigma_0)\Delta T + \frac{\partial\vec{x}}{\partial\sigma}\Delta\sigma = -(\vec{x} - \vec{\zeta}_0)$$

$$(\nabla_\zeta\vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial\vec{x}}{\partial\sigma} \quad \mapsto \textbf{Predictor}$$

# Henrard-Deprit Algorithm I

$$\ddot{\vec{x}} = \vec{F}(x;\sigma) \qquad \vec{x} \equiv \vec{x}(t,\vec{\zeta_0},\sigma_0) = \vec{x}(t+T_0,\vec{\zeta_0},\sigma_0)$$

$$\vec{\zeta_1} = \vec{\zeta_0} + \Delta\vec{\zeta} \quad T_1 = T_0 + \Delta T? \quad \sigma_1 = \sigma_0 + \Delta\sigma$$

$$\vec{0} = x(T_1,\vec{\zeta_1},\sigma_1) - \vec{\zeta_1}$$

$$x(T_0+\Delta T,\vec{\zeta_0}+\Delta\vec{\zeta},\sigma_0+\Delta\sigma) - (\vec{\zeta_0}+\Delta\vec{\zeta}) = \vec{0}$$

$$(\nabla_\zeta\vec{x} - \vec{I})\Delta\vec{\zeta} + \vec{F}(\vec{x},\sigma_0)\Delta T + \frac{\partial\vec{x}}{\partial\sigma}\Delta\sigma = -(\vec{x}-\vec{\zeta_0})$$

$$(\nabla_\zeta\vec{x} - \vec{I})\frac{\Delta\vec{\zeta}}{\Delta\sigma} + \vec{F}(\vec{x},\sigma_0)\frac{\Delta T}{\Delta\sigma} = -\frac{\partial\vec{x}}{\partial\sigma} \quad \mapsto$$ **Predictor**

# Henrard-Deprit Algorithm II

$$x(T_1, \vec{\zeta_1}, \sigma_1) - \vec{\zeta_1} \neq 0$$

$$\Delta\vec{\zeta_1}, \Delta T_1 ? \Rightarrow x(T_1 + \Delta T_1, \vec{\zeta_1} + \Delta\vec{\zeta_1}, \sigma_1) - (\vec{\zeta_1} + \Delta\vec{\zeta_1}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta_1} + \vec{F}(\vec{x}, \sigma_1)\Delta T_1 = -(\vec{x} - \vec{\zeta_1}) \quad \mapsto \textbf{Corrector}$$

- *The predictor requests as a minimum a tangent to the natural family $O(\sigma)$ at the initial condition in the direction of increasing parameter values. The phase state obtained by shifting the initial conditions along a tangent does not generally lie on the natural family. Therefore the corrector stage aims at bringing this estimation closer to the periodic orbit.*

# Henrard-Deprit Algorithm II

$$x(T_1, \vec{\zeta_1}, \sigma_1) - \vec{\zeta_1} \neq 0$$

$$\Delta\vec{\zeta_1}, \Delta T_1? \Rightarrow x(T_1 + \Delta T_1, \vec{\zeta_1} + \Delta\vec{\zeta_1}, \sigma_1) - (\vec{\zeta_1} + \Delta\vec{\zeta_1}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta_1} + \vec{F}(\vec{x}, \sigma_1)\Delta T_1 = -(\vec{x} - \vec{\zeta_1}) \quad \mapsto \textbf{Corrector}$$

- *The predictor requests as a minimum a tangent to the natural family $O(\sigma)$ at the initial condition in the direction of increasing parameter values. The phase state obtained by shifting the initial conditions along a tangent does not generally lie on the natural family. Therefore the corrector stage aims at bringing this estimation closer to the periodic orbit.*

# Henrard-Deprit Algorithm II

$$x(T_1, \vec{\zeta_1}, \sigma_1) - \vec{\zeta_1} \neq 0$$

$$\Delta\vec{\zeta_1}, \Delta T_1? \Rightarrow x(T_1 + \Delta T_1, \vec{\zeta_1} + \Delta\vec{\zeta_1}, \sigma_1) - (\vec{\zeta_1} + \Delta\vec{\zeta_1}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta_1} + \vec{F}(\vec{x}, \sigma_1)\Delta T_1 = -(\vec{x} - \vec{\zeta_1}) \quad \mapsto \textbf{Corrector}$$

- *The predictor requests as a minimum a tangent to the natural family $O(\sigma)$ at the initial condition in the direction of increasing parameter values. The phase state obtained by shifting the initial conditions along a tangent does not generally lie on the natural family. Therefore the corrector stage aims at bringing this estimation closer to the periodic orbit.*

# Henrard-Deprit Algorithm II

$$x(T_1, \vec{\zeta_1}, \sigma_1) - \vec{\zeta_1} \neq 0$$

$$\Delta\vec{\zeta_1}, \Delta T_1? \Rightarrow x(T_1 + \Delta T_1, \vec{\zeta_1} + \Delta\vec{\zeta_1}, \sigma_1) - (\vec{\zeta_1} + \Delta\vec{\zeta_1}) = \vec{0}$$

$$(\nabla_\zeta \vec{x} - \vec{I})\Delta\vec{\zeta_1} + \vec{F}(\vec{x}, \sigma_1)\Delta T_1 = -(\vec{x} - \vec{\zeta_1}) \quad \mapsto \textbf{Corrector}$$

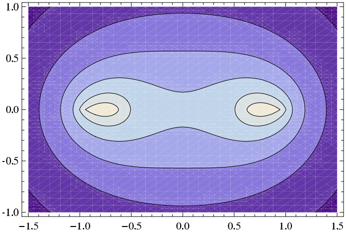
- *The predictor requests as a minimum a tangent to the natural family $O(\sigma)$ at the initial condition in the direction of increasing parameter values. The phase state obtained by shifting the initial conditions along a tangent does not generally lie on the natural family. Therefore the corrector stage aims at bringing this estimation closer to the periodic orbit.*

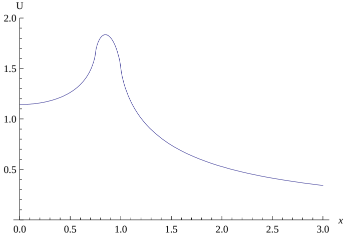
$$M_1 = \begin{pmatrix} \bar{0}_{2x1} & \vec{I}_2 & \bar{0}_{2x1} & \bar{0}_{2x2} \\ \bar{0}_{2x1} & \bar{0}_{2x2} & \bar{0}_{2x1} & \vec{I}_2 \end{pmatrix} \quad \vec{D}P_1 = \begin{pmatrix} \bar{0}^t_{2x1} & \bar{0}^t_{2x1} \\ \vec{I}_2 & \bar{0}_{2x2} \\ \partial_x y_1^t & \partial_y y_1^t \\ \bar{0}_{2x2} & \vec{I}_2 \end{pmatrix}$$

$$\partial_x y_2 = \begin{pmatrix} \frac{\partial y_2}{\partial x_1} \\ \frac{\partial y_2}{\partial x_3} \end{pmatrix} \qquad \partial_y y_2 = \begin{pmatrix} \frac{\partial y_2}{\partial y_1} \\ \frac{\partial y_2}{\partial y_3} \end{pmatrix}$$

$$Df(\tilde{z}) = \begin{pmatrix} \bar{0}_{2x2} & \vec{I}_{2x2} \\ \bar{A}_{2x2} & \bar{0}_{2x2} \end{pmatrix} \quad A = \begin{pmatrix} U_{xx} & U_{xy} \\ U_{yx} & U_{yy} \end{pmatrix}$$

# Periodic orbits around an Annular Ring

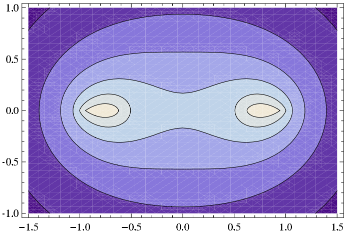


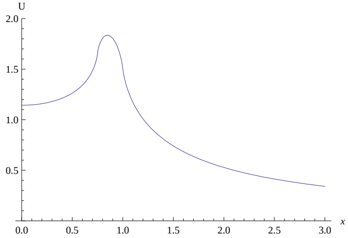




Cylindrical coordinates: $(r, \lambda, z)$

$$r'' = -W'(r), \quad W(r) = U(r) + \frac{\Lambda^2}{2r^2}$$

# Periodic orbits around an Annular Ring

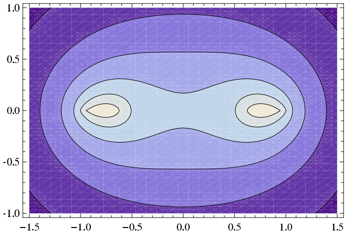


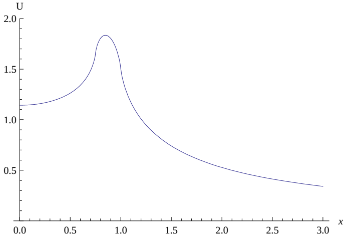




Cylindrical coordinates: $(r, \lambda, z)$

$$r'' = -W'(r), \quad W(r) = U(r) + \frac{\Lambda^2}{2r^2}$$

# Periodic orbits around an Annular Ring

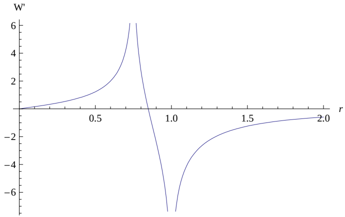


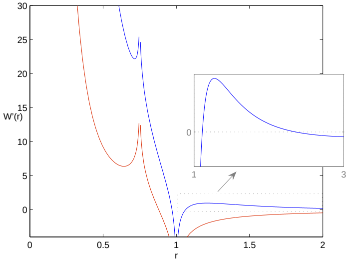




Cylindrical coordinates: $(r, \lambda, z)$

$$r'' = -W'(r), \quad W(r) = U(r) + \frac{\Lambda^2}{2r^2}$$

# Periodic orbits around an Annular Ring




- The origin is the unique equilibrium solution in case of $\Lambda = 0$
- For $\Lambda > \Lambda^*(a, b) > 1$ there are two critical points corresponding to a maximum and minimum of $U_{eff}$ while for smaller values there is no critical points.