22, 23 y 24 de Octubre de 2008. El Escorial (Madrid)

Cuarta reunión de la red temática Dance

Ddays 2008

Dance (Dinámica, Atractores y No linealidad. Caos y Estabilidad)

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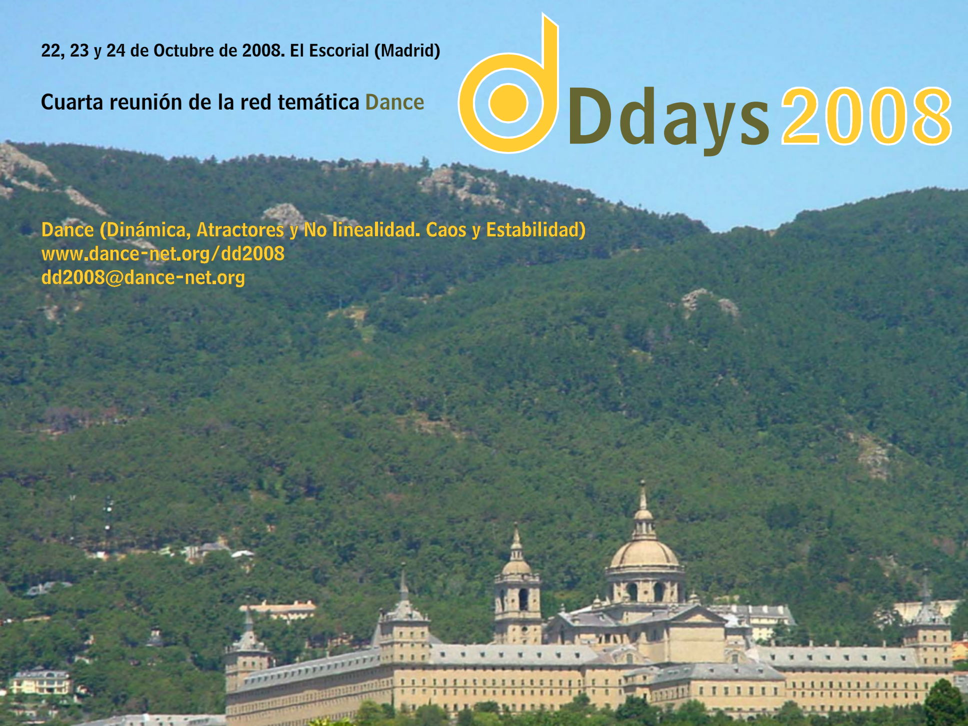

Complexity, branch points and sensitive dependence

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DDays - Red Dance
El Escorial 22 October 2008

Tres ideas

$$\dot{\mathbf{x}} = F(\mathbf{x}), \qquad \mathbf{x} = \mathbf{x}(t)$$

1. Las singularidades de $\mathbf{x}(t)$ en el plano t complejo son importantes.
2. Un sistema puede ser *integrable* y tener *dependencia sensitiva* sobre condiciones iniciales.
3. El estudio de la superficie de Riemann de $\mathbf{x}(t)$ proporciona mucha información sobre la dinámica (que resulta difícil de obtener por otros métodos)

Ejemplo 1

Example 1: A system in $\mathbb{C}^3$

$$\dot{z}_1 = -\mathrm{i}\,\omega\, z_1 + \frac{g_{12}}{z_1 - z_2} + \frac{g_{13}}{z_1 - z_3}\ ,$$
$$\dot{z}_2 = -\mathrm{i}\,\omega\, z_2 + \frac{g_{21}}{z_2 - z_1} + \frac{g_{23}}{z_2 - z_3}\ ,$$
$$\dot{z}_3 = -\mathrm{i}\,\omega\, z_3 + \frac{g_{31}}{z_3 - z_1} + \frac{g_{32}}{z_3 - z_2}\ .$$

$$z_i = z_i(t), \quad z_i \in \mathbb{C}, \quad t \in \mathbb{R}$$

Coupling constants $g_{ij} = g_{ji} \in \mathbb{R}$

$$\omega = 2\pi \Rightarrow T = 1$$

The complex ODEs

$$
\begin{aligned}
\zeta_1' &= \frac{g_{12}}{\zeta_1 - \zeta_2} + \frac{g_{13}}{\zeta_1 - \zeta_3}\,, \\
\zeta_2' &= \frac{g_{12}}{\zeta_2 - \zeta_1} + \frac{g_{13}}{\zeta_2 - \zeta_3}\,, \\
\zeta_3' &= \frac{g_{13}}{\zeta_3 - \zeta_1} + \frac{g_{23}}{\zeta_3 - \zeta_2}\,,
\end{aligned}
$$

Change of variables

$$
\left.\begin{aligned}
\tau(t) &= \tfrac{1}{2\mathrm{i}\omega}\mathrm{e}^{2\mathrm{i}\omega t} - 1 \\
z_n(t) &= \mathrm{e}^{-\mathrm{i}\omega t}\zeta_n(\tau)
\end{aligned}\right\} \Rightarrow
\begin{aligned}
\dot z_1 &= -\mathrm{i}\omega z_1 + \tfrac{g_{12}}{z_1 - z_2} + \tfrac{g_{13}}{z_1 - z_3} \\
\dot z_2 &= -\mathrm{i}\omega z_2 + \tfrac{g_{21}}{z_2 - z_1} + \tfrac{g_{23}}{z_2 - z_3} \\
\dot z_3 &= -\mathrm{i}\omega z_3 + \tfrac{g_{31}}{z_3 - z_1} + \tfrac{g_{32}}{z_3 - z_2}
\end{aligned}
$$

Reducing the general solution to quadratures

$$\zeta_1' = \frac{g_{12}}{\zeta_1 - \zeta_2} + \frac{g_{13}}{\zeta_1 - \zeta_3} \,,$$
$$\zeta_2' = \frac{g_{12}}{\zeta_2 - \zeta_1} + \frac{g_{13}}{\zeta_2 - \zeta_3} \,,$$
$$\zeta_3' = \frac{g_{13}}{\zeta_3 - \zeta_1} + \frac{g_{23}}{\zeta_3 - \zeta_2} \,,$$

- Translational invariance: CM motion is conserved.

$$Z(\tau) = \frac{1}{3}(\zeta_1 + \zeta_2 + \zeta_3) \Rightarrow Z' = 0 \Rightarrow Z(\tau) = Z(0)$$

- Another conserved quantity

$$\zeta_1' \zeta_1 + \zeta_2' \zeta_2 + \zeta_3' \zeta_3 = g_{12} + g_{23} + g_{13}$$

Reducing the general solution to quadratures

$$\zeta_1' = \frac{g_{12}}{\zeta_1 - \zeta_2} + \frac{g_{13}}{\zeta_1 - \zeta_3} \,,$$
$$\zeta_2' = \frac{g_{12}}{\zeta_2 - \zeta_1} + \frac{g_{13}}{\zeta_2 - \zeta_3} \,,$$
$$\zeta_3' = \frac{g_{13}}{\zeta_3 - \zeta_1} + \frac{g_{23}}{\zeta_3 - \zeta_2} \,,$$

- Write $(\zeta_1, \zeta_2, \zeta_3)$ in terms of (Z, ρ, θ):

$$\zeta_1(\tau) = Z - \sqrt{\frac{2}{3}}\,\rho \cos\left(\theta + \frac{2\pi}{3}\right),$$
$$\zeta_2(\tau) = Z - \sqrt{\frac{2}{3}}\,\rho \cos\left(\theta - \frac{2\pi}{3}\right),$$
$$\zeta_3(\tau) = Z - \sqrt{\frac{2}{3}}\,\rho \cos\theta,$$

Reducing the general solution to quadratures

$$
\begin{aligned}
\zeta_1(\tau) &= Z - \sqrt{\frac{2}{3}}\,\rho\cos\left(\theta + \frac{2\pi}{3}\right),\\
\zeta_2(\tau) &= Z - \sqrt{\frac{2}{3}}\,\rho\cos\left(\theta - \frac{2\pi}{3}\right),\\
\zeta_3(\tau) &= Z - \sqrt{\frac{2}{3}}\,\rho\cos\theta,
\end{aligned}
$$

- Using the previous conserved quantity and trigonometric identities

$$\zeta_1^2 + \zeta_2^2 + \zeta_3^2 = 3Z^2 + \rho^2 \Rightarrow$$

$$\rho^2(\tau) = 2(g_{12} + g_{13} + g_{23})(\tau - \tau_1)$$

Reducing the general solution to quadratures

$$\zeta_1' = \frac{g_{12}}{\zeta_1 - \zeta_2} + \frac{g_{13}}{\zeta_1 - \zeta_3}\,,$$
$$\zeta_2' = \frac{g_{12}}{\zeta_2 - \zeta_1} + \frac{g_{13}}{\zeta_2 - \zeta_3}\,,$$
$$\zeta_3' = \frac{g_{13}}{\zeta_3 - \zeta_1} + \frac{g_{23}}{\zeta_3 - \zeta_2}\,,$$

- The evolution of $Z(\tau)$ y $\rho(\tau)$ is integrated. Need only an equation for $\theta(\tau)$:

$$\rho^2\,(\cos\theta)'\,\left(4\,\cos^2\theta - 1\right) = (4\,g_{12} + 4\,g_{13} + g_{23})\,\cos\theta$$
$$-4\,(g_{12} + g_{13} + g_{23})\,\cos^3\theta + \sqrt{3}\,(g_{23} - g_{13})\,\sin\theta$$

- Call $u = \cos\theta$ and assume that $g_{23} = g_{13} = g$, $g_{12} = f$:

$$\rho^2 u' = \frac{(f + 8g)u - 4(f + 2g)u^3}{4u^2 - 1}$$

Reducing the general solution to quadratures

$$\rho^2 u' = \frac{(f+8g)u - 4(f+2g)u^3}{4u^2-1}$$

Since ρ^2 is linear in τ we have

$$\int \frac{d\tau}{2(2g+f)(\tau-\tau_1)} = \int du \frac{4u^2-1}{(f+8g)u+(4f+8g)u^3}$$

which can be integrated explicitly as

$$u^{-2\mu}\left(u^2 - \frac{1}{4\mu}\right)^{\mu-1} = K(\tau-\tau_1),$$

where K is the integration constant and

$$\mu = \frac{f+2g}{f+8g}$$

Reducing the general solution to quadratures

One last change of variables

$$\xi = \frac{K(\tau - \tau_1)}{4\mu}$$
$$w = 4\mu\, u^2$$

transforms the equation

$$u^{-2\mu} \left(u^2 - \frac{1}{4\mu} \right)^{\mu - 1} = K(\tau - \tau_1),$$

into

$$(w-1)^{\mu-1}\, w^{-\mu} = \xi, \qquad \mu = \frac{f + 2g}{f + 8g}$$

Now work everything back to the original variables...

General solution

$$z_1(t) = Z\mathrm{e}^{-\mathrm{i}\omega t} - \frac{1}{2}\left(\frac{f+8g}{6\mathrm{i}\omega}\right)^{1/2}(1+\eta\mathrm{e}^{-2\mathrm{i}\omega t})^{1/2}\left[\check{w}(t)^{1/2} + (12\mu - 3\check{w}(t))^{1/2}\right]$$

$$z_2(t) = Z\mathrm{e}^{-\mathrm{i}\omega t} - \frac{1}{2}\left(\frac{f+8g}{6\mathrm{i}\omega}\right)^{1/2}(1+\eta\mathrm{e}^{-2\mathrm{i}\omega t})^{1/2}\left[\check{w}(t)^{1/2} - (12\mu - 3\check{w}(t))^{1/2}\right]$$

$$z_3(t) = Z\mathrm{e}^{-\mathrm{i}\omega t} - \frac{1}{2}\left(\frac{f+8g}{6\mathrm{i}\omega}\right)^{1/2}(1+\eta\mathrm{e}^{-2\mathrm{i}\omega t})^{1/2}\ \check{w}(t)^{1/2}$$

$$\check{w}(t) = w[\xi(t)] \quad \begin{cases} (w-1)^{\mu-1}\, w^{-\mu} = \xi \\ \xi(t) = \overline{\xi} + R\,\mathrm{e}^{2\mathrm{i}\omega t} \end{cases}$$

- The complex constants Z, η, R y $\overline{\xi}$ are fixed by the initial data (only 3 of them are independent).
- f, g, ω and μ are parameters of the problem.
- The function $\check{w}(t)$ has all the “juicy” part of the dynamics: we need to follow by continuity one of the many solutions of $(w-1)^{\mu-1}\, w^{-\mu} = \xi(t)$ as t evolves.

General solution

Constants Z, η, R y $\bar{\xi}$ are fixed by initial data :

$$
\begin{aligned}
Z &= \frac{z_1(0) + z_2(0) + z_3(0)}{3} \,, \\
R &= \frac{3\,(f + 8\,g)}{2\,\mathrm{i}\,\omega\,\left[2\,z_3(0) - z_1(0) - z_2(0)\right]^2} \left[1 - \frac{1}{\check{w}(0)}\right]^{\mu - 1} \,, \\
\eta &= \frac{\mathrm{i}\,\omega\,\left\{\left[z_1(0) - z_2(0)\right]^2 + \left[z_2(0) - z_3(0)\right]^2 + \left[z_3(0) - z_1(0)\right]^2\right\}}{3\,(f + 2\,g)} - 1 \,, \\
\check{w}(0) &= \frac{2\,\mu\,\left[2\,z_3(0) - z_1(0) - z_2(0)\right]^2}{\left[z_1(0) - z_2(0)\right]^2 + \left[z_2(0) - z_3(0)\right]^2 + \left[z_3(0) - z_1(0)\right]^2} \,, \\
\bar{\xi} &= R\,\eta \,.
\end{aligned}
$$

(note only 3 of them are independent)

Riemann surface of the solution

We must study the Riemann surface of the function $w(\xi)$ defined implicitly by

$$\Gamma = \{(\xi, w) \mid (w-1)^{\mu-1}\, w^{-\mu} = \xi\}$$

$\mu = p/q$ rational

- Γ is a finitely-sheeted covering of the extended ξ-plane $\mathbb{C} \cup \infty$.
- Almost all solutions are periodic, stable and isochronous.
- A set of null measure of singular orbits (collision manifolds).
- Closed formulas for the period can be written (using mostly combinatorial arguments).

μ irrational

- Γ is an infinitely-sheeted covering of the extended ξ-plane $\mathbb{C} \cup \infty$.
- Some orbits are periodic, some are aperiodic.
- Collision manifolds are dense (but null measure) on an open set of phase space.
- Sensitive dependence on initial data.

Description of the Riemann surface

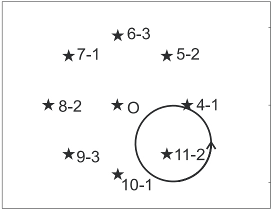
$$\mu = p/q, \quad p, q, \in \mathbb{Z} \text{ coprimes}$$

The RS is different in the two cases $0 < \mu < 1$ and $\mu > 1$.

Case $\mu > 1$: Γ has $p + q$ sheets, with ramification points:

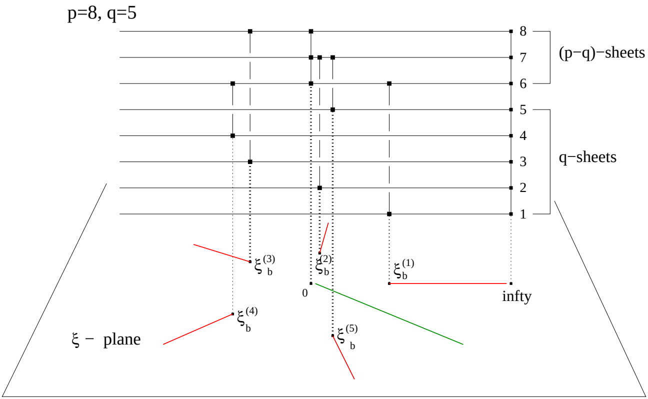
- O is a branch point order $p - q$.
- There are q branch points $(\xi_b^{(j)}, \mu)$ of order 2 at

$$\xi_b^{(j)} = r_b \exp\left[\mathrm{i} \frac{2\pi j p}{q} \right], \quad j = 1, 2, ..., q$$



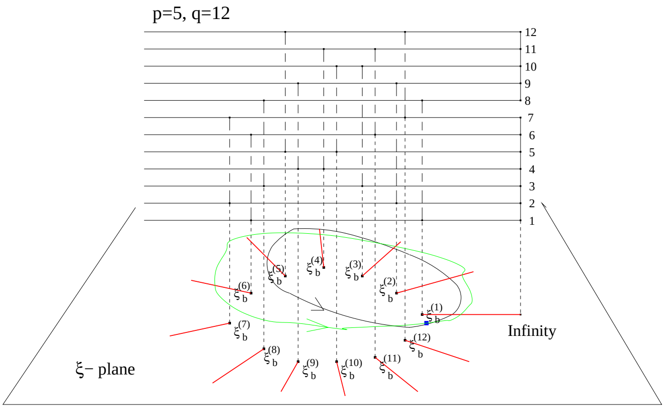
$p = 11,\ q = 8$

Description of the Riemann surface



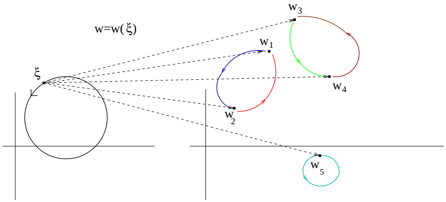
$\mu = 8/5$

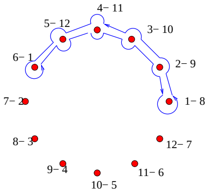
Description of the Riemann surface



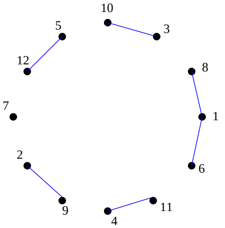
$\mu = 5/12$

Ferrers diagrams



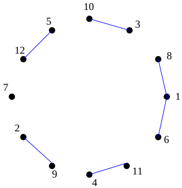


Branch point configuration (p=5, q=12)

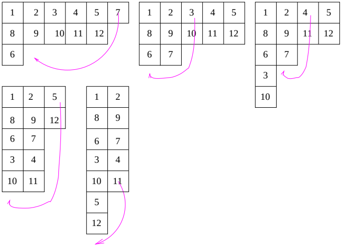


1	2	3	4	5	7
8	9	10	11	12	
6					

Bumping algorithm



1	2	3	4	5	7
8	9	10	11	12	
6					

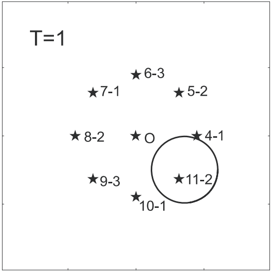


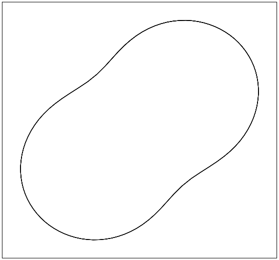
- Closed formulas for the period can be given after some combinatorics and modular arithmetic.
- In some cases, the formulas depend on the coefficients of the continued fraction expansion of μ.

Motions for $\mu = 11/8$

$$f = 12, g = -0.5 \Rightarrow \mu = 11/8$$

Initial data: $z_1(0) = (0,0), \quad z_2(0) = (-0.5, 1), \quad z_3(0) = (1,1)$

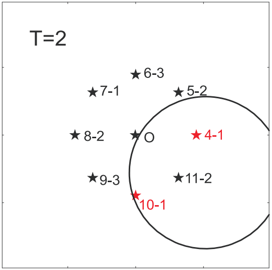


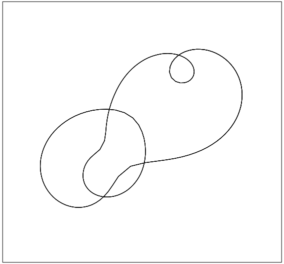


Motions for $\mu = 11/8$

$$f = 12, g = -0.5 \Rightarrow \mu = 11/8$$

Initial data: $z_1(0) = (0, 0)$, $z_2(0) = (0.4, 1)$, $z_3(0) = (1, 1)$

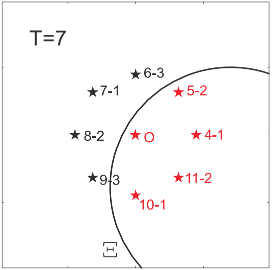


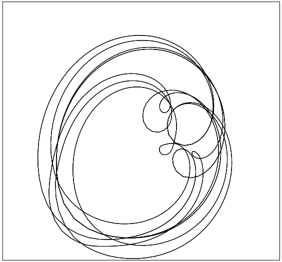


Motions for $\mu = 11/8$

$$f = 12, g = -0.5 \Rightarrow \mu = 11/8$$

Initial data: $z_1(0) = (0, 0)$, $z_2(0) = (0.7, 1)$, $z_3(0) = (1, 1)$

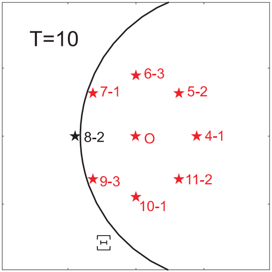


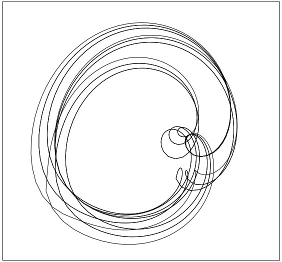


Motions for $\mu = 11/8$

$$f = 12, g = -0.5 \Rightarrow \mu = 11/8$$

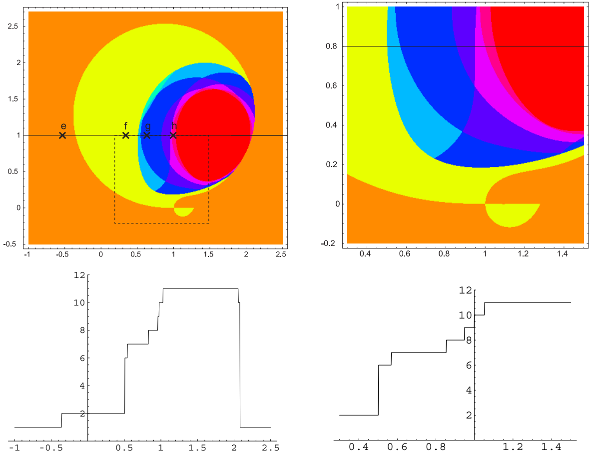
Initial data: $z_1(0) = (0,0)$, $z_2(0) = (1,1)$, $z_3(0) = (1,1)$





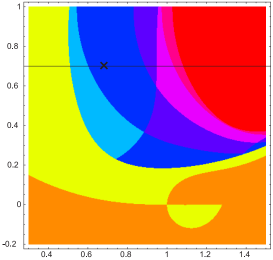
Dependence of the period on initial data

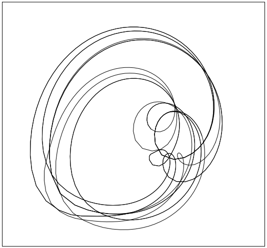
$z_1(0) = (0, 0), \quad z_2(0) = (x, y), \quad z_3(0) = (1, 1)$

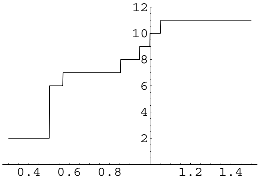


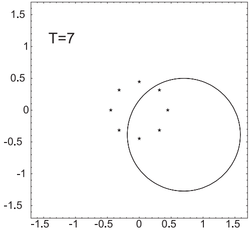
Sensitive dependence

$\mu = 11/8$



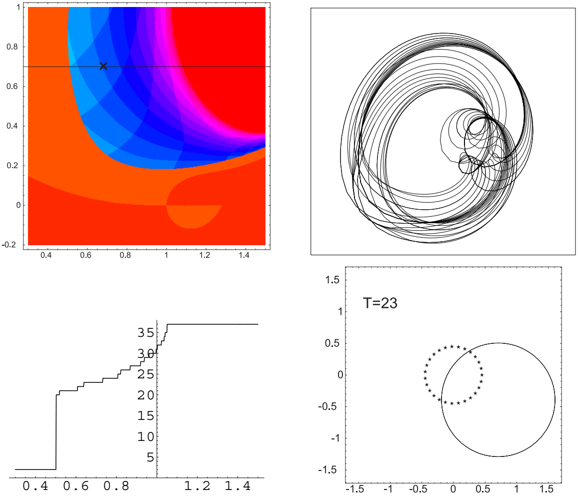






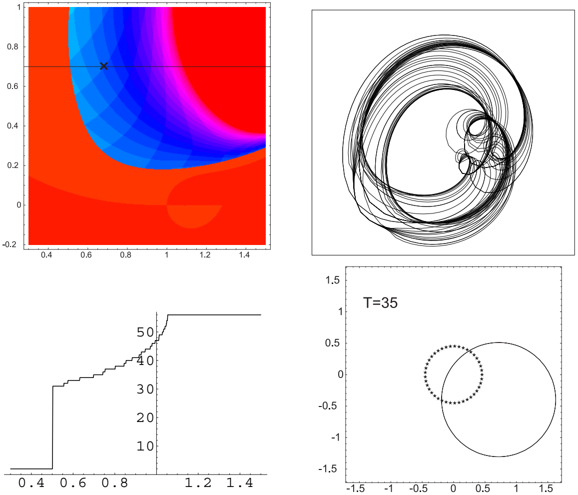
Sensitive dependence

$\mu = 37/27$



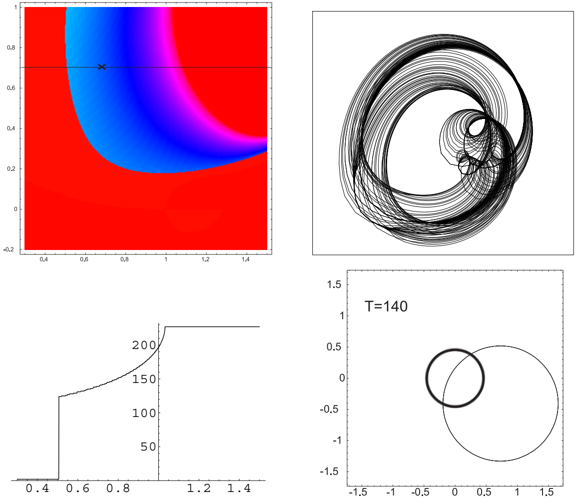
Sensitive dependence

$\mu = 56/41$

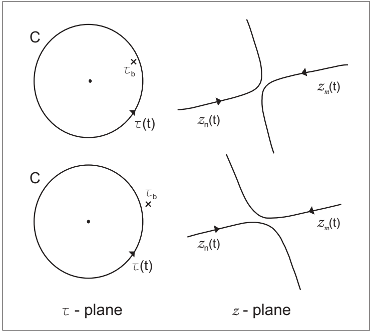


Sensitive dependence

$\mu = 227/167$



- Two nearby trajectories that pass on different sides of a BP separate (path jumps to different sheets of the RS). In the physical variables an almost collision occurs.



- c.f. analogy with polygonal billiards

Example 2

Example 2

$$\frac{d^2\zeta}{d\tau^2} = P(\zeta) = -\frac{d\,V(\zeta)}{d\zeta}, \qquad \tau, \zeta \in \mathbb{C}. \qquad P(\zeta), V(\zeta) \text{ are polynomials in } \zeta$$

- Circular motion: $\tau(t) = \tau_0 + \frac{1}{\mathrm{i}\omega}\left(\mathrm{e}^{\mathrm{i}\omega t} - 1\right)$
- Rectilinear motion: $\tau(t) = at + b, \quad a, b \in \mathbb{C}$

$$\text{First integral } E = \frac{1}{2}(\zeta')^2 + V(\zeta)$$

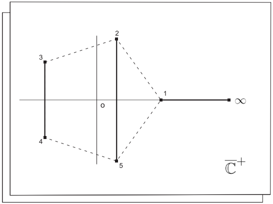
Solution is formally obtained by inverting

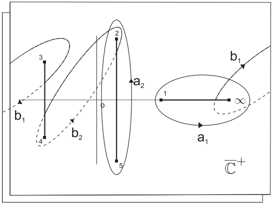
$$\tau - \tau_0 = \int_{\zeta_0}^{\zeta} \frac{d\eta}{\sqrt{2(E - V(\eta))}}, \qquad \zeta_0 = \zeta(\tau_0)$$

- $\deg V \leq 4 \Rightarrow \zeta(\tau)$ is meromorphic (single-valued)
- $\deg V > 4 \Rightarrow \zeta(\tau)$ is (in general) infinitely multiple-valued

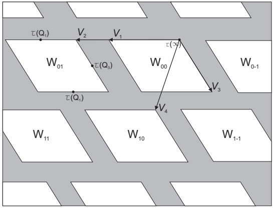
Inverting hyper-elliptic integrals

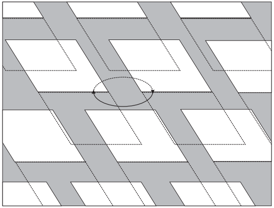
$$\Gamma = \{(\eta, \mu) | \, \mu^2 = P_5(\eta)\},$$





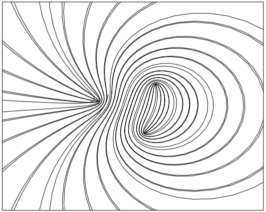
$$V_1 = \oint_{a_1} \frac{d\eta}{\sqrt{2}\,\mu}, \quad V_2 = \oint_{a_2} \frac{d\eta}{\sqrt{2}\,\mu}, \quad V_3 = \oint_{b_1} \frac{d\eta}{\sqrt{2}\,\mu}, \quad V_4 = \oint_{b_2} \frac{d\eta}{\sqrt{2}\,\mu}.$$



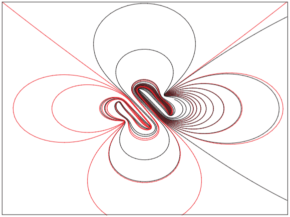


Rectilinear motion $\tau(t) = t$

$$\ddot{z} = -(k+1)z^k, \qquad z = z(t) \in \mathbb{C},\ t \in \mathbb{R}$$



$k < 4 \Rightarrow$ quasi-periodic motion
(c.f. irrational covering of $\mathbb{T}^2$)

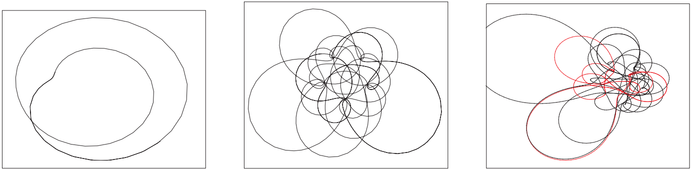


$k \geq 4 \Rightarrow$ sensitive dependence
(c.f. polygonal billiard)

Dynamical systems on hyper-elliptic Riemann surfaces

Circular motion $\tau(t) = \tau_0 + \frac{1}{\mathrm{i}\omega}\left(\mathrm{e}^{\mathrm{i}\omega t} - 1\right)$

$$\ddot{z} + \mathrm{i}\left(\frac{3+k}{1-k}\right)\omega\dot{z} - \frac{2+2k}{(1-k)^2}\,\omega^2 z = -(k+1)z^k, \qquad k \in \mathbb{Z}$$



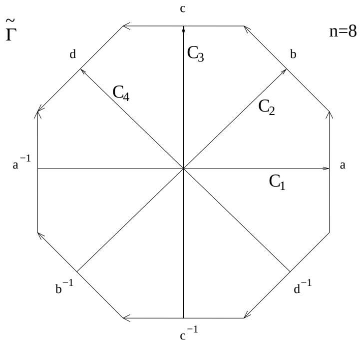
$k < 4$	$k = 4$	$k = 5$
elliptic	hyperelliptic	“hyperelliptic”
periodic $T = 1$	periodic T unbounded	periodic $T = 1, 2$

Surfaces of higher genus

- Consider the Riemann Surface corresponding to the curve

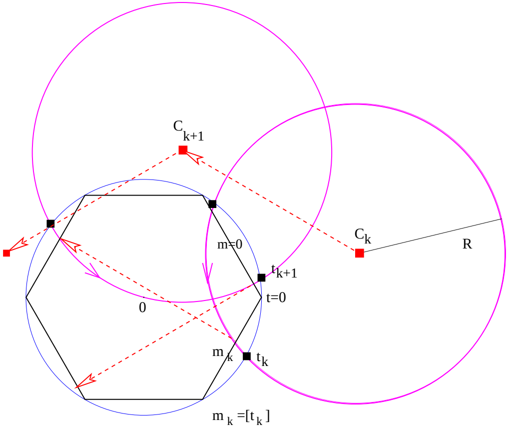
$$\Gamma_{2n} = \{(\eta, \mu) | \mu^2 = E - \eta^{2n}\}$$

We can view this surface as the result of identifying the opposite sides of a regular n-gon (genus is $n - 1$).

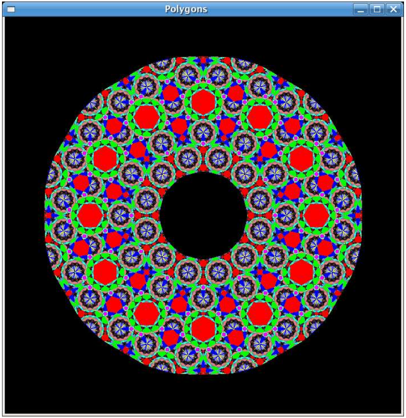


The center map

- The center map is a discrete 2-dim map depending on two parameters that captures the essential dynamics of the differential equation.

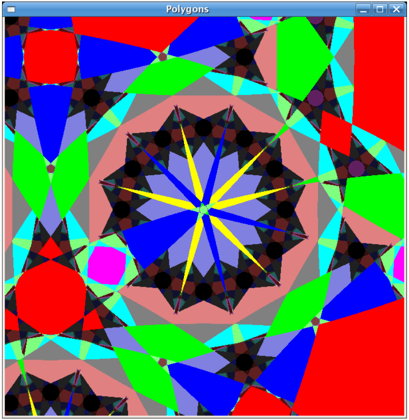


Results for $n = 12$ and $R = 1.8$



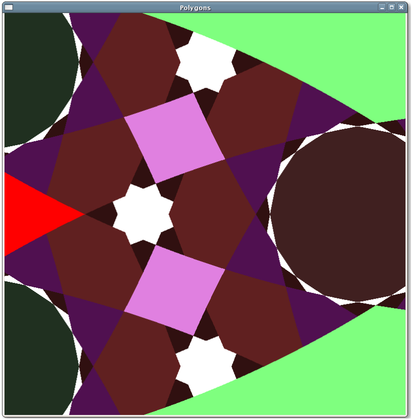
The center map $n = 12$

Results for $n = 12$ and $R = 1.8$: $10\times$ magnification



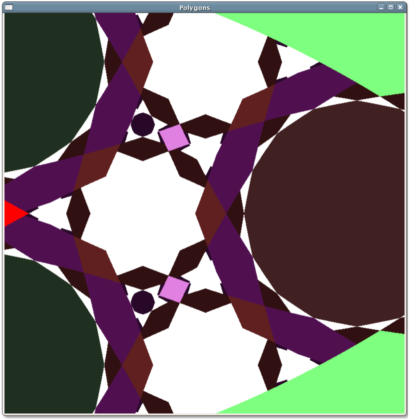
All regions have smooth boundaries and finite area $\Rightarrow$ Local isochronocity

Results for $n = 14$ and $R = 0.22$: $100\times$ magnification



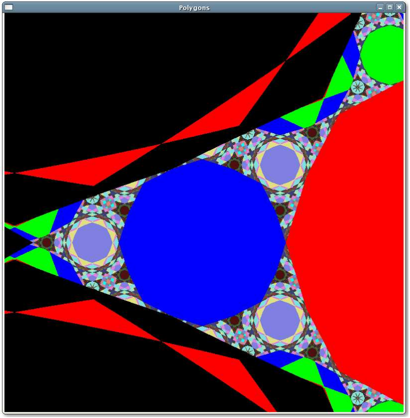
The center map $n = 14$: Periodic regime

Results for $n = 14$ and $R = 0.225$: $100\times$ magnification



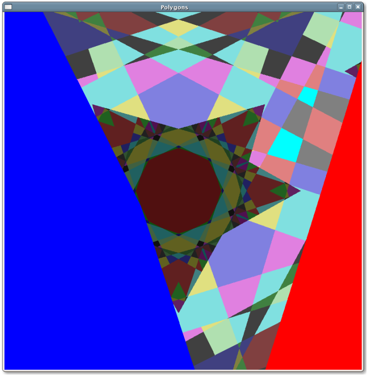
The center map $n = 14$: Periodic regime

Results for $n = 14$ and $R = 0.228$: $300\times$ magnification



The center map $n = 14$: Periodic regime

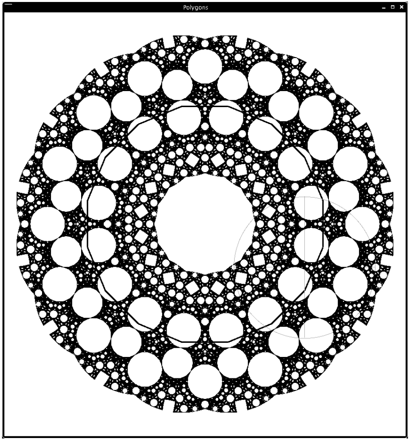
Results for $n = 14$ and $R = 0.228$: $3000\times$ magnification



0	64	128	192	256	320	384	448
512	576	640	704	768	832	896	960
1024	1088	1152	1216	1280	1344	1408	1472
1536	1600	1664	1728	1792	1856	1920	1984
2048	2112	2176	2240	2304	2368	2432	2496
2560	2624	2688	2752	2816	2880	2944	3008
3072	3136	3200	3264	3328	3392	3456	3520
3584	3648	3712	3776	3840	3904	3968	4032
4096	4160	4224	4288	4352	4416	4480	4544
4608	4672	4736	4800	4864	4928	4992	

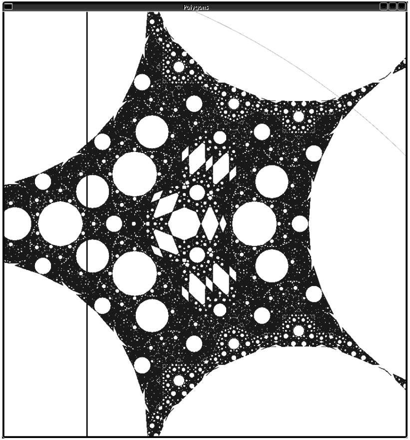
The center map $n = 14$: Chaotic regime

Results for $n = 14$ and $R = 0.358$, initial center $C_0 = (1.1610, -0.4037)$, $2 \cdot 10^9$ iterations



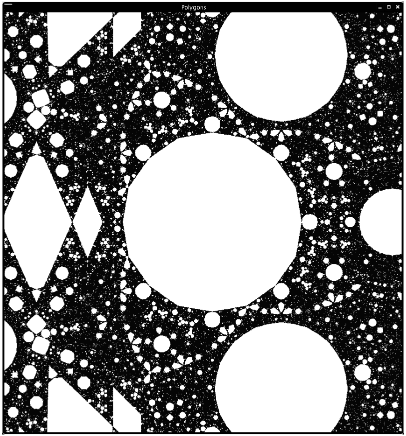
The center map

A region of the previous plot, magnified 10 times



The center map

A region of the previous plot, magnified 4 times more



Understanding the numerics

- For $n \leq 12$ and any radius
  - all orbits are periodic
  - period can become arbitrarily large (grows exponentially with R/l)
  - Periods of order 10^8 have been observed for $R/l \sim 20$
- For $n \geq 14$ two different behaviours:
  1. For $R < R_c$, periodic behaviour (same as above).
  2. For $R > R_c$ aperiodic and irregular fractal behaviour

Question

Why is there a critical genus between $n = 12$ and $n = 14$?

A similar situation is observed in the theory of pattern formation. . .

The ergodic hypothesis

- The total shift of the center after N iterations of the map is given by

$$C_N = C_0 + \sum_{i=1}^{N} m_j V_j$$

V_j : the possible shifts along the sides of the polygon (period vectors)
m_j : net number of shifts along direction V_j.

- For aperiodic trajectories, the numerical behaviour for large N of C_N is

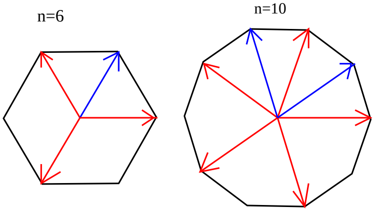
$$C_N \sim N^{\lambda} \vec{v}(N), \qquad \lambda = 1/2$$

$\vec{v}(N)$: a random vector of order 1.

- $\lambda = 1/2$ can be interpreted as an algebraic Lyapunov exponent
The motion is a random walk on an effective lattice.

The effective lattice

- Not all shifts are independent over the integers



so the dimension of the effective lattice is lower than k for a regular polygon of $2k$ sides.

  - If $n = 2k$ with k odd: the alternated sum of the periods is zero.
  - If $n = 4k$ with k odd, the symmetry of the polygon implies extra relations

Dimension of the effective lattice

- With this in mind, the effective dimension of the lattice as a function of the number of sides of the polygon for the first few cases is:

n	num. indep. shifts	Eff. lattice dim.
8	4	2
10	4	2
12	4	2
14	6	4
16	8	6

1. A random walk on a 2-dim lattice comes back to the initial point with probability 1 (i.e. with probability 1 the orbit will be periodic)
2. A random walk on a lattice of dimension $d > 2$ there is a non-zero probability of not returning to the initial point (i.e. there will be aperiodic orbits).

Summary and outlook

- The solutions of many systems of ODEs are infinitely valued functions of time.
- Information on their Riemann Surface is relevant to understanding the dynamics.
- Sensitive dependence can be understood in terms of clustering of branch points.
- Singularities on the complex time plane play a role even if time travels on the real axis.

Outlook:

- Understand how this new notion of chaos relates to existing theories
- In particular, introduce Lyapunov exponents related to this theory.
- Elaborate on Universality of this mechanism, built on the understanding of these examples.