

22, 23 y 24 de Octubre de 2008. El Escorial (Madrid)

Cuarta reunión de la red temática Dance

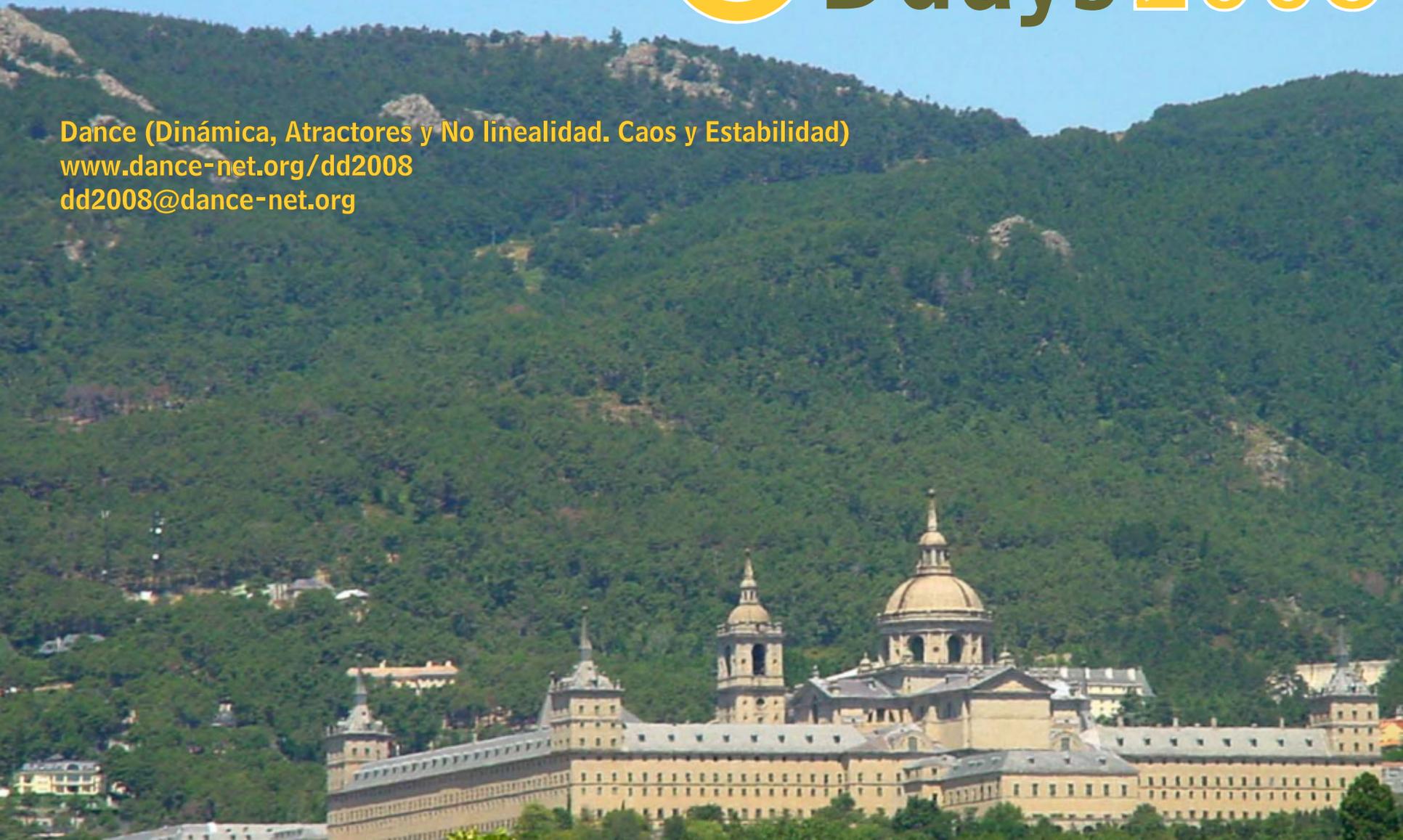
The logo for Ddays 2008 features a stylized lowercase 'd' in yellow with a white outline. The 'd' is composed of a vertical bar on the right and a circular shape on the left. Inside the circle, there are two concentric circles: an outer one in white and an inner one in yellow. To the right of the 'd', the word 'Ddays' is written in a bold, dark grey sans-serif font, and '2008' is written in a yellow sans-serif font with a white outline.

Ddays 2008

Dance (Dinámica, Atractores y No linealidad. Caos y Estabilidad)

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Eventos poco frecuentes en sistemas de reacción-difusión

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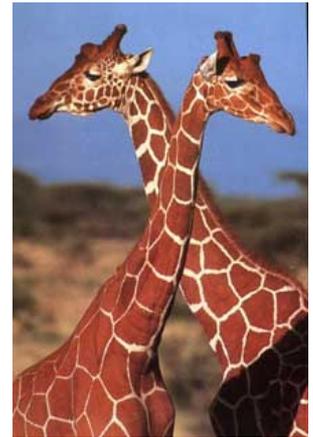
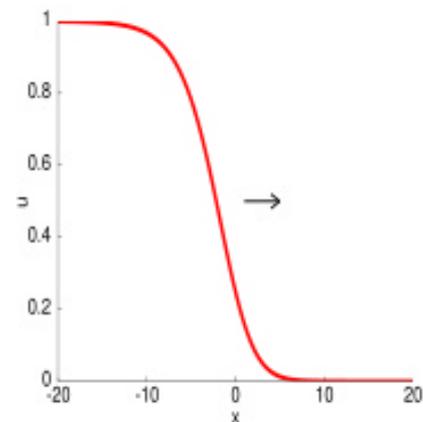
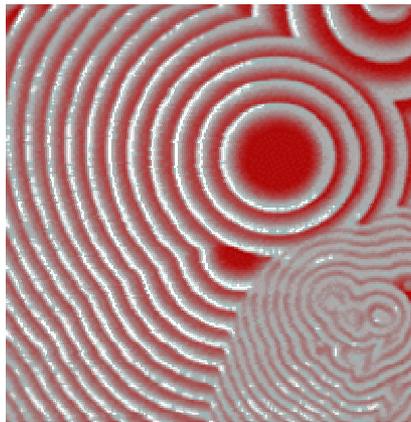
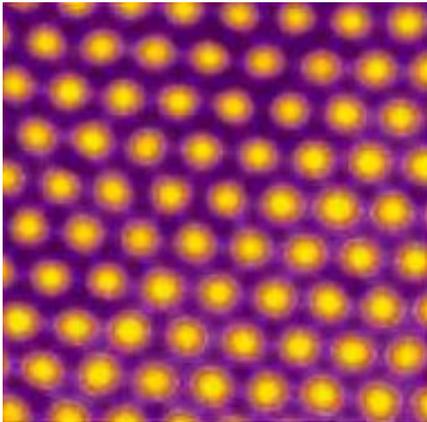
Sistemas de Reacción-Difusión

- Son sistemas cuyos constituyentes se transforman localmente unos en otros por medio de reacciones químicas y son transportados por difusión espacial.
- Una descripción matemática usual son los sistemas de ecuaciones parabólicas semi-lineales:

$$\partial_t \vec{u} = \vec{D} \Delta \vec{u} + \vec{R}(\vec{u})$$

donde el vector $\vec{u}(\vec{x}, t)$ describe las concentraciones de las diferentes sustancias y \vec{D} es la matriz de las constantes de difusión.

- Los sistemas de reacción-difusión se usan como modelos en numerosos campos: combustión, descargas eléctricas en gases, morfogénesis, coagulación sanguínea,...

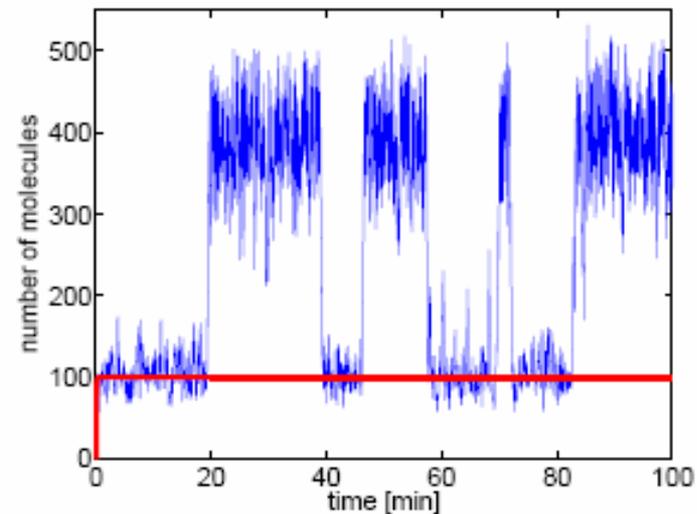
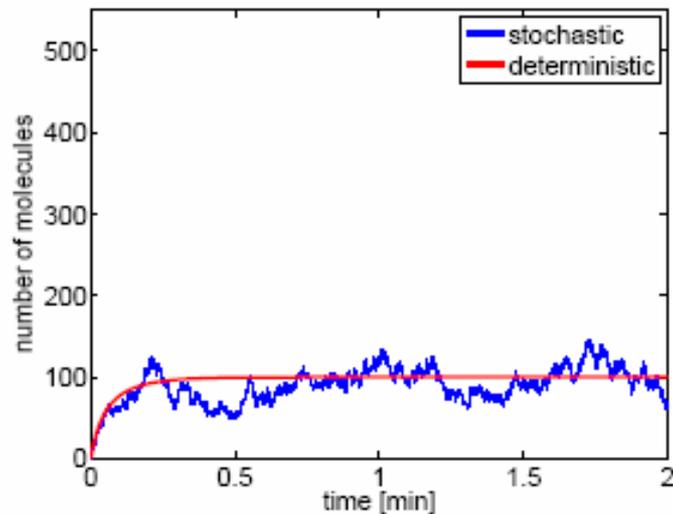
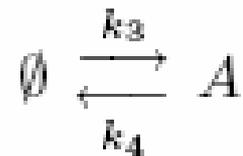
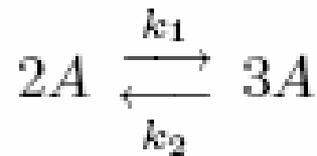


La ecuación maestra

- Los modelos deterministas sólo son fiables a tiempos cortos.
- La dinámica a tiempos a largos está dominada por eventos poco frecuentes.
- Asumiendo que el estado del sistema es markoviano, entonces podemos describirlo por medio de una ecuación maestra:

$$\frac{dP_k}{dt} = \sum_{\ell} (T_{k\ell}P_{\ell} - T_{\ell k}P_k)$$

El modelo de Schlögl



$$\frac{da}{dt} = -k_2 a^3 + k_1 a^2 - k_4 a + k_3$$

La aproximación WKB

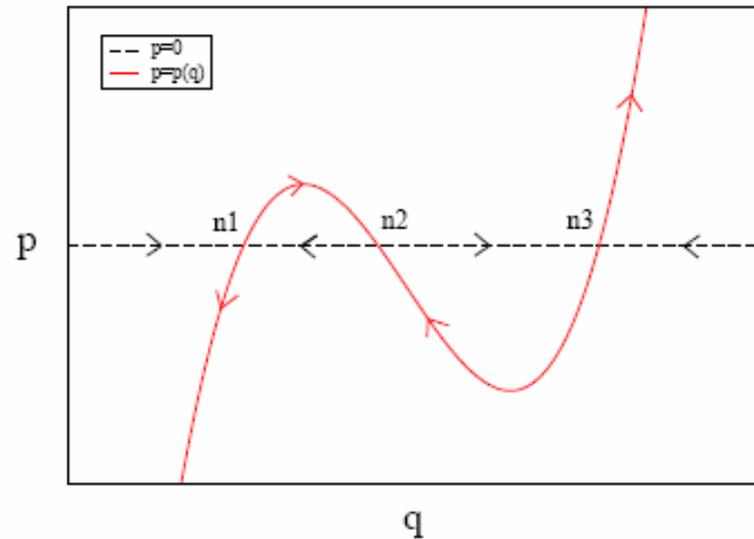
$$\begin{aligned}\partial_t P_n(t) &= \sum_r [W_r(n-r)P_{n-r}(t) - W_r(n)P_n(t)] \\ &= \sum_r (e^{-r\partial_n} - 1) W_r(n)P_n(t)\end{aligned}$$

$$W_r(n) = Nw_r(q) + u_r(q) + O(1/N) \quad q = n/N$$

$$\pi(q) = \exp\{-NS(q) - S_1(q)\}$$

$$S' = p$$

$$H(q, p) = \sum_r w_r(q) (e^{rP} - 1)$$



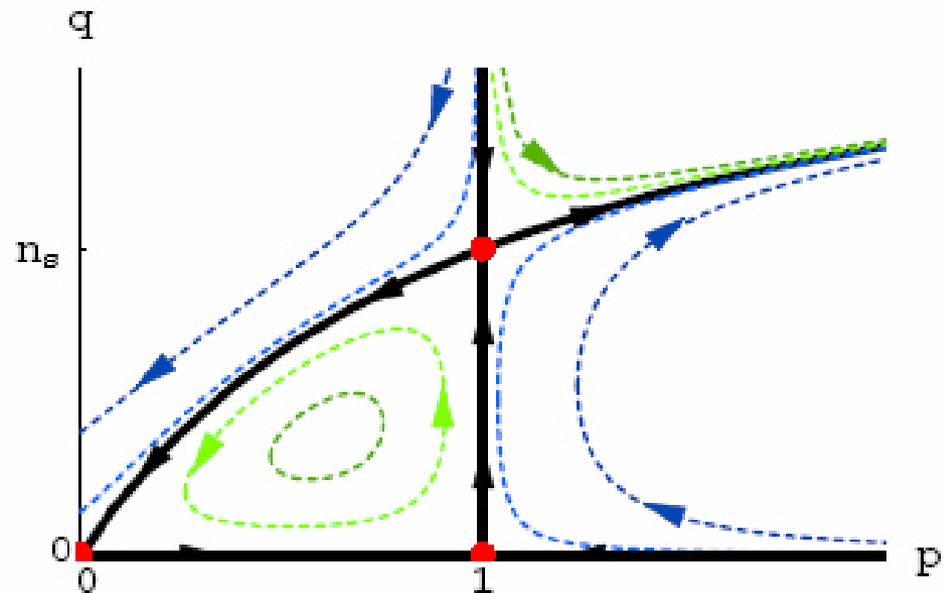
$$0 = \sum_r \left(w_r + \frac{1}{N} u_r \right) \\ \times \left(e^{rS'} \left[1 - \frac{r^2}{2N} S'' + \frac{r}{N} S'_1 - \frac{r}{N} \frac{w'_r}{w_r} \right] - 1 \right)$$

$$S(q) = \int^q dq p_\alpha(q)$$

$$S_1(q) = \int^q dq \frac{p'_\alpha H_{pp} + 2H_{qp} - 2 \sum_r u_r (e^{rp_\alpha} - 1)}{2H_p}$$

$$\tau = \frac{4\pi}{H_{pp}(q_s, 0)} \frac{e^{S_1(q_s) - S_1(q_0)}}{\sqrt{|S''(q_s)| S''(q_0)}} e^{N[S(q_s) - S(q_0)]}$$

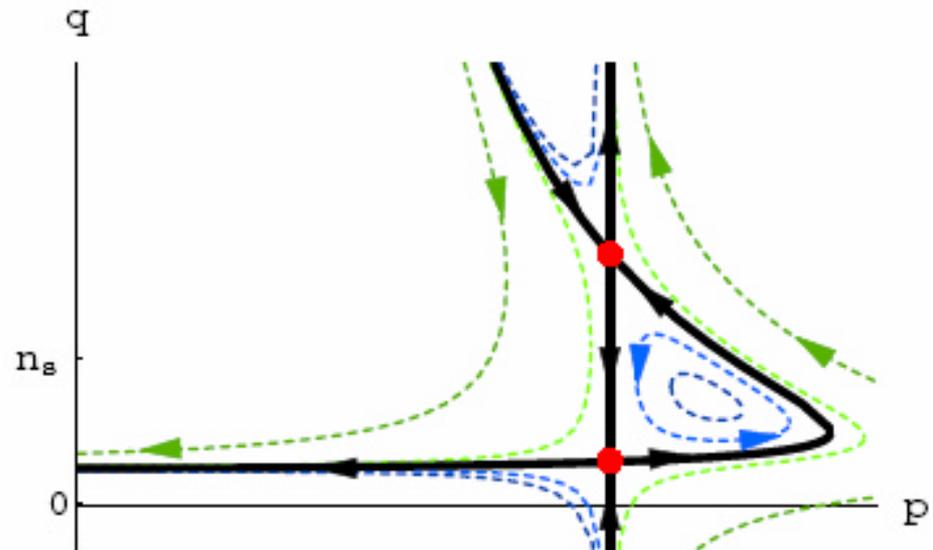
C. Escudero and A. Kamenev, preprint (2008).



$$\begin{aligned} \frac{d}{dt} P_n(t) &= \frac{\lambda}{2} [(n+2)(n+1)P_{n+2}(t) - n(n-1)P_n(t)] \\ &+ \sigma [(n-1)P_{n-1}(t) - nP_n(t)] \end{aligned}$$

$$\hat{H}(\hat{p}, \hat{q}) = \frac{\lambda}{2} (\hat{p}^2 - 1) \hat{q}^2 - \sigma (\hat{p} - 1) \hat{p} \hat{q}$$

V. Elgart and A. Kamenev, Phys. Rev. E **70**, 041106 (2004).



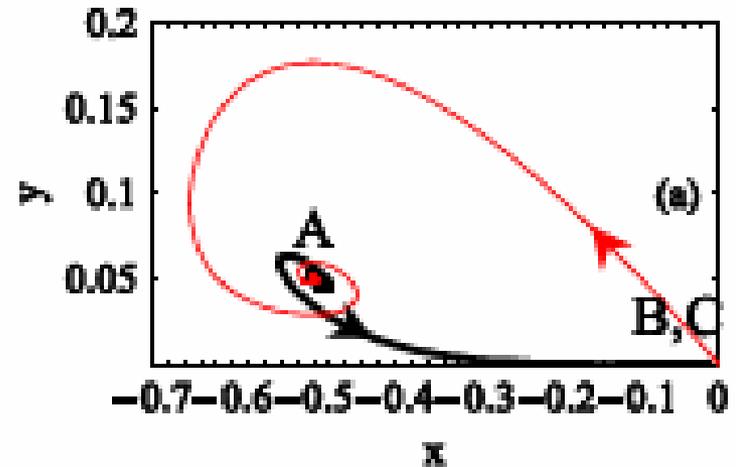
$$\hat{H}_0(\hat{p}, \hat{q}) = \frac{\lambda}{2}(\hat{p}^2 - \hat{p}^3)\hat{q}^2 + \sigma(\hat{p} - 1)\hat{q} + \mu(1 - \hat{p})$$

$$\begin{aligned} \frac{dP_n}{dt} = & \lambda \left[\frac{(n-1)(n-2)}{2} P_{n-1} - \frac{n(n-1)}{2} P_n \right] \\ & + \sigma [(n+1)P_{n+1} - nP_n] + \mu [P_{n-1} - P_n] \end{aligned}$$

V. Elgart and A. Kamenev, Phys. Rev. E **70**, 041106 (2004).

Modelos SIR

$$\begin{aligned}\dot{S} &= \mu N - \mu S - (\beta/N) S I \\ \dot{I} &= -\mu_I I - \gamma I + (\beta/N) S I \\ \dot{R} &= -\mu_R R + \gamma I\end{aligned}$$



$$\begin{aligned}\dot{P}_{n,m} &= \mu [N(P_{n-1,m} - P_{n,m}) + (n+1)P_{n+1,m} - nP_{n,m}] \\ &\quad + \Gamma [(m+1)P_{n,m+1} - mP_{n,m}] \\ &\quad + (\beta/N) [(n+1)(m-1)P_{n+1,m-1} - nmP_{n,m}]\end{aligned}$$

$$H = \mu(p_S - 1)(N - S) - \Gamma(p_I - 1)I - (\beta/N)(p_S - p_I)p_I SI$$

A. Kamenev and B. Meerson, Phys. Rev. E **77**, 061107 (2008).

Dependencia temporal explícita

$$\mathcal{H} = \sum_{m \neq n} \frac{\beta_{mn}(t)}{m!} (p^n - p^m) q^m, \quad m, n = 0, 1, 2, \dots$$

$$\dot{p} = \sum_{m \neq n} \frac{\beta_{mn}(t)}{(m-1)!} (p^m - p^n) q^{m-1}, \quad m, n = 0, 1, 2, \dots$$

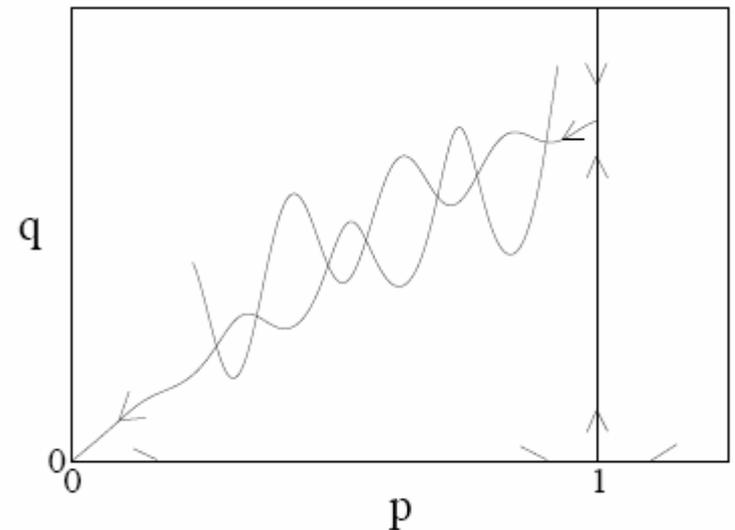
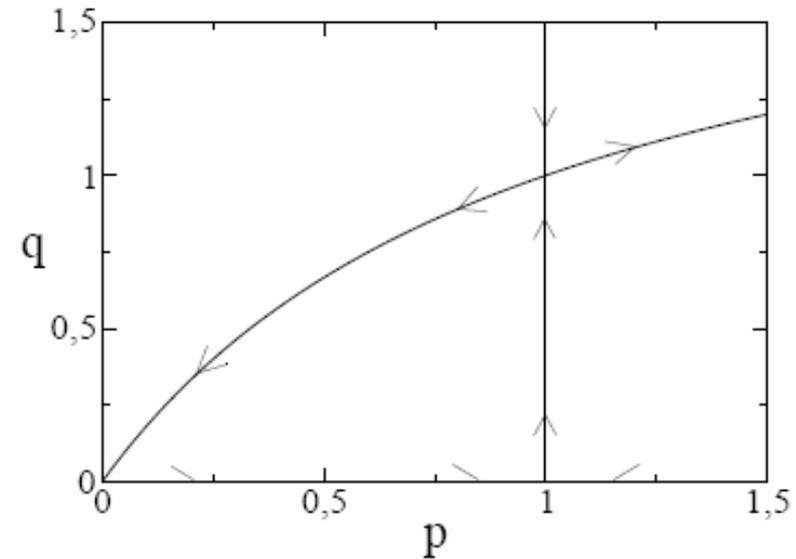
$$\dot{q} = \sum_{m \neq n} \frac{\beta_{mn}(t)}{m!} (np^{n-1} - mp^{m-1}) q^m, \quad m, n = 0, 1, 2, \dots$$

$$\beta_{mn}(t) = \beta_{mn}^0 + \epsilon \beta_{mn}^1(t)$$

$$M(t_0) = \int_{-\infty}^{\infty} \{ \mathcal{H}_1(\mathbf{x}_h(t-t_0), t), \mathcal{H}_0(\mathbf{x}_h(t-t_0)) \} dt$$

En ausencia de degeneración la conexión persiste.

C. Escudero and J. Á. Rodríguez,
Phys. Rev. E **77**, 011130 (2008).



Un caso degenerado

$$A \xrightarrow{\gamma} A + A \quad A \xrightarrow{\gamma} \emptyset$$

$$\partial_t G = \gamma(p - 1)^2 \partial_p G$$

$$H = \gamma(p - 1)^2 q$$

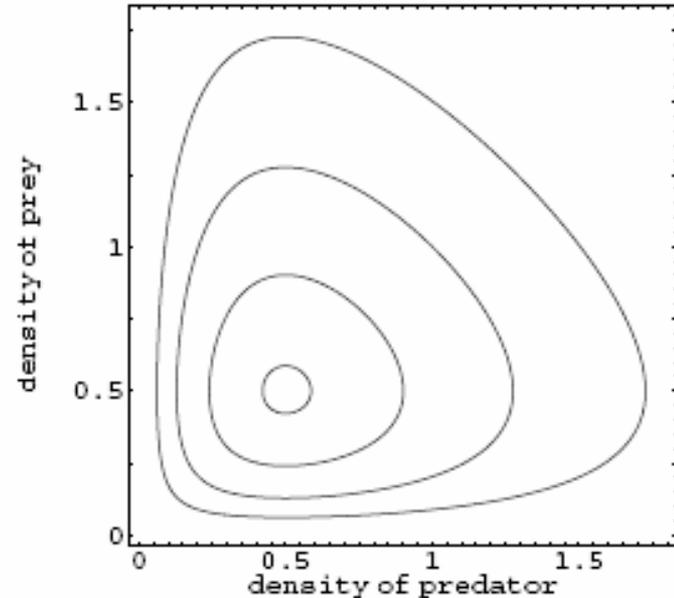


$$\lim_{t \rightarrow \infty} G(p, t) = \lim_{t \rightarrow \infty} G_0 \left[1 - \frac{1 - p}{1 + (1 - p)\gamma t} \right] = G_0(1) = 1$$

$$d\rho = \sqrt{2\gamma\rho} dW$$

W. R. Young, A. J. Roberts, and
G. Stuhne, *Nature* **412**, 328 (2001).

Lotka-Volterra



$$H = \mu a^* a - \sigma(1 + b^*)b^*b + \lambda(a^* + 1)(b^* - a^*)ab$$

$$\begin{aligned} \frac{dP_{m,n}}{dt} &= -(\mu m + \sigma n + \lambda mn)P_{m,n} \\ &+ \mu(m+1)P_{m+1,n} + \sigma(n-1)P_{m,n-1} \\ &+ \lambda(m-1)(n+1)P_{m-1,n+1} \end{aligned}$$

Un problema espacial

$$\frac{\partial \rho(r, t)}{\partial t} = D \nabla^2 \rho(r, t) + (\sigma_1 - \sigma_2) \rho(r, t) - \lambda \rho(r, t)^2$$

$$\rho\left(-\frac{L}{2}, t\right) = \rho\left(\frac{L}{2}, t\right) = 0$$

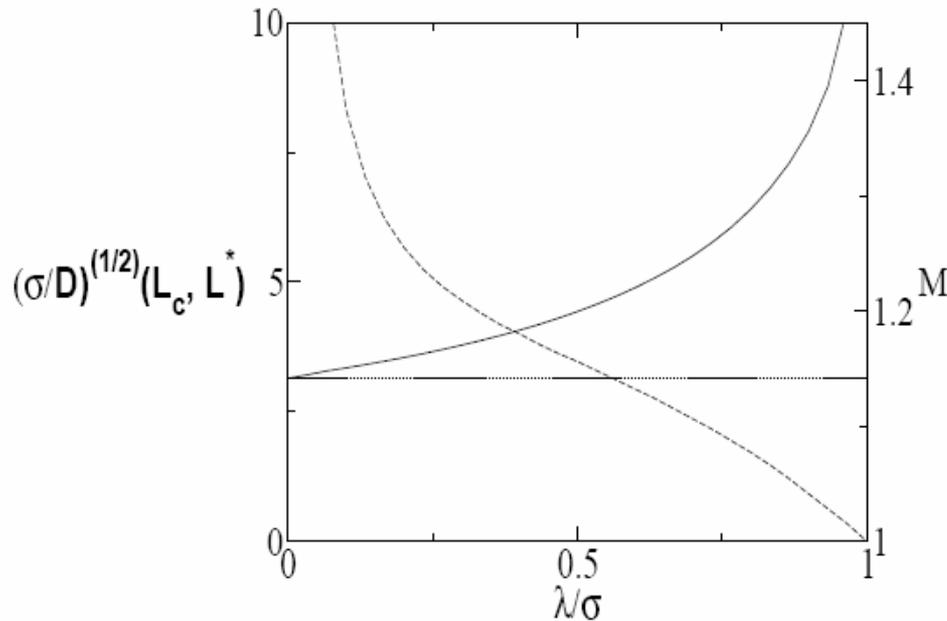
$$\rho(x, t) = \sum_n a_n \rho_n(x, t)$$

$$\rho_n(x, t) = \exp\left[\left(\sigma - \frac{n^2 \pi^2}{L^2} D\right) t\right] \sin\left[\frac{n\pi}{L} \left(x + \frac{L}{2}\right)\right]$$

$$L_c = \pi \sqrt{\frac{D}{\sigma}}$$

J. G. Skellam, *Biometrika* **38**, 196-218 (1951).

$$\frac{\partial \psi(x, t)}{\partial t} = D \nabla^2 \psi(x, t) + \sigma \psi(x, t) + \sqrt{2\sigma_1 \psi(x, t)} \xi(x, t)$$



C. Escudero, J. Buceta, F. J. de la Rubia, and K. Lindenberg,
 Phys. Rev. E **69**, 021908 (2004).

$$\langle \rho(x, t) \rangle^2 = \langle \rho^2(x, t) \rangle - \langle \rho(x, t) \rangle^2 \quad \langle \psi(x, t) \rangle^2 = \langle \psi^2(x, t) \rangle + \langle \psi(x, t) \rangle - \langle \psi(x, t) \rangle^2$$

$$\frac{\partial \langle \psi(x, t) \rangle}{\partial t} = D \nabla^2 \langle \psi(x, t) \rangle + \sigma \langle \psi(x, t) \rangle - \lambda \langle \psi(x, t) \rangle^2$$

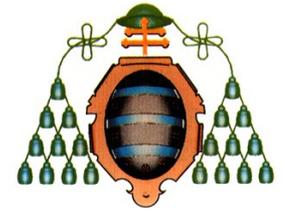
$$\frac{\partial \langle \psi(x, t) \rangle}{\partial t} = D \nabla^2 \langle \psi(x, t) \rangle + (\sigma + \lambda) \langle \psi(x, t) \rangle - 2\lambda \langle \psi(x, t) \rangle^2$$

Conclusiones

- La dinámica estocástica puede ser descrita en la aproximación eikonal por medio de un sistema dinámico hamiltoniano.
- Los puntos fijos describen estados cuasi-estacionarios y las conexiones heteroclínicas los caminos óptimos entre ellos.
- La ausencia de conexiones implica la existencia de un proceso difusivo que comunica los estados cuasi-estacionarios y la desaparición del tiempo de espera exponencialmente largo.

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