22, 23 y 24 de Octubre de 2008. El Escorial (Madrid)

Cuarta reunión de la red temática Dance

Ddays 2008

Dance (Dinámica, Atractores y No linealidad. Caos y Estabilidad)

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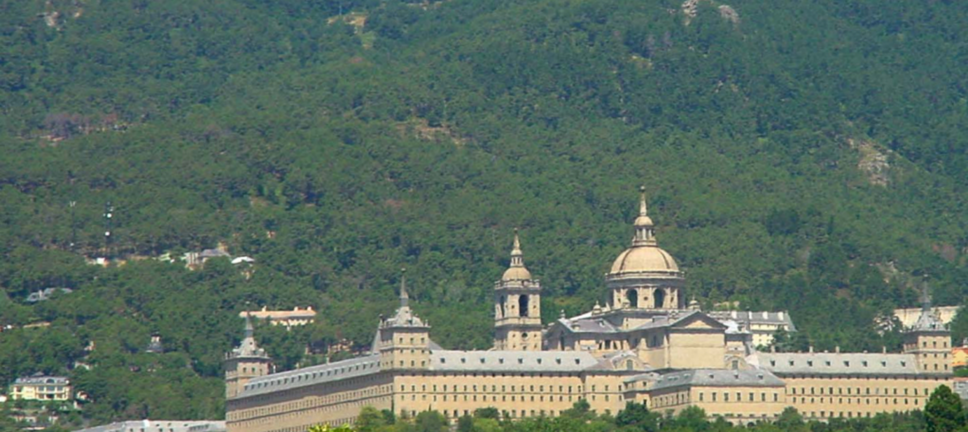

Manifolds on the verge of a hyperbolicity breakdown

Breakdown of invariant tori in qp systems

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El Escorial, October 2008

- **The long term behavior of dynamical systems is organized by the invariant objects.**
- **It is important to understand which invariant objects persist under modifications of the system.**
- An invariant manifold persists under perturbations if and only if it is normally hyperbolic. [HirschP69][Fenichel71][Mane78]
- There are spectral characterizations of hyperbolicity. [Mather68][HirschPS77][Swanson83]

In this talk:

- Non-autonomous systems in which the external forcing is quasi-periodic.
- Existence of (normally hyperbolic) invariant tori for quasi-periodic systems, persistence of those tori under perturbations and regularity.
- Phenomena that happen at the breakdown of exponential dichotomies (loss of reducibility).
- Quantitative laws. (Empirically conjectured scaling properties.)

- A quasi-periodic map with irrational frequency vector $\omega \in \mathbb{R}^d$ is a skew product in $\mathbb{R}^n \times \mathbb{T}^d$

$$\left\{ \begin{array}{l} \bar{x} = F(x,\theta) \\ \bar{\theta} = \theta + \omega \quad (\text{mod } 1) \end{array} \right. , \tag{1}$$

where $F : \mathbb{R}^n \times \mathbb{T}^d \to \mathbb{R}^n$.

- A solution $K : \mathbb{T}^d \to \mathbb{R}^n$ of

$$F(K(\theta),\theta) = K(\theta+\omega) \ , \tag{2}$$

parameterizes an invariant torus for (1)

$$\mathcal{K} = \{(K(\theta),\theta) \mid \theta \in \mathbb{T}^d\}$$

whose dynamics is a rotation.

The linearization around the torus $K : \mathbb{T}^d \to \mathbb{R}^n$, induces:

- a linear skew product (cocycle) in $\mathbb{R}^n$ over $\mathbb{T}^d$,

$$\begin{cases} \bar{v} = M(\theta)v \\ \bar{\theta} = \theta + \omega \end{cases} ; \tag{3}$$

- a transfer operator $\mathcal{M}_\omega$ acting on bounded sections $v : \mathbb{T}^d \to \mathbb{C}^n$ by

$$\mathcal{M}_\omega v(\theta) = M(\theta - \omega)v(\theta - \omega) \, . \tag{4}$$

The functional analysis properties (4) *are closely related to the dynamical properties of* (3).

Mather, Sacker, Sell, Palmer, Hirsch, Pugh, Shub, Mañé, Chicone, Swanson, Johnson, Latushkin, Stëpin, de la Llave, ...

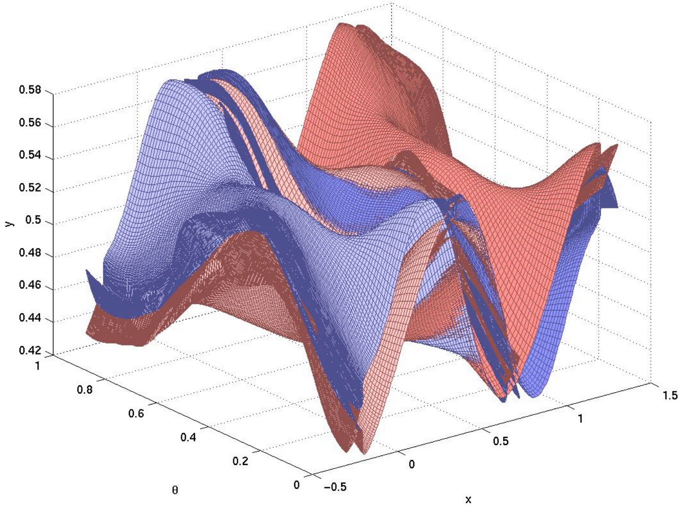
Theorem (Spectral Theorem)

There is a spectral gap in the annulus of radii $0 < \lambda_- < \lambda_+$ *if and only if there is an invariant and continuous splitting* $\mathbb{R}^n = E_\theta^- \oplus E_\theta^+$ *characterized by the rates of growth*

$$\begin{aligned} v \in E_\theta^- \quad &\Leftrightarrow \quad |M^{+m}(\theta)v| \leq C(\lambda_-)^{+m}|v|, \; m \geq 0; \\ v \in E_\theta^+ \quad &\Leftrightarrow \quad |M^{-m}(\theta)v| \leq C(\lambda_+)^{-m}|v|, \; m \geq 0. \end{aligned} \tag{5}$$

(exponential dichotomy)

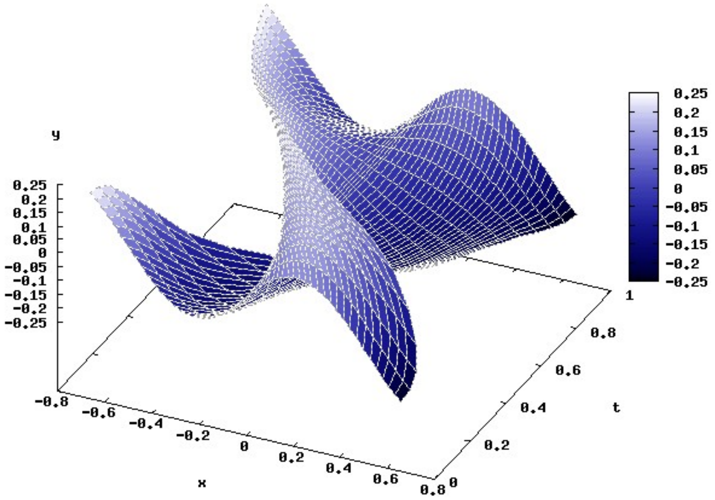
- The spectrum of $\mathcal{M}_\omega$ is a set of annuli, centered at 0.
- The torus is (fiberwise) hyperbolic if and only if the corresponding transfer operator is hyperbolic (i.e. 1 is not in the spectrum)
- Under non-resonance conditions, one can attach invariant manifolds to the invariant bundles.



0.58
0.56
0.54
0.52
0.5
0.48
0.46
0.44
0.42
y
1
0.8
0.6
0.4
0.2
0
θ
−0.5
0
0.5
1
1.5
x

Invariant tori

Whiskers



- The torus is reducible if its linearization $M(\theta)$ is reducible to constants, i.e.

$$M(\theta)P(\theta) = P(\theta + \omega)\Lambda \tag{6}$$

  for suitable $P(\theta)$ and constant matrix Λ.
- In such a case, the spectrum is a set of circles, one for each eigenvalue of Λ.
- The modulus of the eigenvalues are the Lyapunov multipliers.
- Reducibility is a desirable property. Unfortunately, it does not always hold.

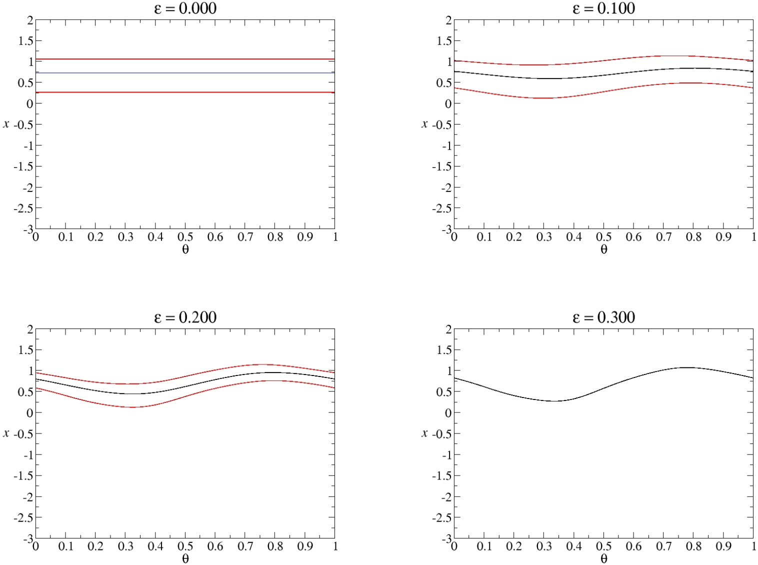
$$\begin{cases} \bar{x} = 1 + y - a\,x^2 + \varepsilon \cos(2\pi\theta) \\ \bar{y} = bx \\ \bar{\theta} = \theta + \omega \qquad (\text{mod } 1) \end{cases}$$

- a is the nonlinear parameter ($a = 0.68$);
- b is the dissipative parameter ($b = 0.1$);
- ε is the quasi-periodic parameter;
- $\omega = \frac{1}{2}(\sqrt{5} - 1)$ is the frequency of the forcing.

[Krauskopf,Osinga 98][Feudel,Osinga 00]

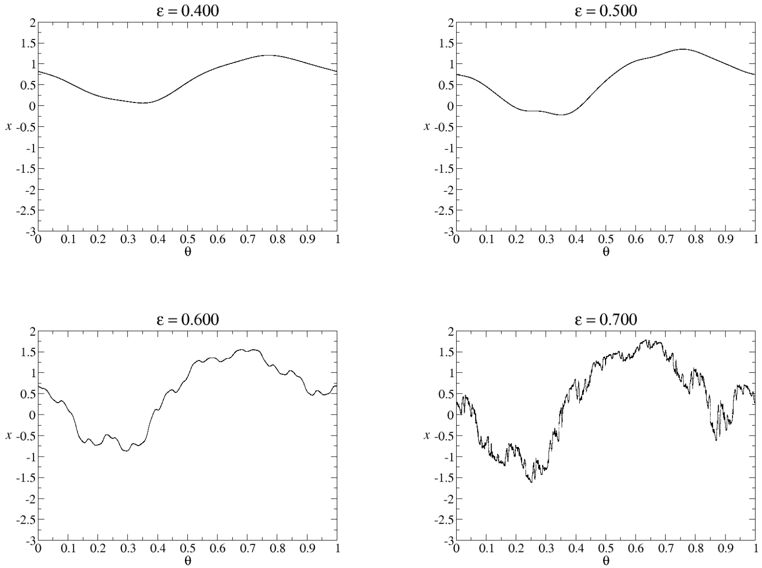
Continuation of an invariant torus

(I) Period “halving” (from saddle to attracting-node)



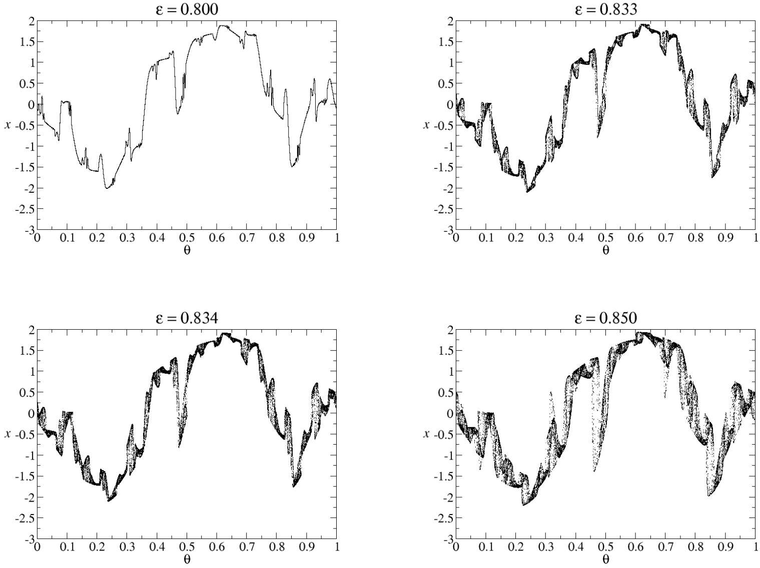
Continuation of an invariant torus

(II) Continuation of an attracting torus

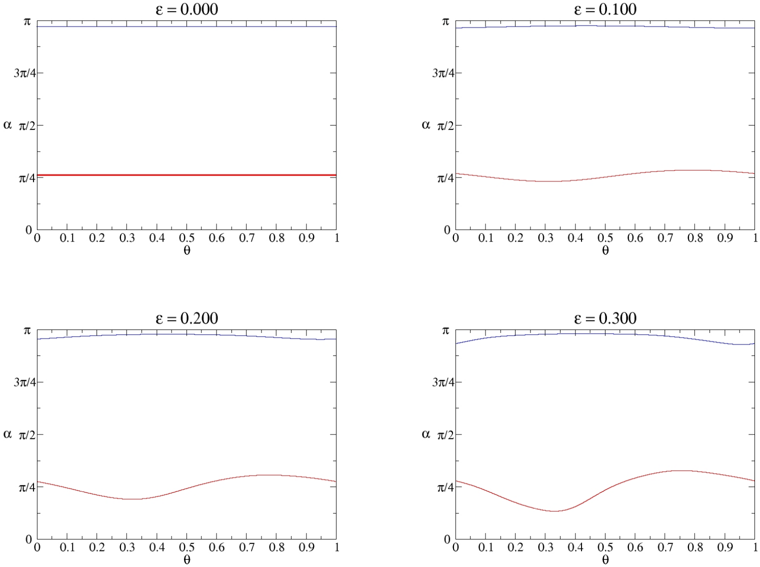


Continuation of an invariant torus

(III) Fractalization of the torus



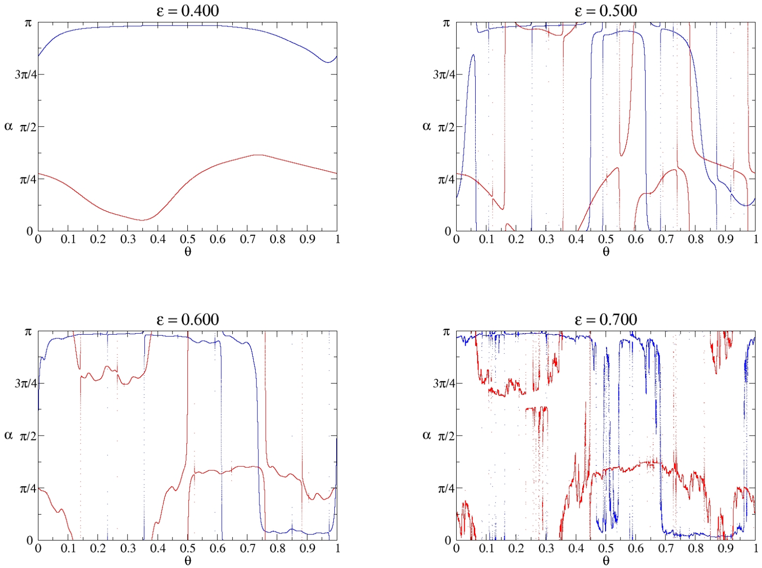
Invariant bundles (projectivization)
(I) Unstable bundle becomes a slow stable bundle



Invariant bundles (projectivization)

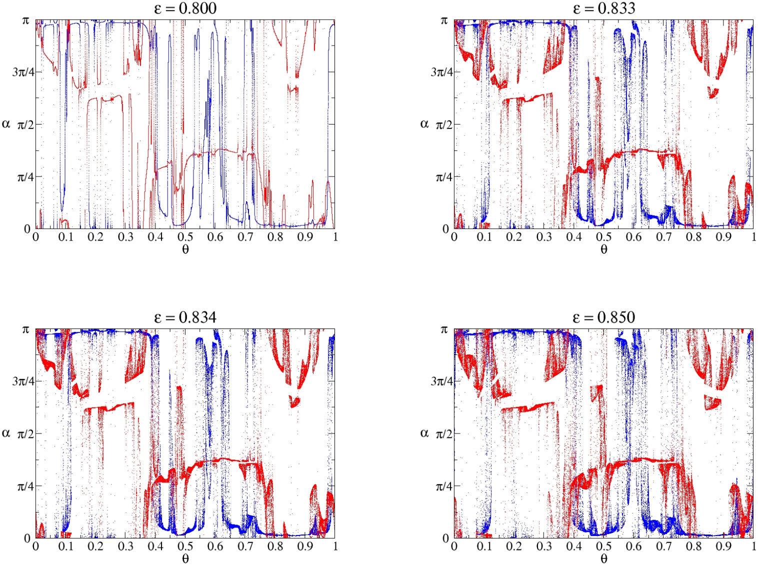
(II) Merging of bundles (collision of curves, SNA)

See also [Jalnine,Osbaldestin 05]



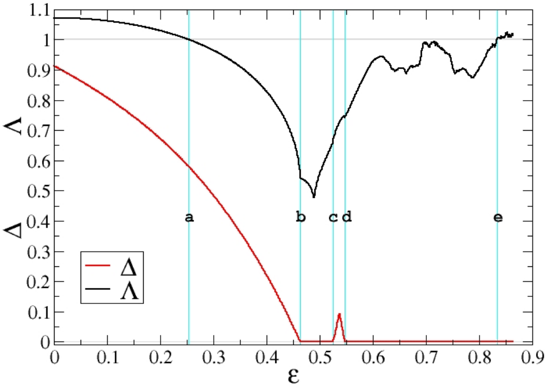
Invariant directions (projectivized bundles)

(III) Invariant directions for the fractalization of the torus



Description of the bifurcations

Observables: Λ (maximal Lyapunov multiplier)
Δ (distance beween bundles)



a) Period halving bifurcation.

b, c, d) Bundle merging bifurcation, SNAs in the projective dynamics.

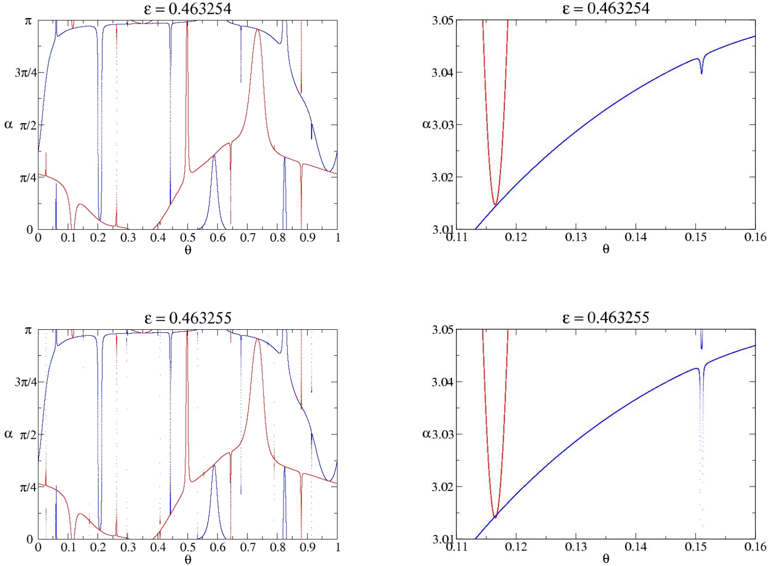
e) Fractalization of the torus, a phenomenon not well understood.

Rotating Hénon map: a= 0.68, b= 0.1

ε	eigenvalues	error	nfm
0.000	-1.0721039594 , 0.0932745366	9.6e-21	100
0.200	-1.0297559933 , 0.0971103841	8.3e-21	100
0.400	-0.8288693291 , 0.1206462786	9.6e-20	100
0.450	-0.6721643269 , 0.1487731437	9.9e-13	100
0.460	-0.6034304995 , 0.1657191675	2.9e-14	300
0.461	-0.5925812920 , 0.1687532181	2.7e-12	300
0.462	-0.5792054526 , 0.1726503084	2.3e-13	400
0.463	-0.5584521519 , 0.1790663706	9.1e-10	6800

The bundle merging bifurcation

Visual verification (zooms)



- For $\varepsilon = 0.460$, the torus is attracting and the cocycle is reducible to a constant diagonal matrix

$$\mathrm{diag}(-0.6034304995, 0.1657191675).$$

- For $\varepsilon = 0.530$, the torus is attracting and the cocycle is reducible to a constant diagonal matrix

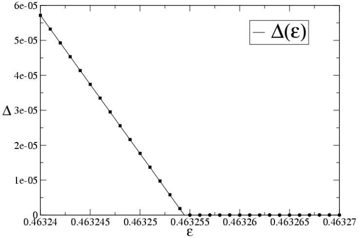
$$\mathrm{diag}(0.6945467500, -0.1439787890).$$

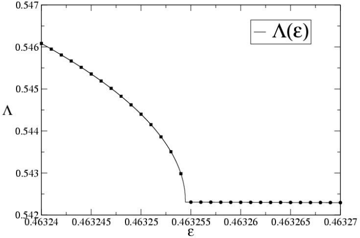
- Since the Lyapunov multipliers are different during the continuation,

the cocycle can not be reducible during the whole continuation!

The bundle merging bifurcation

Quantitative estimates (universal laws)





$$\begin{cases} \Delta_\varepsilon \sim \alpha(\varepsilon_{\rm b} - \varepsilon)^\beta & \text{if } \varepsilon \leq \varepsilon_{\rm b} \\ \Delta_\varepsilon \approx 0 & \text{if } \varepsilon \geq \varepsilon_{\rm b} \end{cases}$$

$$\begin{cases} \Lambda_\varepsilon \sim \Lambda_{\rm b} + A(\varepsilon_{\rm b} - \varepsilon)^B & \text{if } \varepsilon \leq \varepsilon_{\rm b} \\ \Lambda_\varepsilon \approx \Lambda_{\rm b} + \bar{A}(\varepsilon - \varepsilon_{\rm b})^{\bar{B}} & \text{if } \varepsilon \geq \varepsilon_{\rm b} \end{cases}$$

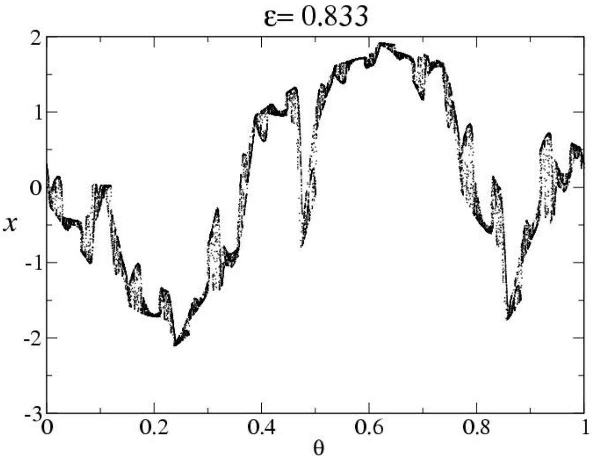
$\varepsilon_b = 0.46325447112$
$\alpha = 3.94933$
$\beta = 0.999979 \approx 1$

Bjerklov and Saprykina, 08!

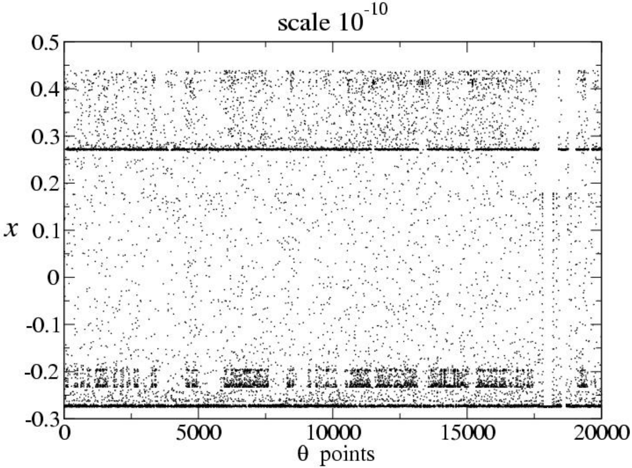
$\Lambda_b = 0.5423122$
$A = 1.015$
$B = 0.5020 \approx 0.5$
$\bar{A} = -0.7409$
$\bar{B} = 1.00035 \approx 1$

The fractalization route

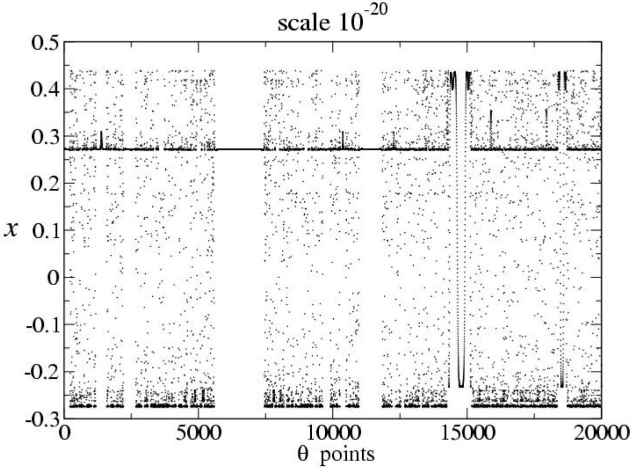
Is this a SNA?



The fractalization route
Zooming

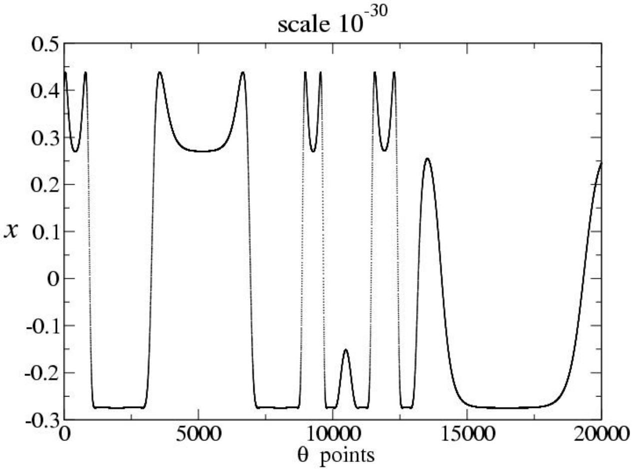


The fractalization route
Zooming again



The fractalization route

It is a regular curve!



[Haro,Simó], [Broer,Simó,Vitolo 05][Jorba,Tatjer 05] for qp logistic map, and Simó at DDAYS'03!!

Summary

- The formation of a strange non chaotic attractor for the linearized dynamics of an attracting torus produces a sudden growth of the spectrum (breakdown of exponential dichotomies and loss of reducibility).
- There are quantitative regularities for Λ and Δ, and some of them have been proved in specific models.
- This seems to be the prelude of the destruction of the torus when the upper Lyapunov multiplier (the external radius of the spectrum) crosses 1.
- Does it produces a formation of a strange chaotic attractor? (playing with parameters)

The rotating standard map

$$\begin{cases} \bar{x} = x + \bar{y} \pmod 1 \\ \bar{y} = y - \dfrac{\sin(2\pi x)}{2\pi}(K + \varepsilon\cos(2\pi\theta)) \\ \bar{\theta} = \theta + \omega \pmod 1 \end{cases}$$

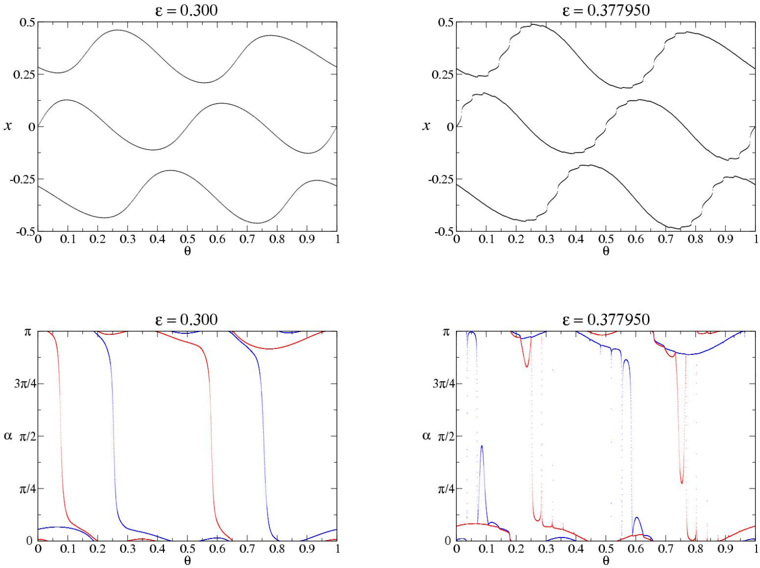
- K is the parameter of the standard map ($K = 0.2$);
- ε is the quasi-periodic parameter;
- ω is an algebraic number of order 3:

$$\omega = \sqrt[3]{\frac{19}{27} + \sqrt{\frac{11}{27}}} + \sqrt[3]{\frac{19}{27} - \sqrt{\frac{11}{27}}} - \frac{2}{3}.$$

[Artuso et al 91, Tompaidis 96, Haro 98]

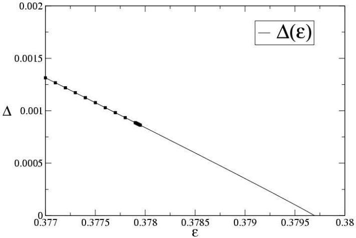
Bundle merging causing breakdown

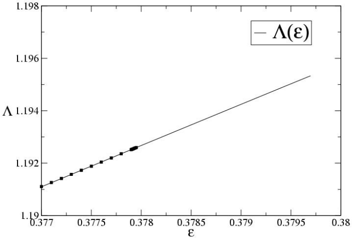
A 3-periodic torus close to breakdown, and projectivized bundles



Bundle merging causing breakdown

Quantitative estimates (universal laws)





$$\Delta_\varepsilon \sim \alpha(\varepsilon_c - \varepsilon)^\beta \text{ if } \varepsilon \leq \varepsilon_c$$

$$\Lambda_\varepsilon \sim \Lambda_c + A(\varepsilon_c - \varepsilon)^B \text{ if } \varepsilon \leq \varepsilon_c$$

$\varepsilon_c = 0.3796965$
$\alpha = 0.4063$
$\beta = 0.9693 \approx 1$

$\Lambda_c = 1.19533$
$A = -1.6$
$B = 1.00 \approx 1$

Conclusions

- The formation of a SNA in the linearized dynamics of a saddle type torus produces the sudden growth of the spectrum.
- Since at the collapse, 1 is inside the spectrum, the torus is not normally hyperbolic and it breaks down.
- There are some conjectured regularities in the behavior of the observables Δ and Λ, but no proofs!

The bundle merging bifurcation
Description and consequences

The invariant bundles approach each other when $\varepsilon < \varepsilon_c$, and collapse for $\varepsilon = \varepsilon_c$, while the Lyapunov multipliers $\Lambda_\varepsilon^- < \Lambda_\varepsilon^+$ remain different.

	$\varepsilon < \varepsilon_c$	$\varepsilon = \varepsilon_c$
Linear dynamics: Invariant bundles	Continuous	Measurable [Oseledets 68]
Projective dynamics: Invariant curves	Continuous (attracting / repelling)	Measurable (SNA / SNR)
Spectrum	Two circles of radii $\Lambda_\varepsilon^\pm$	Annulus of radii $\Lambda_\varepsilon^\pm$
Reducibility (ω Diophantine)	Yes	No
If $\Lambda_\varepsilon^- < \Lambda_\varepsilon^+ < 1$	Attracting-node torus	The torus survives
If $\Lambda_\varepsilon^- < 1 < \Lambda_\varepsilon^+$	Saddle torus	The torus is destroyed

Some papers ...

by Àlex Haro and Rafael de la Llave

- A parameterization method for the computation of invariant tori and their whiskers in quasi periodic maps: explorations and mechanisms for the breakdown of hyperbolicity. (SIADS, 2007)

- Manifolds on the verge of a hyperbolicity breakdown. (Chaos, 2006)

... and some movies
by Pedro Almodóvar