Critical points and periodic orbits of planar differential equations

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We will deal with

$$\begin{cases} \dot{x} = P(x,y), \\ \dot{y} = Q(x,y), \end{cases} \quad (1)$$

where $P(x,y)$ and $Q(x,y)$ are analytic functions.

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$\gamma(t)$ is a **periodic orbit** of (1) if it is a non-constant solution and there exists $T \in \mathbb{R}^+$ such that $\gamma(0) = \gamma(T)$.

An isolated periodic orbit is called **limit cycle**.

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   - Stability and bifurcations from monodromic nilpotent critical points via
     - generalized Lyapunov constants
     - normal forms
   - A k–blow up algorithm to desingularize critical points

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   - Stability and bifurcations from monodromic nilpotent critical points via
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2. Periodic orbits
   - Trigonometric Abel equation
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Critical points: Introduction

Let $DX(0,0)$ the jacobian matrix at the critical point $(0,0)$ of (1)

$$DX(0,0) = \begin{pmatrix} \frac{\partial P}{\partial x}(0,0) & \frac{\partial P}{\partial y}(0,0) \\ \\ \frac{\partial Q}{\partial x}(0,0) & \frac{\partial Q}{\partial y}(0,0) \end{pmatrix}$$

and λ_1, λ_2 its eigenvalues.

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and λ_1, λ_2 its eigenvalues.

- If $\lambda_1\lambda_2 \neq 0$, $(0,0)$ is an **elementary** critical point.
- If one, and only one, of λ_i is zero, $(0,0)$ is a **semi-elementary** critical point.
- If $\lambda_1 = \lambda_2 = 0$, $(0,0)$ is a degenerate critical point and
  - if $DX(0,0)$ is not identically null, $(0,0)$ is called **nilpotent**,
  - If $DX(0,0)$ is identically null, $(0,0)$ is called **linearly zero**.

Topological classification of critical points

Except for the center-focus problem:

- Hartman-Grobman Theorem (1964) topologically classifies elementary critical points.
- In [ALGM, 1973], topologically classified semi-elementary critical points.
- Andreev Theorem (1953) topologically classifies nilpotent critical points.
- Each linearly zero case can be studied separately.

Center-Focus Problem

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The origin of system (1) is **monodromic** if there exists a neighborhood of it where all the orbits turn around it.

The **monodromy problem** consists in determining when a critical point is monodromic.

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Il'yashenko and Ecalle (1991) proved that, if $P(x,y)$ and $Q(x,y)$ are analytic and the origin is monodromic then it is either a focus or a center.

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Il'yashenko and Ecalle (1991) proved that, if $P(x,y)$ and $Q(x,y)$ are analytic and the origin is monodromic then it is either a focus or a center.

The **center-focus problem** consists in distinguishing if a monodromic singular point is a focus or a center.

Center-focus problem

- Solved by Poincaré (1881) for elementary critical points. The Lyapunov constants can be used to bifurcate limit cycles from this kind of critical points.
- Solved by Moussu (1982) for nilpotent critical points.

Center-focus problem

- Solved by Poincaré (1881) for elementary critical points. The Lyapunov constants can be used to bifurcate limit cycles from this kind of critical points.
- Solved by Moussu (1982) for nilpotent critical points.

The goal of the following results is to compute the generalized Lyapunov constants of a general monodromic nilpotent critical point and use them to bifurcate limit cycles from it.

Normal form for nilpotent critical points

Lemma [Andreev, 1953]. An analytic vector field with the origin being an isolated nilpotent monodromic singularity can be written

$$\begin{cases} \dot{x} = y(-1 + X_1(x,y)), \\ \dot{y} = f(x) + \phi(x)y + Y_0(x,y)y^2, \end{cases} \tag{2}$$

where

$$X_1(x,y) = \sum_{i+j\geq 1} d_{ij}x^i y^j, \qquad f(x) = x^{2n-1} + \sum_{i\geq 0} a_i x^{2n+i},$$

$$\phi(x) = bx^\beta + \sum_{i\geq 1} b_i x^{\beta+i}, \qquad Y_0(x,y) = \sum_{i+j\geq 0} e_{ij}x^i y^j,$$

satisfying $\phi(x) \equiv 0$ or $\beta \geq n-1$. Furthermore if $\beta = n-1$ then $b^2 - 4n < 0$.

Stability for monodromic nilpotent critical points

$$\begin{cases} \dot{x} = y(-1 + X_1(x,y)), \\ \dot{y} = x^{2n-1} + O(x^{2n}) + (bx^\beta + O(x^{\beta+1}))y + Y_0(x,y)y^2. \end{cases} \tag{2}$$

Theorem A. The origin of system (2) is a stable (resp. unstable) monodromic critical point when $\Delta < 0$ (resp. $\Delta > 0$), where:

(a) $\Delta = b$, if $\beta \in \{n-1, n, n+1\}$ and β is $even$;

(b) $\Delta = (2n+1)b_1 + \big(-3e_{00} + (n-1)d_{10} - (n+2)a_0 \big) b$,
if $\beta = n$ and β is odd;

(c) $\Delta = (2n+1)b_1 + \big(-5e_{00} + (n-2)d_{10} - (n+3)a_0 \big) b +$
$+5\big(d_{11} + 3e_{01} + d_{01}d_{10} + 2d_{01}e_{00}\big)\mathcal{X}_{\{n=2\}}$,
if $\beta = n+1$ and β is odd.

Some remarks

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Remarks:

- The theorem only covers the cases $\beta \in \{n-1, n, n+1\}$. The method used to prove the theorem could also be utilized to cover the other cases satisfying $\beta > n+1$.

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Remarks:

- One case has resisted by using this approach: $\beta = n-1$, $b^2 - 4n < 0$ and β is odd.

Analytic normal form for nilpotent critical points

Theorem [Stróżyna & Żoładek, 2002]. Consider an analytic planar system having the origin as a nilpotent critical point. Then there exists an analytic change of variables such that, in a neighborhood of the origin, it writes as

$$\begin{cases} \dot{x} = -y, \\ \dot{y} = x^{2n-1} + y b(x), \end{cases} \tag{3}$$

being $b(x) \equiv 0$ or $b(x) = \sum_{j \geq \beta} b_j x^j$, with $b_\beta \neq 0$.

Stability for monodromic nilpotent critical points

Theorem B. Consider system (3), $\begin{cases} \dot{x} = -y, \\ \dot{y} = x^{2n-1} + yb(x), \end{cases}$

having a monodromic nilpotent critical point at the origin. Then

(i) If $b(x) = b^o(x) + x^{2\ell}\left(b_{2\ell} + O(x)\right)$, with $b_{2\ell} \neq 0$, being $b^o(x) := (b(x) - b(-x))/2$, then its first significant generalized Lyapunov constant is

  (I) $V_{2-n+2\ell} = K b_{2\ell}$ when either $\beta > n-1$, or $\beta = n-1$ and β is odd. Here $K = K(n, \ell, b_{n-1})$ is a positive constant given in the proof.

  (II) $V_1 = \exp\left(\frac{2b_\beta \pi}{n\sqrt{4n - b_\beta^2}}\right)$ when $\beta = 2\ell = n-1$.

(ii) The origin is a center if and only if $b^e(x) := b(x) - b^o(x) \equiv 0$.

Bifurcations from monodromic nilpotent critical point

Theorem. Consider next system of Kukles type

$$\begin{cases} \dot{x} &= -y, & (*) \\ \dot{y} &= a_{11}xy + a_{02}y^2 + a_{30}x^3 + a_{21}x^2y + a_{12}xy^2 + a_{03}y^3, \end{cases}$$

with $a_{30} > 0$ and $a_{11}^2 - 8a_{30} < 0$ (the monodromy conditions).

The only families inside $(*)$ with a center at the origin are $a_{21} = a_{03} = a_{11}a_{02} = 0$.

Moreover, there exist systems of the form $(*)$ with at least 3 limit cycles around the origin.

Abel equation

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Systems with homogeneous nonlinearities

$$\begin{cases} \dot{x} &= -y + P_n(x,y), \\ \dot{y} &= x + Q_n(x,y), \end{cases}$$

by passing to polar coordinates and applying Cherkas' transformation, $x = \frac{r^{n-1}}{1+r^{n-1}g(\theta)}$, they are transformed into equation (4).

Trigonometric Abel equation

Consider the Abel equation

$$\dot{x} = A(t)x^3 + B(t)x^2, \qquad (4)$$

where $A(t)$ and $B(t)$ are trigonometric polynomials of degrees n and m, respectively.

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Version of Hilbert's 16th Problem for equation (4): Does there exist an upper bound, depending only on n and m, $H(n,m)$, for the number of periodic orbits of (4)?

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Version of Hilbert's 16th Problem for equation (4): Does there exist an upper bound, depending only on n and m, $H(n, m)$, for the number of periodic orbits of (4)?

The goal of the following result is to give a lower bound for $H(n, m)$ in three simple cases.

Trigonometric Abel equation

$$\dot{x} = A(t)x^3 + B(t)x^2. \tag{4}$$

Theorem C. Set $H(n, m)$ for the number of isolated periodic orbits of equation (4). Then

(a) $H(n, 0) = H(0, m) = 2$,

(b) $H(n, 1) \geq n + 2$,

(c) $H(1, m) \geq 2m + 1$.

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Remarks:

- Statement (a) can be easily proved by using the results of [Lins-Neto, 1980] and [Gasull & Llibre, 1990].

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(a) $H(n,0) = H(0,m) = 2$,

(b) $H(n,1) \geq n+2$,

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Remarks:

- Statement (b) was also proved by Lins-Neto.
  Both proofs are based on first order Melnikov functions.
  Lins gives a lower bound for the number of zeroes while we compute the Abelian integral explicitly and give a sharp upper bound.

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Remarks:

- Statement (c) is also based on a first order Melnikov function. This case is much more difficult than the previous one because the Abelian integral involves elliptic functions. We get a lower and an upper bound for its number of zeroes. But we have not been able to prove that our upper bound is sharp.

Abel equation on a strip

Abel equation on a strip

$$\dot{x} = A(t)x^3 + B(t)x^2, \tag{5}$$

defined on the strip $\mathcal{S} = \{(t, x) \, : \, t \in [0, 1], \; x \in \mathbb{R}\}$.

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If $A(t)$ and $B(t)$ are 1-periodic, the Abel equation is a differential equation defined on a cylinder and equation (5) is equation (4).

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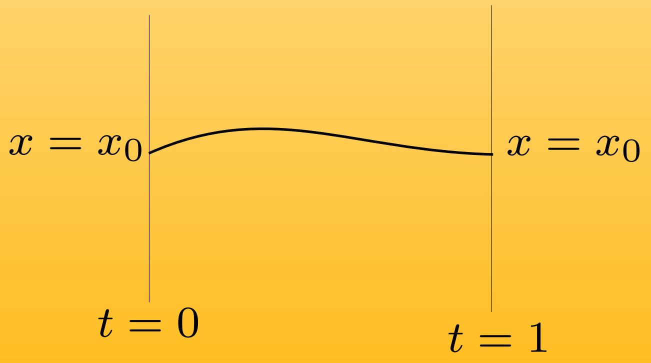
A **periodic orbit** is a solution starting on $t = 0$ at $x = x_0$ and arriving to $t = 1$ with $x = x_0$.



$x = 0$ is always a periodic orbit.

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A periodic orbit is **hyperbolic** if the Poincaré map between $t = 0$ and $t = 1$ has derivative different from one at the initial condition of the periodic orbit.
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$x = 0$ is always a non-hyperbolic periodic orbit.

The goal of the following result is to find a new criterion to bound the number of periodic orbits of equation (5).

Existing criteria

$$\dot{x} = A(t)x^3 + B(t)x^2. \tag{5}$$

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- In [Gasull & Llibre, 1990]: If $B(t)$ does not change sign, the Abel equation (5) has, at most, 1 non-zero periodic orbit.

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- In [Gasull & Llibre, 1990]: If $B(t)$ does not change sign, the Abel equation (5) has, at most, 1 non-zero periodic orbit.

In both cases, if the periodic orbits exists, it is hyperbolic.

Abel equation on a strip

Theorem D. Consider the Abel equation

$$\dot{x} = A(t)x^3 + B(t)x^2. \tag{5}$$

Assume that there exist $a, b \in \mathbb{R}$ such that $aA(t) + bB(t)$ does not vanish identically and does not change sign in $[0, 1]$. Then equation (5) has at most one non-zero periodic orbit.

Furthermore, when this periodic orbit exists, it is hyperbolic.

Abel equation on a strip

Theorem D. Consider the Abel equation

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Assume that there exist $a, b \in \mathbb{R}$ such that $aA(t) + bB(t)$ does not vanish identically and does not change sign in $[0, 1]$. Then equation (5) has at most one non-zero periodic orbit.

Furthermore, when this periodic orbit exists, it is hyperbolic.

Remark: This criterion generalizes the two first ones, that are obtained through ours setting $ab = 0$.

Systems on the cylinder

Consider the system on the cylinder

$$\begin{cases} \frac{dr}{dt} = \alpha(\theta)\, r + \beta(\theta)\, r^{k+1} + \gamma(\theta)\, r^{2k+1}, \\ \frac{d\theta}{dt} = b(\theta) + c(\theta)\, r^{k}, \end{cases} \qquad (6)$$

where t is real, $k \in I\!N^+$ and all the above functions are real, smooth and $2\pi-$periodic.

System (6) has two types of periodic orbits: **contractible**, the ones that can be deformed continuously to a point, and **non-contractible**, the ones that can not.

Systems on the cylinder

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where t is real, $k \in I\!N^+$ and all the above functions are real, smooth and $2\pi-$periodic.

System (6) has two types of periodic orbits: **contractible**, the ones that can be deformed continuously to a point, and **non-contractible**, the ones that can not.

The goal of the following result is to find new criteria to bound the number of non-contractible periodic orbits of system (6).

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And if $b(\theta) \neq 0$ for all θ then, applying Cherkas' transformation, $x = \frac{r}{b(\theta)+c(\theta)r}$, system (6) writes as the complete Abel equation

$$\frac{dx}{d\theta} = A(\theta)x^3 + B(\theta)x^2 + C(\theta)x.$$

What happens if $b(\theta^*) = 0$ for some $\theta^* \in [0, 2\pi]$?

Systems on the cylinder

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We define the functions

$$\mathbf{A}(\theta) = k(c(\theta)^2\alpha(\theta) + b(\theta)^2\gamma(\theta) - b(\theta)\beta(\theta)c(\theta)),$$

$$\mathbf{B}(\theta) = -2kc(\theta)\alpha(\theta) + kb(\theta)\beta(\theta) + c(\theta)b'(\theta) - b(\theta)c'(\theta).$$

Systems on the cylinder

Theorem E. Consider system (6) on the cylinder and suppose that the function $b(\theta)$ vanishes. Define the functions $\mathbf{A}(\theta)$ and $\mathbf{B}(\theta)$ as before. Then

(a) If one of the functions $\mathbf{A}(\theta)$ or $\mathbf{B}(\theta)$ does not change sign then system (6) has at most 2 non-contractible limit cycles if k is odd, or 4 non-contractible limit cycles if k is even. Furthermore both bounds are sharp.

(b) If one of the functions $b(\theta)\mathbf{A}(\theta)$ or $b(\theta)\mathbf{B}(\theta)$ does not change sign then system (6) has at most 3 non-contractible limit cycles if k is odd, or 6 non-contractible limit cycles if k is even.

Cubic systems with symmetry of order 4

Consider the equation

$$\dot{z} = \varepsilon z + p\, z^2 \bar{z} - \bar{z}^3, \tag{7}$$

where z is complex, the time is real and $\varepsilon = \varepsilon_1 + i\varepsilon_2$, $p = p_1 + ip_2$ are complex parameters.

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In [Arnold, 1980] there is a general study of the whole family, for all q.

In [CLW, 1994] there is a deep study on the limit cycles, for all q.

Cubic systems with symmetry of order 4

$$\dot{z} = \varepsilon z + p\, z^2 \bar{z} - \bar{z}^3, \qquad (7)$$

The study of equation (7) can be split in three cases:

(I) Equation (7) has a unique critical point, the origin.

(II) Equation (7) has five critical points, the origin and four saddle-nodes.

(III) Equation (7) has nine critical points, the origin, four saddle points and four critical points of index $+1$.

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Cubic systems with symmetry of order 4

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When the infinity has no critical points, how many limit cycles can exist surrounding the origin, and eventually other 4 or 8 critical points? Can they coexist with the 4 limit cycles that do not surround the origin?

Cubic systems with symmetry of order 4

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Theorem F. (a) Consider equation (7) with $\varepsilon_2 \neq 0$, $p_2 > 1$ and :

$$\Sigma_A^{\pm} = \frac{\varepsilon_2 p_1 p_2 \pm \sqrt{\varepsilon_2^2 (p_1^2 + p_2^2 - 1)}}{p_2^2 - 1},$$

$$\Sigma_B^{\pm} = \frac{\varepsilon_2 p_1 p_2 \pm \sqrt{\varepsilon_2^2 (p_1^2 + 9p_2^2 - 9)}}{2(p_2^2 - 1)}.$$

If either (i) $\varepsilon_1 \notin (\Sigma_A^-, \Sigma_A^+)$, or (ii) $\varepsilon_1 \notin (\Sigma_B^-, \Sigma_B^+)$, then equation (7) has at most one limit cycle surrounding the origin. Furthermore, when it exists it is hyperbolic.

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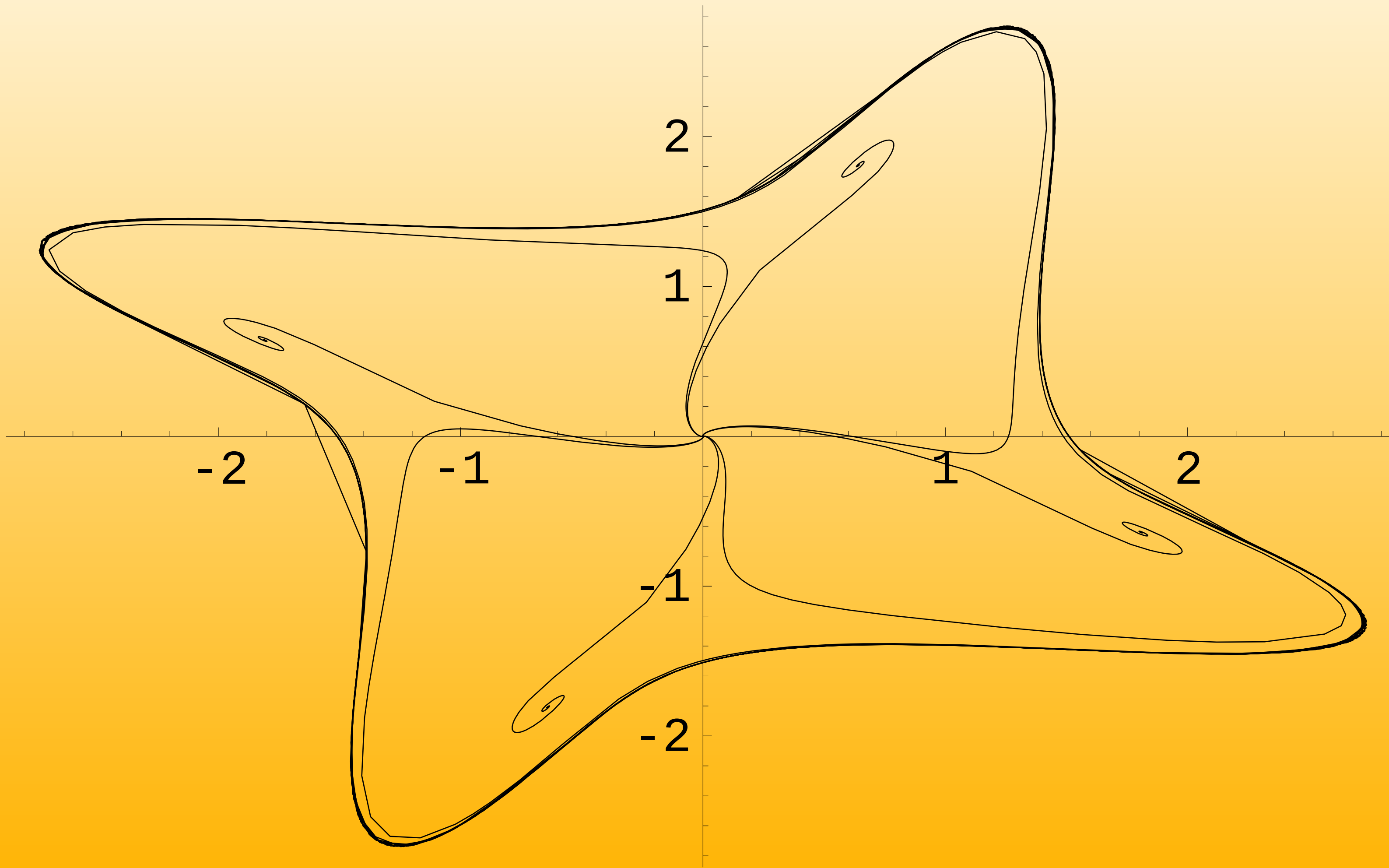
(b) There are equations (7) under condition (i) having exactly one hyperbolic limit cycle surrounding either 1 or 5 critical points and equations under condition (ii) having exactly one limit cycle surrounding either 1, 5 or 9 critical points.

Cubic systems with symmetry of order 4

Numerical example of a system (7) with 9 critical points and a limit cycle surrounding them.



Thanks for your attention!