Conjuntos ω-límite y Entropía Topológica de Aplicaciones Triangulares Bidimensionales

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Introducción

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$\mathrm{Tra}_\psi(x) = \{\psi^n(x)\}_{n=0}^{\infty}$, $\psi^n = \psi^{n-1} \circ \psi$ si $n > 0$ y $\psi^0 = \mathrm{Id}_{\mathbb{X}}$

Definición

Definimos el conjunto ω-límite de un punto $x \in \mathbb{X}$ a través de una aplicación continua ψ como el conjunto:

$$\omega_{\psi}(x) = \bigcap_{m \geq 0} \overline{\bigcup_{n \geq m} \psi^n(x)},$$

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o equivalentemente, como el conjunto de los puntos de acumulación de $Tra_\psi(x)$:

$$\omega_\psi(x) = \{y \in \mathbb{X} : \exists \{n_k\}_{k=1}^{\infty} \subseteq \mathbb{N} \text{ tal que } \lim_{n_k \to \infty} \psi^{n_k}(x) = y\}.$$

Definición

Consideremos $\varepsilon > 0$ y n un entero positivo. Un subconjunto $E \subseteq \mathbb{X}$ se dice (n, ε)-*separado* para ψ, si para cada $x \neq y \in E$ existe $j \in \{0, 1, ..., n-1\}$ tal que $d(\psi^j(x), \psi^j(y)) > \varepsilon$.

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$$h(\psi) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \cdot log \left(\sup_E card(E) \right)$$

donde el supremo se toma con relación a todos los subconjuntos (n, ε)-separados E de $\mathbb{X}$.

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$$\mathcal{W}(\psi) := \{\omega_\psi(x) : x \in \mathbb{X}\} \text{ y } \mathcal{W}_{\mathcal{F}} := \{\omega_\psi(x) : \psi \in \mathcal{F} \text{ y } x \in \mathbb{X}\}.$$

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(P1) Dado un elemento $\psi \in \mathcal{F}$, describir completamente la familia $\mathcal{W}(\psi)$ de todos los conjuntos ω-límite producidos por la aplicación ψ.

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(P2) Analizar la existencia de elementos universales y d-universales (universales en sentido débil) en la clase de aplicaciones $\mathcal{F}$ para la familia de conjuntos $\mathcal{W}_{\mathcal{F}}$. Definiciones

Problemas

(P3) Estudiar las propiedades de clausura en el espacio de todos los subconjuntos compactos y no vacíos de $\mathbb{X}$, $\mathcal{K}(\mathbb{X})$, dotado con la topología métrica de Hausdorff, d_H, de la familia de conjuntos ω-límite de generados por un elemento $\psi \in \mathcal{F}$.

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(P4) Obtener para los elementos de $\mathcal{F}$ caracterizaciones de la propiedad de poseer entropía topológica cero.

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Definición

Una aplicación $F \in \mathcal{C}(\mathbb{I}^2)$ se llama triangular si es de la forma $F : (x, y) \to (f(x), g(x, y))$.

Guide of Chapters

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- Chapter 5$\longrightarrow$ Problem (P4)

Chapter 1. Notation and Preliminary Results

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- Recurrent Points. $\text{Rec}(\psi) = \{x \in \mathbb{X} : x \in \omega_\psi(x)\}$

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The topological entropy.

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Measures of Chaos

The topological entropy.
The Chaos in the sense of Li and Yorke.

Definition

Let $\mathbb{X}$ be a compact metric space and $\psi \in \mathcal{C}(\mathbb{X})$. A pair of points $\{x, y\} \subset \mathbb{X}$ is called a Li-Yorke pair of ψ, if simultaneously is held:

$$\liminf_{n\to\infty} d(\psi^n(x), \psi^n(y)) = 0 \text{ and } \limsup_{n\to\infty} d(\psi^n(x), \psi^n(y)) > 0.$$

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Definition

Let $\mathbb{X}$ be a compact metric space, $\psi \in \mathcal{C}(\mathbb{X})$ and $A \subseteq \mathbb{X}$. The map $\psi|_A$ is chaotic (in the sense of Li and Yorke) if A contains at least a Li-Yorke pair of ψ.

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Is true that the existence of a Li-Yorke pair implies the existence of an uncountable number of them?

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The system $(\mathbb{I}, f)$ is chaotic iff there exists an uncountable set S (called scrambled) of Li-Yorke pairs.

Question 1

Is true that the existence of a Li-Yorke pair implies the existence of an uncountable number of them?
The answer in general is NOT, Forti, Paganoni and Smítal presented a map defined on $\mathbb{I}^2$ (triangular) generating only Li-Yorke pairs, i.e., the maximal cardinality of its scrambled sets is two.

Question 2. Appendix

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Theorem 5.4.2.(joint with M. Lampart)

There exists a continuous selfmap defined on a Cantor set space (closed, without isolated points and totally disconnected), such that every scrambled sets has exactly two points.

Theorem 5.4.4.(joint with M. Lampart)

There exists a continuum $\mathbb{X} \subset \mathbb{R}^2$ (nonempty compact and connected space) with empty interior and an homeomorphism h on $\mathbb{X}$, such that every scrambled sets has exactly two points.

ω-limit sets on $\mathcal{C}_\Delta(\mathbb{I}^2)$

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- As a consequence of their special morphology, triangular maps have some similarities with interval ones.

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- As a consequence of their special morphology, triangular maps have some similarities with interval ones.
  For example, the Sharkovskiĭ´s theorem concerning periodic structure is true on $\mathcal{C}_\Delta(\mathbb{I}^2)$, there exists a result by Kolyada which links the special points sets of the basis map and the triangular map, there exists some formulas and bounds by Bowen to compute the topological entropy,...

Why to use triangular maps?

- Concerning ω-limit sets for the elements of $\mathcal{C}_{\Delta}(\mathbb{I}^2)$ we have two extra results due to by Kolyada and Snoha which complete the results of Agronski and Ceder for transformations on $\mathbb{I}^n$.

Chapter 2. Describing the family $\mathcal{W}(F)$

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1. $Per(F) = Fix(F) = I_0$ ($I_0 = \{0\} \times \mathbb{I}$),
2. $\omega_F(1,1) = I_0$,
3. $h(F) = 0$.

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For instance in [Arteaga,JMAA 94], is established that the non-wandering points set of F, $\Omega(F)$, is different to the periodic points set. ($x \in \Omega(F)$ if for each neighborhood $\mathcal{U}(x)$, there exists a positive integer n such that $F^n(\mathcal{U}) \cap \mathcal{U} \neq \emptyset$)

Theorem 2.1.7. (joint with Balibrea and Muñoz)

Let F be the Kolyada´s triangular map. Then the following property holds:

$$\omega_F(a,b) = I_0 \text{ for each } (a,b) \in \mathbb{I}^2 \backslash I_0, \text{ where } I_0 = \mathrm{Per}(F) = \mathrm{Fix}(F).$$

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Corollary 2.1.7.1.

Let F be the Kolyada´s triangular map. Then we completely describe the family $\mathcal{W}(F)$:

$$\mathcal{W}(F) = I_0 \cup \{(0,c)\}_{c\in\mathbb{I}}.$$

($\bar{P}1$) for a map on $\mathcal{C}_{\Delta}(\mathbb{I}^3)$

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In 2003, Balibrea, Reich and Smítal (see [BRS, IJBC 03]) stated the following problem:

Problem

Find a triangular map explicitly defined on $\mathbb{I}^3$ (i.e., a continuous map of the form $(x, y, z) \to (f(x), g(x, y), h(x, y, z))$) such that the ω-limit sets of all its points except those of a face composed by fixed points (which can assume without loss of generality $I_0^2 = \{0\} \times \mathbb{I}^2$) is equal to I_0^2.

Theorem 2.2.12 (joint with Balibrea and Muñoz)

There exists a map F on $\mathbb{I}^3$ (triangular) holding:

$$\omega_F(x,y,z) = I_0^2 \text{ if } (x,y,z) \notin I_0^2$$

and $\text{Fix}(F) = I_0^2$.

Main ideas of the construction

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Main ideas of the construction

We define a map of the form:

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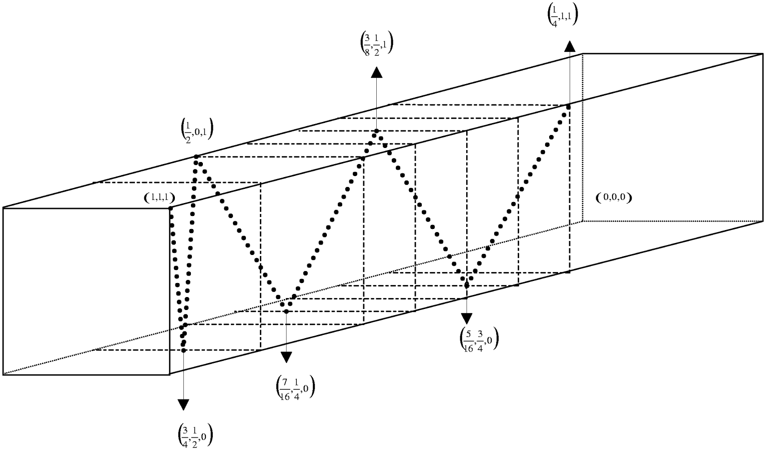
- The basis map $f(x)$ is a basis of Kolyada type, for adequate parameters (we obtain less speed for going to the left),
- The maps g and h are stylized modifications of the fiber map introduces by Kolyada, in order to adjust and synchronize the movement of each coordinated for obtaining the result.

The Kolyada´s triangular map
($\bar{\mathrm{P}}1$) for a map on $\mathcal{C}_{\Delta}(\mathbb{I}^3)$



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Types of Universality

- $\psi \in \mathcal{F}$ is universal regarding the sets of $\mathcal{W}_{\mathcal{F}}$ iff $\mathcal{W}(\psi) = \mathcal{W}_{\mathcal{F}}$.

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Types of Universality

- $\psi \in \mathcal{F}$ is universal regarding the sets of $\mathcal{W}_{\mathcal{F}}$ iff $\mathcal{W}(\psi) = \mathcal{W}_{\mathcal{F}}$.
- $\psi \in \mathcal{F}$ is w-universal regarding the sets of $\mathcal{W}_{\mathcal{F}}$ iff $\mathcal{W}(\psi) = \mathcal{W}_{\mathcal{F}}$ "up to homeomorphism".

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Theorem (Pokluda-Smítal)

There exists a w-universal maps on $\mathcal{C}(\mathbb{I})$ for the elements of $\mathcal{W}_{\mathcal{C}(\mathbb{I})}$.

Universality

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Definition

Let $\mathbb{X}$ be a compact metric space. We say that $\mathbb{X}$ is ω-degenerated (resp. ω-degenerated for $\mathcal{W}_{\mathcal{F}}$) if $\mathcal{W}_{\mathcal{C}(\mathbb{X})} = \{\{x\} : x \in \mathbb{X}\}$ (resp. $\mathcal{W}_{\mathcal{F}} = \{\{x\} : x \in \mathbb{X}\}$).

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Theorem 3.1.3. (with Chudziak, Snoha and Spitalský)

Let $\mathbb{X}$ be a compact metric space. The sets of $\mathcal{W}_{\mathcal{C}(\mathbb{X})}$ (resp. $\mathcal{W}_{\mathcal{F}}$) admits a universal map if and only if $\mathbb{X}$ is ω-degenerated (resp. for $\mathcal{W}_{\mathcal{F}}$ and $Id_{\mathbb{X}} \in \mathcal{F}$). The only universal map is the identity.

Proposition 3.1.2.

Let $\mathbb{X}$ be a compact metric space. Then

(a) If $\mathbb{X}$ is a rigid continuum (i.e., the only continuous selfmaps are the identity and the constant maps), then $\mathbb{X}$ is ω-degenerated.

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(a) If $\mathbb{X}$ is a rigid continuum (i.e., the only continuous selfmaps are the identity and the constant maps), then $\mathbb{X}$ is ω-degenerated.

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(b) If $\mathbb{X}$ is ω-degenerated, then it is connected.

(c) If $\mathbb{X}$ is ω-degenerated, then for each $f \in \mathcal{C}(\mathbb{X})$ and each $x \in \mathbb{X}$ the sequence $(f^n(x))_{n\geq 1}$ converges to a fixed point.

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What is the situation if we reduce the family of sets with regard to the universality?
For example, we consider for the class of triangular maps the family of interval sets located on a fiber (without loss of generality we consider it as I_0), i.e.,

$$\mathcal{A} = \{\{0\} \times [a, b] : a \leq b, \ a, b \in \mathbb{I}\}.$$

Question

Does exist a map F on $\mathcal{C}_{\Delta}(\mathbb{I}^2)$ such that F is universal for the elements of $\mathcal{A}$, i.e., $\mathcal{A} \subset \mathcal{W}(F)$?

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The answer is YES.

Theorem 3.2.5. (with Balibrea and Muñoz)

There exists a triangular map F on $\mathbb{I}^2$ holding the following conditions:

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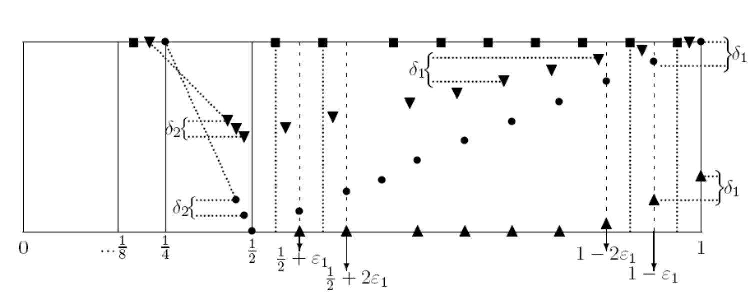
($\mathcal{B}$) Given $(p,q) \in \mathbb{I}^2$, there exists a compact interval $J \subseteq \mathbb{I}$, degenerate or not, such that $\omega_F(p,q) = \{0\} \times J$.

Ideas of the construction



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Corollary 3.3.5.1.

Let $\mathbb{M}$ be a compact m-dimensional manifold with $m \geq 2$. Then, there is no w-universal map for the sets of $\mathcal{W}_{\mathcal{C}(\mathbb{M})}$.

Lemma 3.3.7.

Let $\mathbb{S}^1$ be the unit circle. Then, there is no w-universal map for the sets of $\mathcal{W}_{\mathcal{C}(\mathbb{S}^1)}$.

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Case $\mathbb{X}$ is a *m*-dimensional Manifold:

Corollary 3.3.7.1. (with Chudziak, Snoha and Spitalský)

Let $\mathbb{M}$ be a *m*-dimensional compact manifold ($m \geq 1$). Then, $\mathcal{W}_{\mathcal{C}(\mathbb{M})}$ admits a w-universal map if and only if $\mathbb{M}$ is an arc (and thus $m = 1$).

Case $\mathbb{X}$ is a Graph:

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Theorem 3.3.10. (with Chudziak, Snoha and Spitalský)

Let $\mathbb{X}$ be a graph. Then, $\mathcal{W}_{\mathcal{C}(\mathbb{X})}$ admits a w-universal map if and only if the space $\mathbb{X}$ is an arc.

Chapter 4. Hausdorff closure of $\mathcal{W}(\psi)$

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Given a map $\psi \in \mathcal{F}$, $\mathcal{F} \subseteq \mathcal{C}(\mathbb{X})$, we are going to study if the family $\mathcal{W}(\psi)$ is closed in the space $\mathcal{K}(\mathbb{X})$ of all non-empty compact subsets of $\mathbb{X}$ endowed with the Hausdorff metric topology d_H.

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The case of having the closure property

If $\mathcal{W}(\psi)$ is d_H-closed, then it is possible to construct ω-limit sets of ψ as a limit of d_H-convergent sequences of sets on $\mathcal{W}(\psi)$.

Theorem (Blokh, Bruckner, Humke and Smítal)

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In general, the closure property is not true and easy counterexamples can be constructed.

We center our attention as usual on $\mathbb{I}^2$:

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Counterexample

Let f be the identity map on $\mathbb{I}$ and consider as a fiber map, for each $x \in \mathbb{I}$ a transitive map g_x (i.e., $\exists y \in \mathbb{I}$ s.t. $\omega_{g_x}(y) = \mathbb{I}$) such that $g_x \to Id_{I_0}$ when $x \to 0$.

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Then, the triangular map $F(x,y) = (f(x), g(x,y))$ $(g(x,y) = g_x(y))$ does not holds the closure property.
$(\bigcup_{x\in\mathbb{I}^2} \omega_F(x) = \mathbb{I}^2 \setminus I_0)$

Note that the Kolyada´s triangular map and the map presented in Theorem 3.2.5. fulfill the closure property.

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Question

If by $\mathcal{C}_{\Delta}^{\text{fib}}(\mathbb{I}^2)$ we denote the family of all triangular maps which generate ω-limit sets contained in one fiber.
Is it true the closure property for the elements of $\mathcal{C}_{\Delta}^{\text{fib}}(\mathbb{I}^2)$?

The answer is negative as it is showed in the following result:

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Theorem (with Balibrea and Muñoz)

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There exists a triangular map F on $\mathcal{C}_{\Delta}^{\text{fib}}(\mathbb{I}^2)$ holding the following properties:

($\mathcal{A}$) There exists a sequence of points $\{(p_i, q_i)\}_{i \in 2\mathbb{N}^*} \subset \mathbb{I}^2 \backslash I_0$ such that:

($\mathcal{A}_1$) $\omega_F(p_i, q_i) = \{0\} \times [a_i, b_i]$ with $b_i - a_i = 1 - \frac{1}{2^{10}}$.

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- ($\mathcal{A}_2$) $\lim_{i \to \infty} \omega_F(p_i, q_i) = \{0\} \times [\frac{1}{2^{10}}, 1]$ in the Hausdorff metric topology of $\mathbb{I}^2$.

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($\mathcal{B}$) $\omega_F(p, q) \neq \{0\} \times [\frac{1}{2^{10}}, 1]$ for each $(p, q) \in \mathbb{I}^2$.

Chapter 5. Characterizing $h(F) = 0$ on $\mathcal{C}^*_{\Delta}(I^2)$

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(5) the periods of all periodic points are a power of two ($F \leq 2^{\infty}$).

Theorem

For maps on $\mathcal{C}(\mathbb{I})$ properties (1)-(5) are mutually equivalent.

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For instance, there exists a map F introduced by Kolyada of type 2^∞ with positive topological entropy and with recurrent points which are not uniformly recurrent.
Even in 1995, Balibrea, Esquembre and Linero for every k found an example of class $\mathcal{C}^k$.

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Even under strong hypothesis like all maps g_x are non-decreasing the equivalence is not reached.

The class $\mathcal{C}_\Delta^*(\mathbb{I}^2)$

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3. *Rec(f)=Per(f);*
4. *UR(f)=Per(f).*

Theorem 5.4.1. (with Chudziak)

Let $F \in \mathcal{C}^*_{\Delta}(\mathbb{I}^2)$ with basis map f. Then, conditions $(1) - (5)$ are mutually equivalent properties for F.

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Go to the beginning

Definición

Un elemento $\psi \in \mathcal{F}$ es universal con respecto a los conjuntos ω-límite generados por la clase $\mathcal{F}$, si para cada conjunto $A \in \mathcal{W}_{\mathcal{F}}$ existe un punto $x_A \in \mathbb{X}$ tal que $\omega_{\psi}(x_A) = A$. Es decir, si ocurre $\mathcal{W}(\psi) = \mathcal{W}_{\mathcal{F}}$.

Definición

Un elemento $\psi \in \mathcal{F}$ se llamará universal en sentido débil (d-universal) con respecto a los conjuntos ω-límite generados por la clase $\mathcal{F}$, si para cada conjunto $A \in \mathcal{W}_{\mathcal{F}}$ existe un homeomorfismo $H_A : \mathbb{X} \to \mathbb{X}$ y un punto x_A tal que $\omega_\psi(x_A) = H_A(A)$. Es decir, si ocurre que $H_A(A) \in \mathcal{W}(\psi)$.

Volver