SICONOS

UNIVERSITAT POLITÈCNICA DE CATALUNYA

MODELING AND NUMERICAL STUDY OF NONSMOOTH DYNAMICAL SYSTEMS.

Applications to Mechanical and Power Electronic Systems.

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Supervisors:

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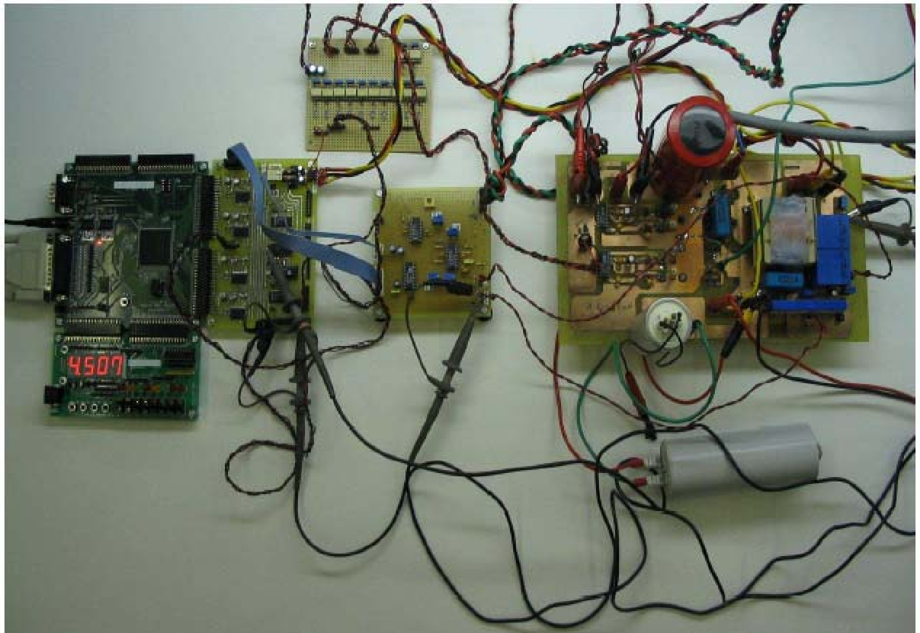
Ddays 2006

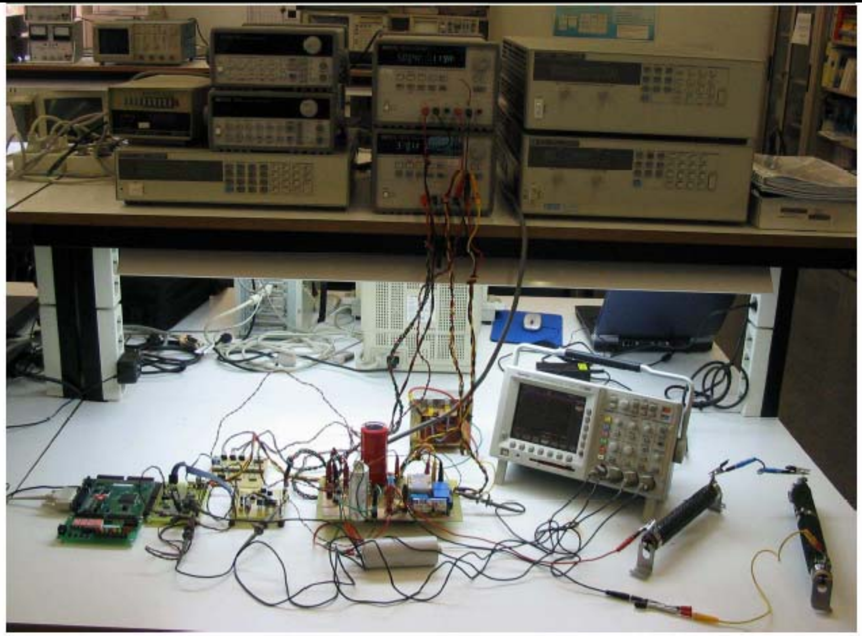
Thesis

Sevilla-Islantilla, October 18-21, 2006

Introduction and motivation

- **Why nonsmooth?**





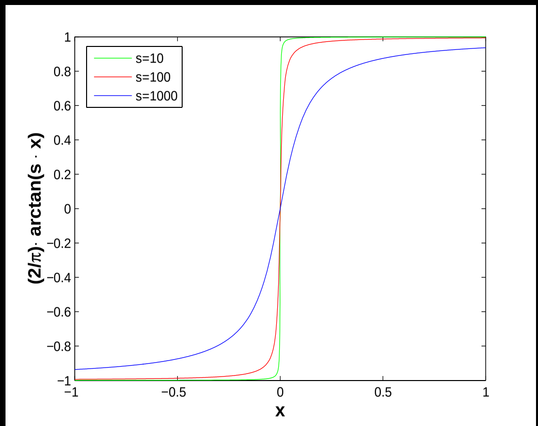
NOT EVERYTHING IS COMPLETELY SMOOTH

Introduction and motivation

- **Why nonsmooth?**
  - There is an established theory for smooth dynamical systems (analysis, bifurcations and control).
  - Many systems are nonsmooth on a macroscopic scale.
  - Think for example of systems with switches, friction, impacts, walking mechanism, etc.
  - There is no consistent theory for those cases where the vector field is nonsmooth and/or non-differentiable.
  - Then what?

Introduction and motivation

- **… Better smooth?**
  - **Typically, nonsmooth characteristic can be smoothed out, e.g.**

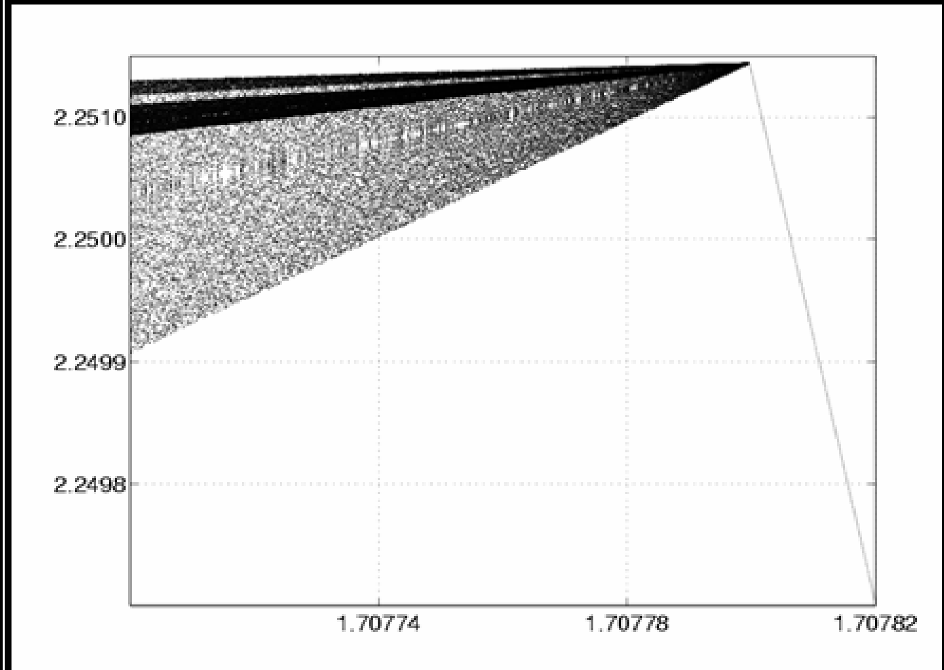


- **Well, this could work but:**
  - ✓ **Usually, eqns become stiff.**
  - ✓ **Time- consuming to simulate.**
  - ✓ **Solution might not be accurate**
- **Take, for example, a block sliding on an inclined plane…**

Introduction and motivation

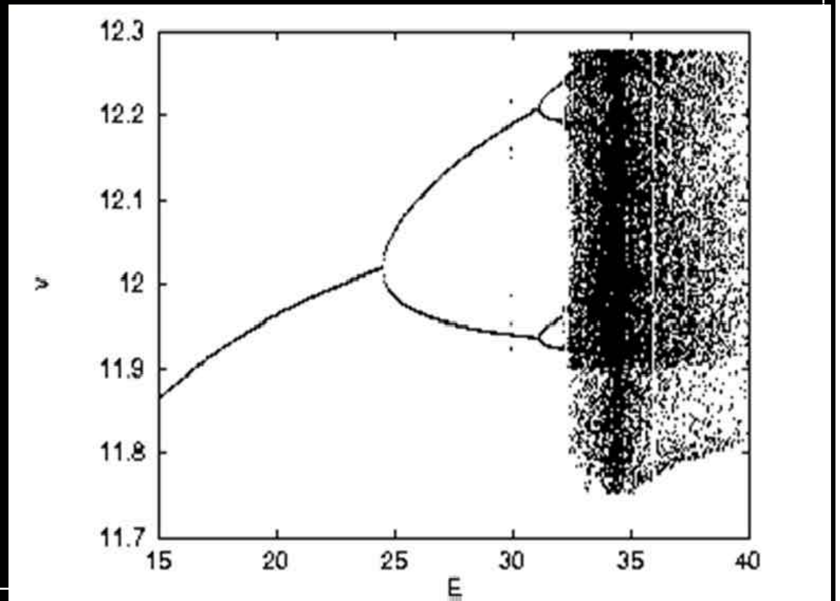
- **Smoothing:**
  - **So, smoothing can be an option but it must be dealt with great care.**
  - **Several techniques can be used but no detailed study/comparison of the techniques has been carried out so far.**
  - **Maybe dealing with the nonsmooth equations can be easier.**
  - **But to do that we need a consistent theory of nonsmooth systems… (Complementarity systems, Hybrid systems,…)**

Introduction and motivation



Friction oscillators

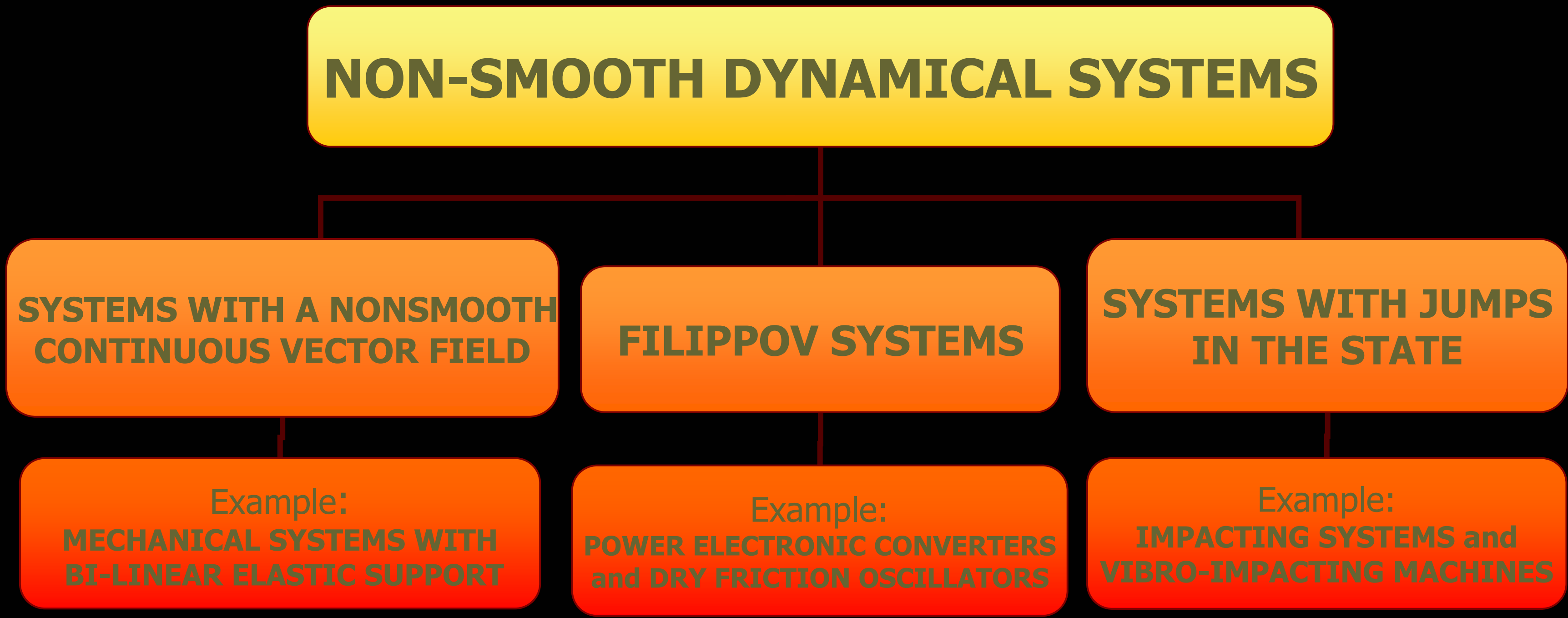
DC-DC buck converter



Background and literature review

- **Classification of nonsmooth dynamical systems.**
- **Modelling using complementarity formalism.**
- **Classification of simulation techniques:**
  - **Event-driven methods.**
  - **Smoothing methods.**
  - **Time-stepping methods.**
- **Bifurcation analysis.**

Classification of nonsmooth dynamical systems



Ref.: **BIFURCATIONS IN NONLINEAR DISCONTINUOUS SYSTEMS. "R.I.LEINE, D.H.VAN CAMPEN and B.L.VAN DE VRANDE".**

Modelling using complementarity formalism

- **Linear Complementarity Problem (LCP):**

Given a matrix $M \in R^{k \times k}$ and a vector $q \in R^k$. The linear complementarity problem LCP(q,M) is to find vectors $u, y \in R^k$ such that

$$y = q + M \cdot u,$$

and satisfying

$$u_i \geq 0,\ y_i \geq 0,\ \text{and}\ \{u_i = 0\ \text{or}\ y_i = 0\},\ \forall\ i \in \{1, \ldots, k\}$$

or to show that no such vector u exists.

The conditions $u \geq 0, y \geq 0, u^T \cdot y = 0$ are called complementarity conditions (CC) and are denoted by $0 \leq u \perp y \geq 0$.

Modelling using complementarity formalism

Theorem 1: If $M \in R^{k \times k}$ is **positive definite**, then the **LCP (q, M)** has a **unique solution** for all $q \in R^k$.

In general, the LCP with a **positive semi-definite matrix** can have **multiple solutions**. For instance, the LCP with:

$$q = \begin{pmatrix} -1 \\ -1 \end{pmatrix} \qquad M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

has solutions

$$u_1 = (1,0),\ u_2 = (0,1),\ u_3 = (\frac{1}{2},\frac{1}{2})$$

Modelling using complementarity formalism

Theorem 1: If $M \in R^{k \times k}$ is **positive definite**, then the **LCP (q, M)** has a **unique solution** for all $q \in R^k$.

Definition 1: A matrix $M \in R^{k \times k}$ is said a **P-matrix** if all its **principal minors** are **positive**. The class of such matrices is denoted P.

Theorem 2: A matrix $M \in R^{k \times k}$ is a **P-matrix** if and only if the **LCP (q, M)** has a **unique solution** for all $q \in R^k$.

Remark: Several pivoting methods have been devoted to solve numerically LCPs. Some examples are Lemke's and Murty's algorithms

Modelling using complementarity formalism

- **Linear Complementarity Systems (LCS).**

Basically, a linear complementarity system (LCS) is a combination of a standard linear system and complementarity conditions.

Therefore, a complementarity system is given by:

$$\dot{x}(t) = A \cdot x(t) + B \cdot u(t) + E$$

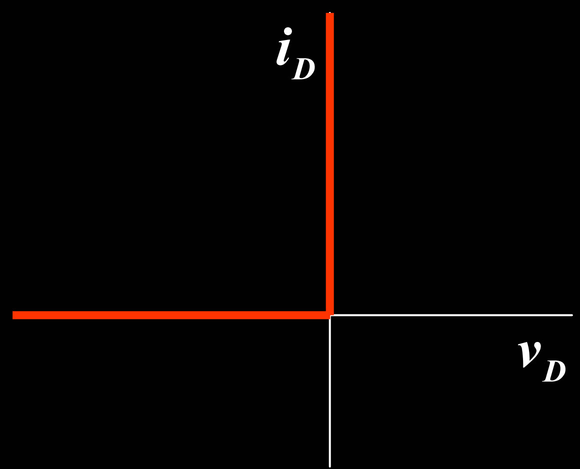
$$y(t) = C \cdot x(t) + D \cdot u(t) + F$$

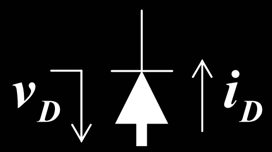
$$0 \leq u(t) \perp y(t) \geq 0, \quad \forall t \in I.$$

Remark: If the positiviness in the complementarity conditions (CC) is relaxed, i.e. the complementarity variables are orthogonal but the positiviness are not required we obtain a **LINEAR CONE COMPLEMENTARITY SYSTEM (LCCS).**

Modelling using complementarity formalism

Characteristic curve of an ideal diode





$$0 \leq -v_D \perp i_D \geq 0$$

In the context of electrical circuits, imposing complementarity conditions simply means that some ports are terminated by ideal diodes, with the current i_D and (minus) the voltage v_D as complementarity variables.

Modelling using complementarity formalism

- **Why Complementarity Systems?**
  - **Complementarity systems are particularly suited to describe systems with unilateral constraints (diodes, impact oscillators, friction, saturations, relays, VSS).**
  - **Systems with a large number of contacts, switches,…**
  - **Routines from optimization can be used.**
  - **The formalism is compact while retaining its physical meaning…**

Modelling using complementarity formalism

- **Thesis results in this chapter:**
  - **We have modelled some basic dc-dc power converters (buck, boost, buck-boost, Çuk).**
  - **After fixing the position of the switches, the system incorporates, in a natural way, the description of generalised discontinuous conduction mode (GDCM), characterised by a reduction of the dimension of the system.**
  - **Analytical state-space conditions have been stated for the presence of GDCM.**
  - **Modelling, analysis and simulation of a parallel resonant converter (PRC).**
  - **Simulation of a boost power converter with sliding mode control, even though a general control theory for complementarity systems is not still developed.**

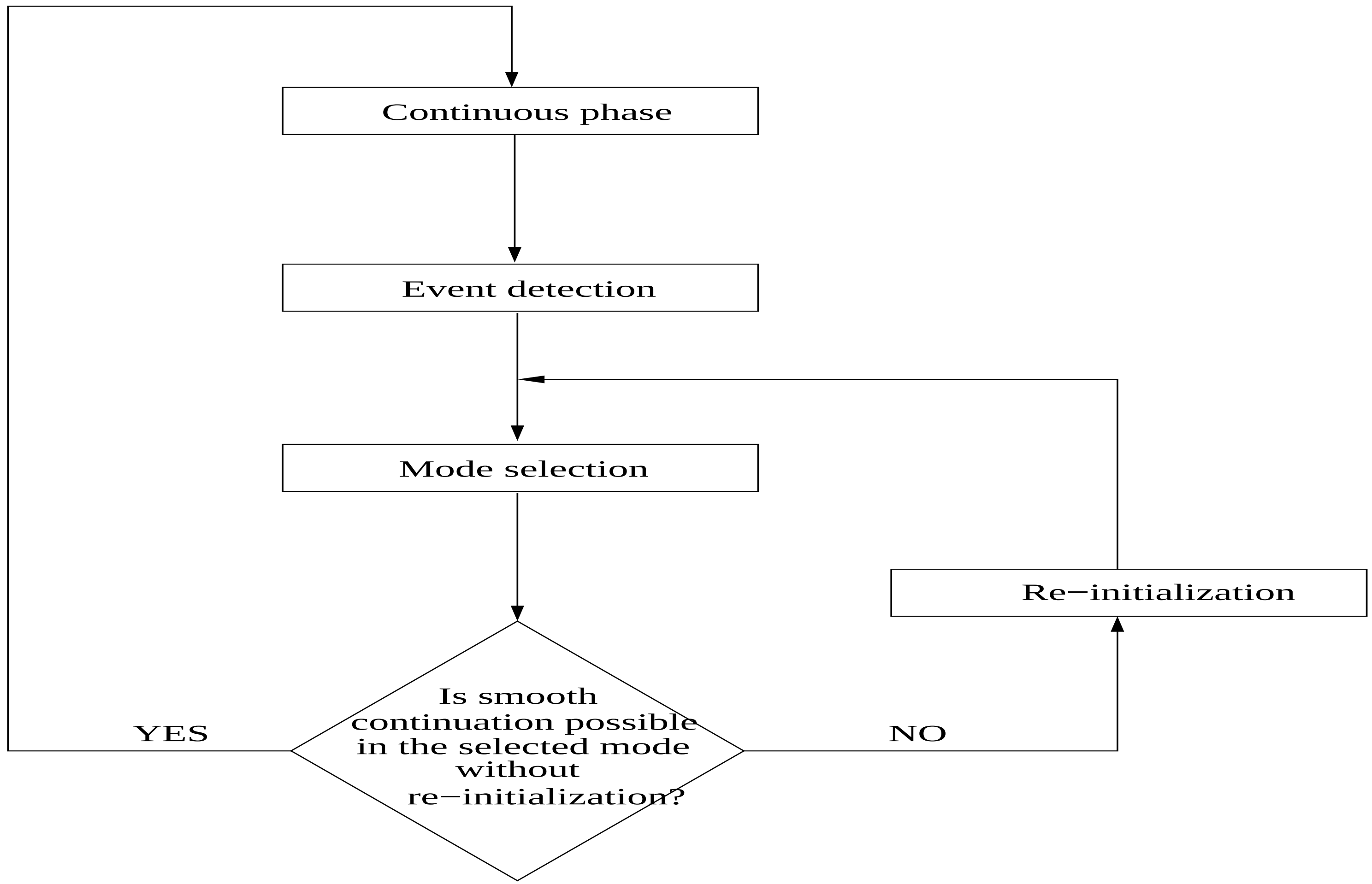
Classification of simulation techniques

Moreau classifies the literature on simulation techniques for rigid body dynamics with collisions into three categories:

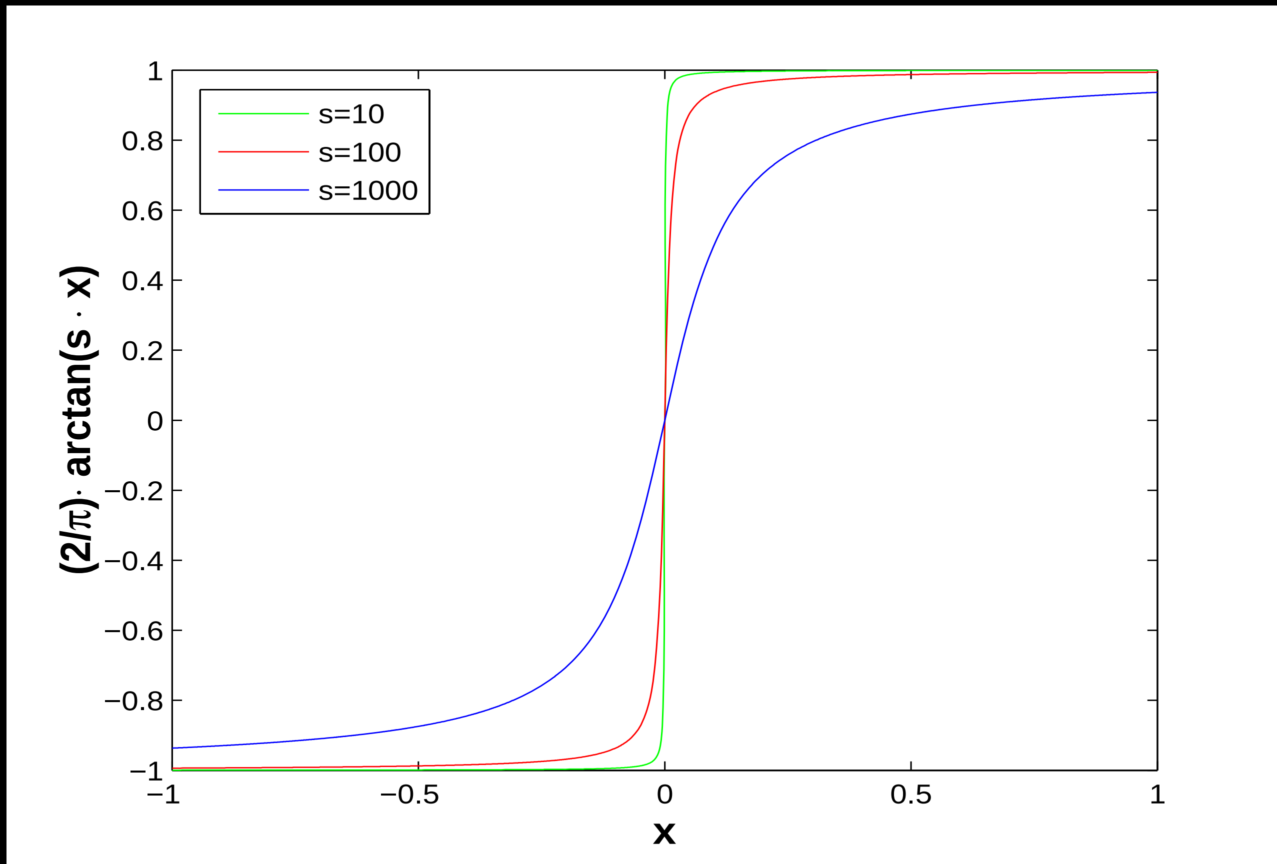
- **Event-driven methods.**
- **Smoothing methods.**
- **Time-stepping methods.**

Heemels in his thesis shows that this classification also applies to possible numerical methods for complementarity systems.

Event-driven method



Smoothing method



Smoothing method

- **Discrete mode do not really exist, so event detection and mode selection are not necessary.**
- **Instantaneous jumps are replaced by (finitely) fast motions, so also the problem of re-initialization disappears.**
- **Drawback: An accurate simulation requires the use of very stiff approximate laws. Then, a very small step-length is needed and possibly artificial terms in the equations are required in order to enforce numerical stability.**

Time-stepping method

- **Time-stepping methods replace the describing equations directly by some “discretized” equivalent.**
- **At each time-step an algebraic problem (called “one-step problem”) involving information obtained from previous time-steps is solved.**

$$\frac{x_{k+1}^h - x_k^h}{h} = A \cdot x_{k+1}^h + B \cdot u_{k+1}^h + E$$

$$y_{k+1}^h = C \cdot x_{k+1}^h + D \cdot u_{k+1}^h + F$$

$$0 \le y_{k+1}^h \perp u_{k+1}^h \ge 0$$

Backward Euler Time-stepping method

Algorithm 1: $(\{u^h_{k+1}\},\{x^h_{k+1}\},\{y^h_{k+1}\}) = LCPsimulator\,(A,B,C,D,T_{end},h,x_0)$

1. $N_h = \left\lfloor \frac{T_{end}}{h} \right\rfloor.$

2. $x^h_{-1} := x_0.$

3. $k:=1.$

4. solve the one-step problem

$$y^h_{k+1} = C\cdot(Id-h\cdot A)^{-1}\cdot x^h_k + \left[D + h\cdot C\cdot(Id-h\cdot A)^{-1}\cdot B\right]u^h_{k+1} + h\cdot(Id-h\cdot A)^{-1}\cdot E + F$$

$$0 \le u^h_{k+1} \perp y^h_{k+1} \ge 0$$

5. $x^h_{k+1} := (Id-h\cdot A)^{-1}\cdot x^h_k + h\cdot(Id-h\cdot A)^{-1}\cdot(B\cdot u^h_{k+1}+E)$

6. $k:=k+1.$

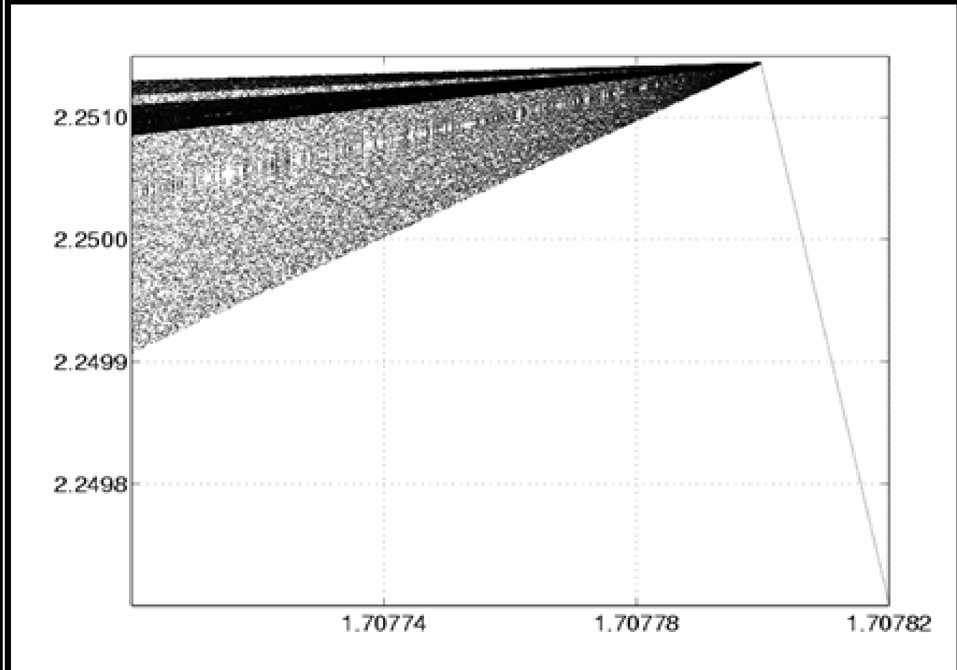
7. if $k<N_h$ go to 4.

8. stop.

Time-stepping method

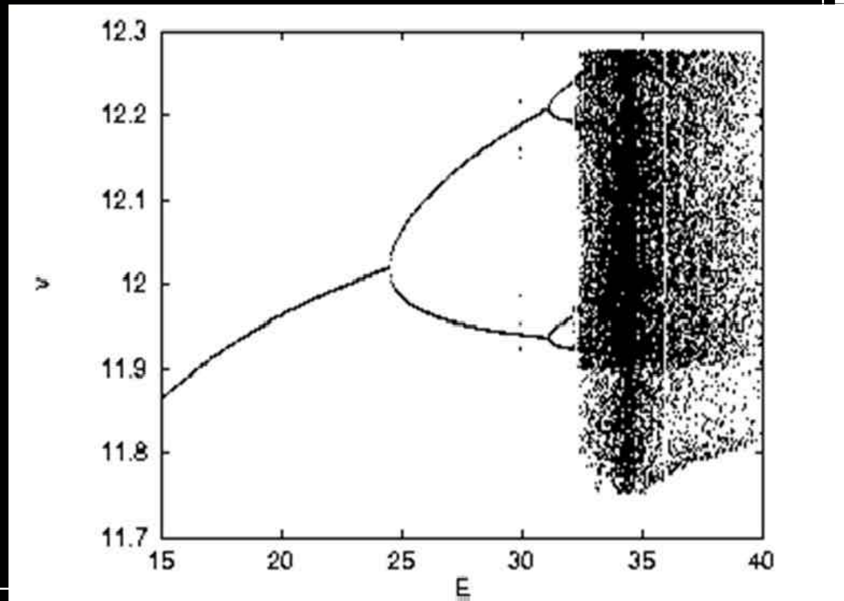
- **Time-stepping methods do not determine the event times accurately, but "overstep" them, which puts the consistency of the method into question.**
- **But, if consistency is proved it is a efficient method for simulation of NSDS with a large number of contacts, diodes,…**
- **For linear relay systems the Backward Euler method is consistent.**
- **<u>Drawback:</u> It's a low order integration method.**

Bifurcation analysis



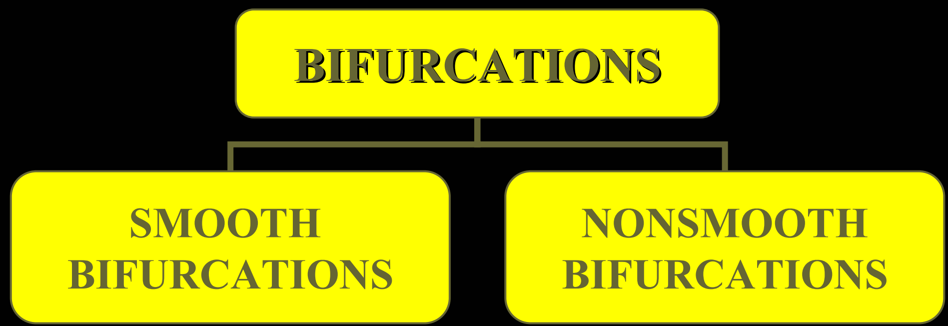
Friction oscillators

DC-DC buck converter



Bifurcation analysis

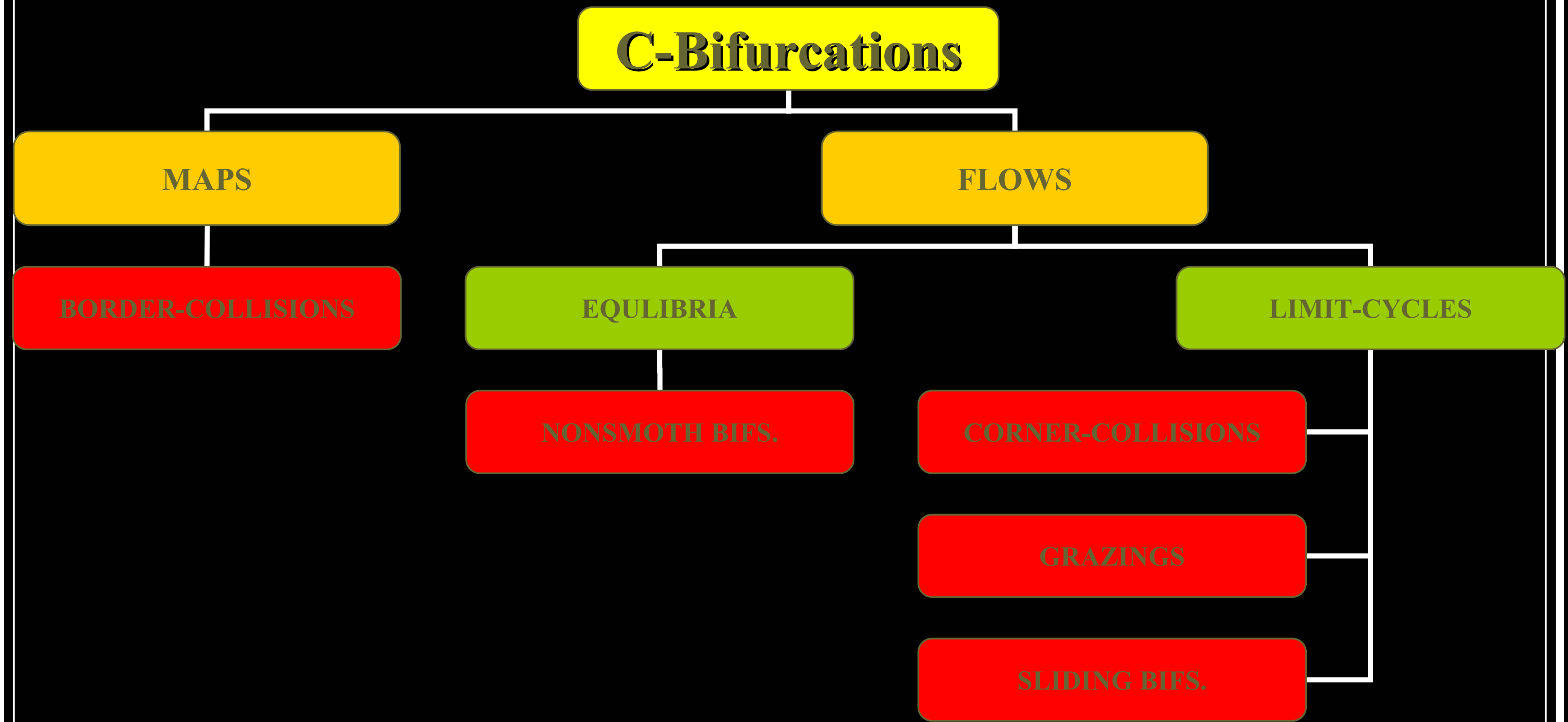
- **Codimension-one nonsmooth bifurcations:**



Definition:

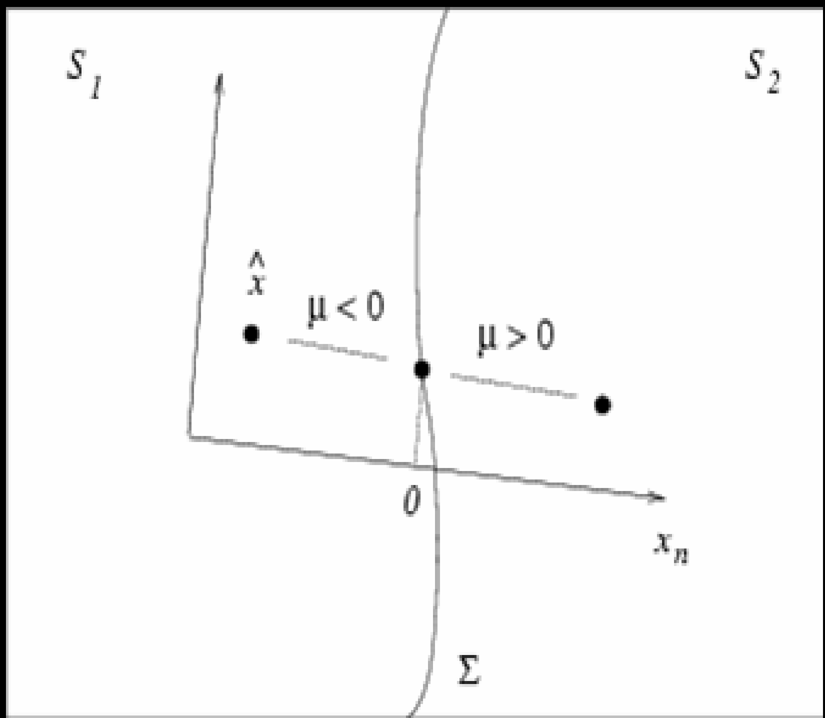
The appearance of a topologically non-equivalent phase portrait under the variation of a parameter is called a bifurcation.

Bifurcation analysis

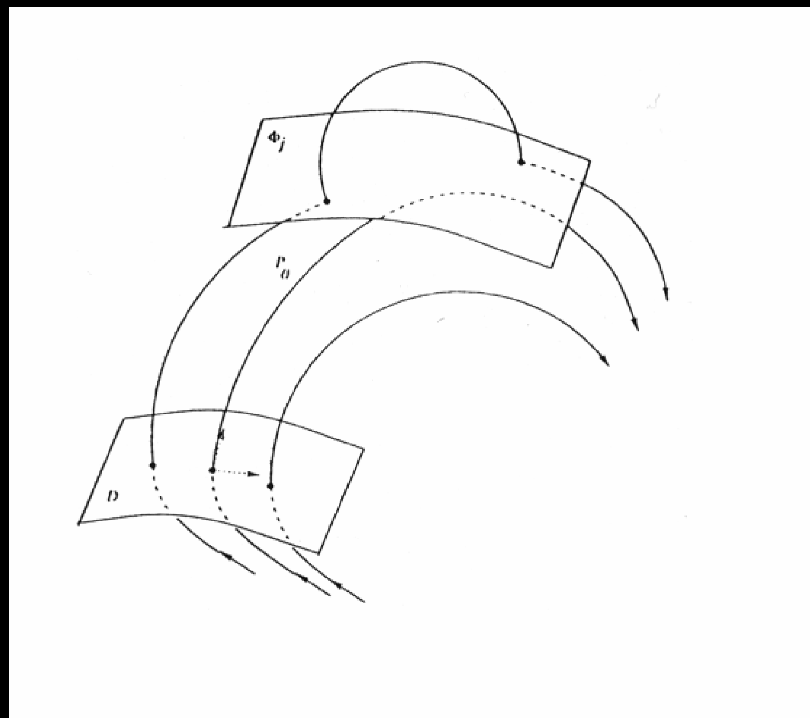


Bifurcation analysis

Border-Collision

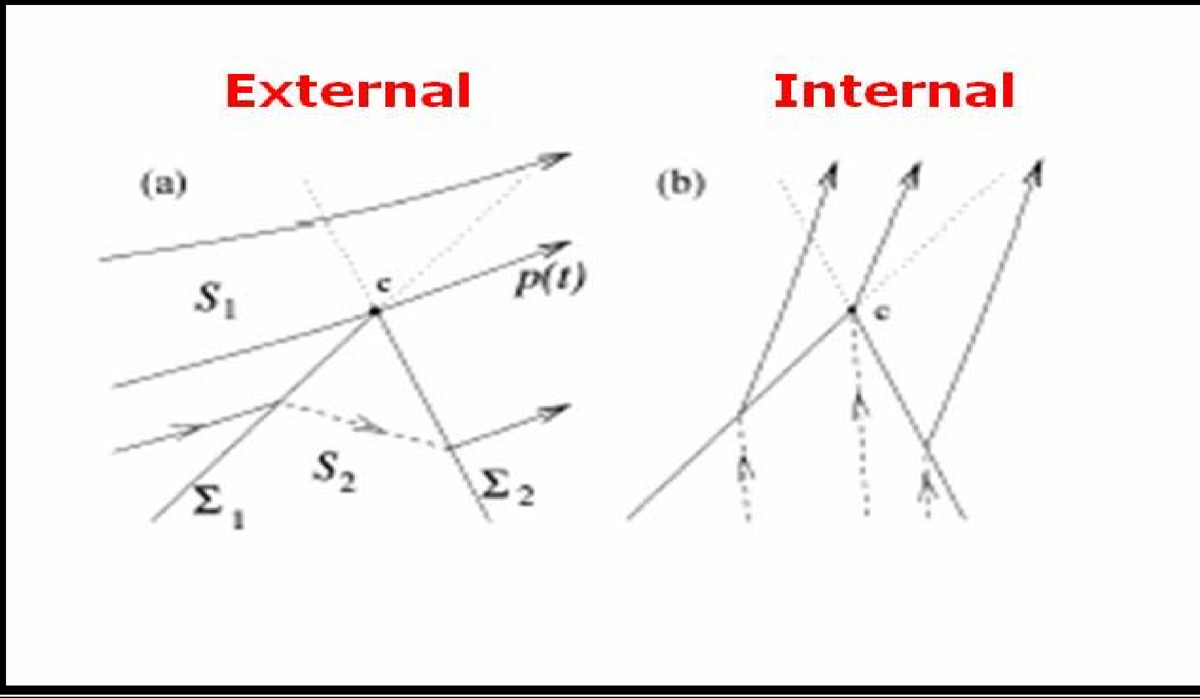


Grazing

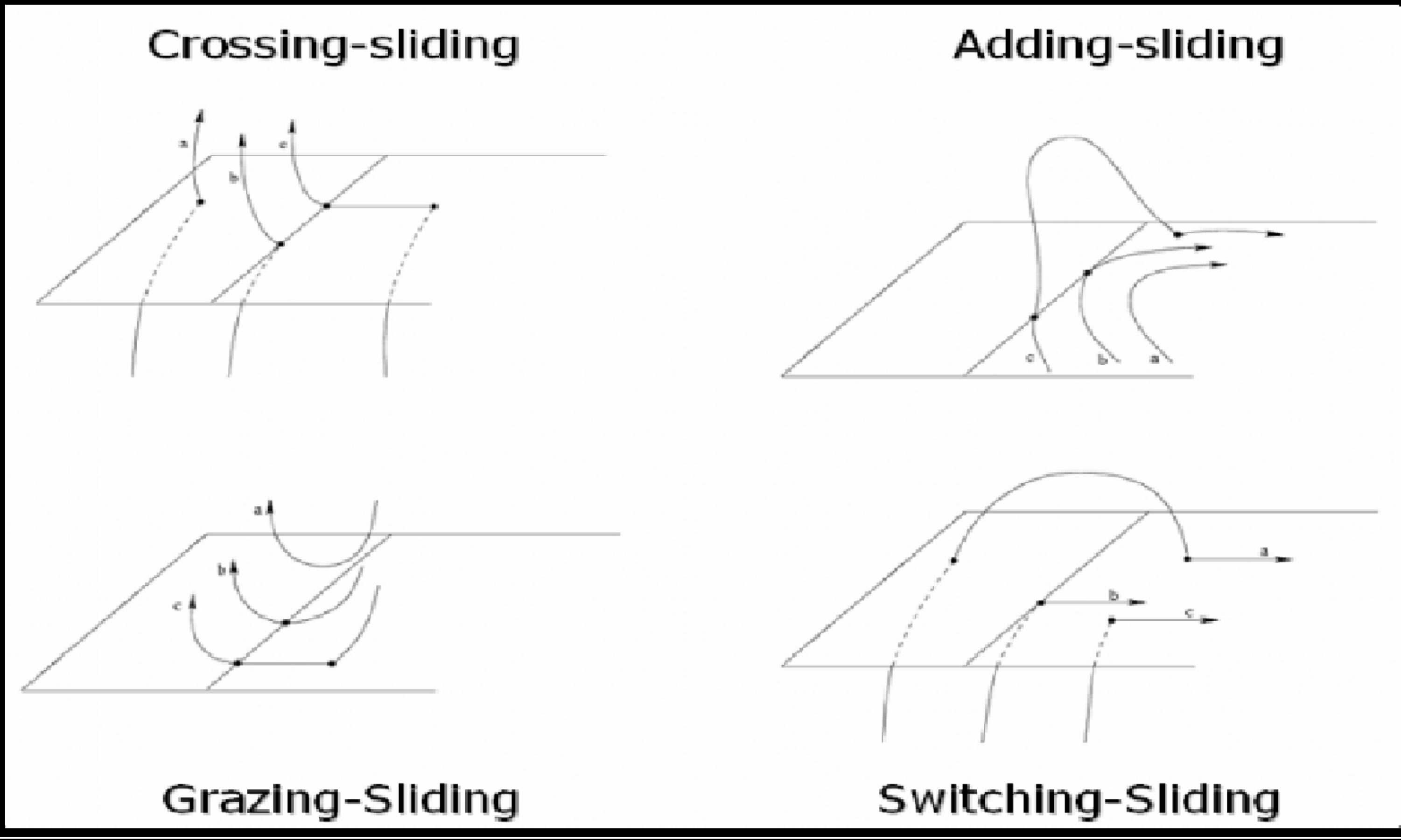


Bifurcation analysis

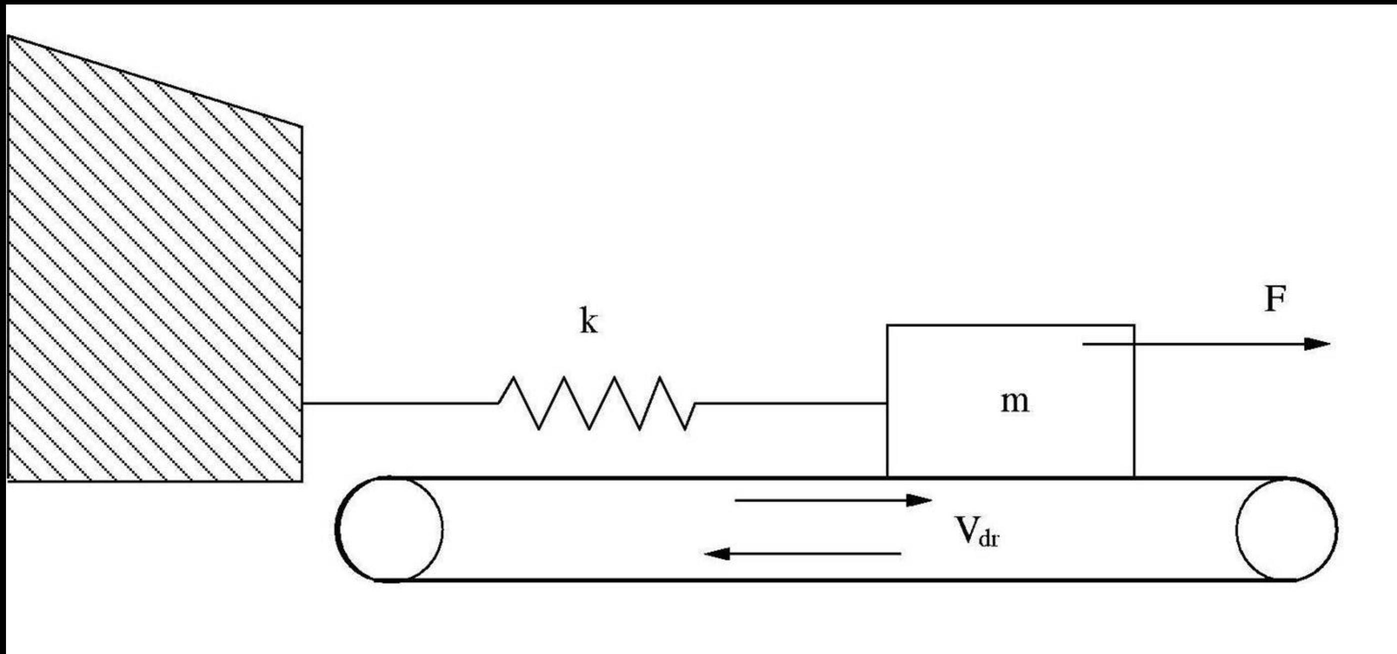
- **We called a Corner-Collision bifurcation if the boundary is itself non-smooth**



Bifurcation analysis



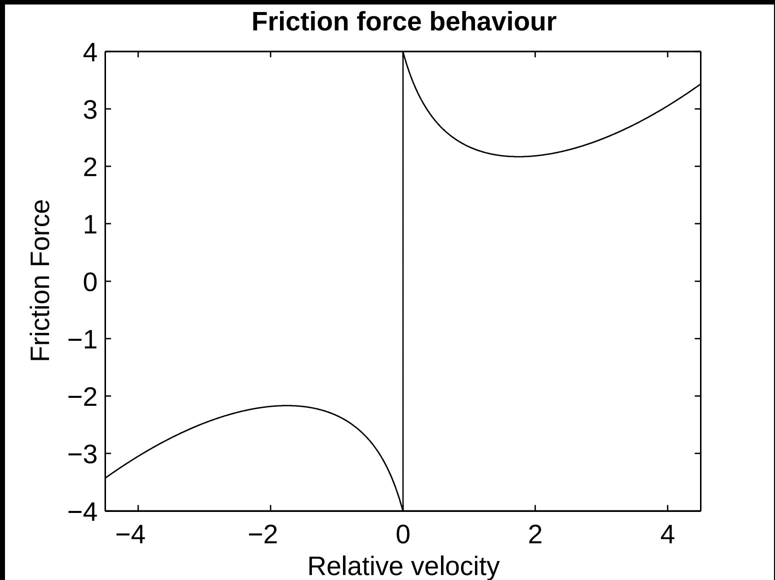
Sliding Bifurcations in a dry friction oscillator



$F = A \cdot \cos(wt)$

Description of the model

- Approximation of a measured friction characteristic introduced by Popp:



$$F_{fr,k} = mg\mu(v_{rel})$$

$$\mu(v_{rel}) = \frac{\alpha}{1+\gamma\cdot|v_{rel}|} + \beta + \eta\cdot v_{rel}^2$$

$$\alpha = 0.3, \gamma = 1.42, \beta = 0.1, \eta = 0.01$$

Filippov system

We consider a discontinuous surface Σ, which is defined by a smooth scalar function $h(x)$:

$$\Sigma = \left\{ x \in \mathbb{R}^3 : h(x) = x_2 - V_{dr} = 0 \right\}$$

Then, we can formulate our system as a Filippov system:

$$\dot{x} = \begin{cases} F_1(x,\omega) & if \quad h(x)<0 \\ F_2(x,\omega) & if \quad h(x)>0 \end{cases}$$

$$F_1 = \begin{cases} x_2 \\ -\frac{k}{m}x_1 + \frac{A}{m}cos(x_3) + \frac{F_{fr,k}}{m} \\ \omega \end{cases}; \quad F_2 = \begin{cases} x_2 \\ -\frac{k}{m}x_1 + \frac{A}{m}cos(x_3) - \frac{F_{fr,k}}{m} \\ \omega \end{cases}.$$

Filippov system

Now, we can define the sliding region as

$$\hat{\Sigma} = \left\{ x \in \Sigma : \left| \frac{-kx_1 + A\cos(\omega t)}{F_{fr,s}} \right| < 1 \right\}$$

with boundaries:

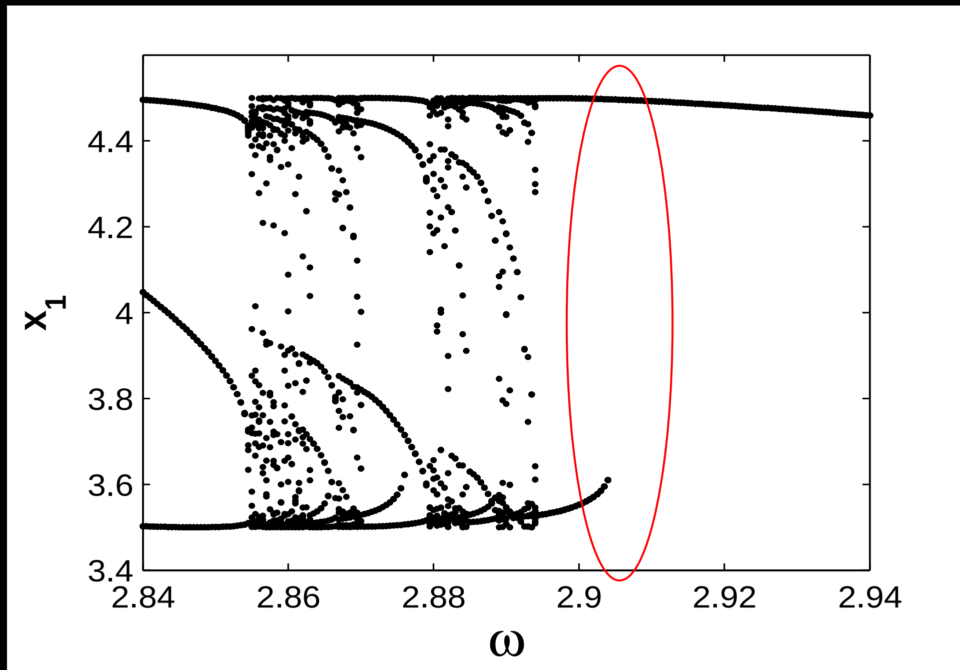
$$\partial\Sigma^+ = \left\{ x \in \Sigma : x_1 = \frac{-F_{fr,s}}{k} + \frac{A}{k}\cos(\omega t) \right\}$$

$$\partial\Sigma^- = \left\{ x \in \Sigma : x_1 = \frac{F_{fr,s}}{k} + \frac{A}{k}\cos(\omega t) \right\}$$

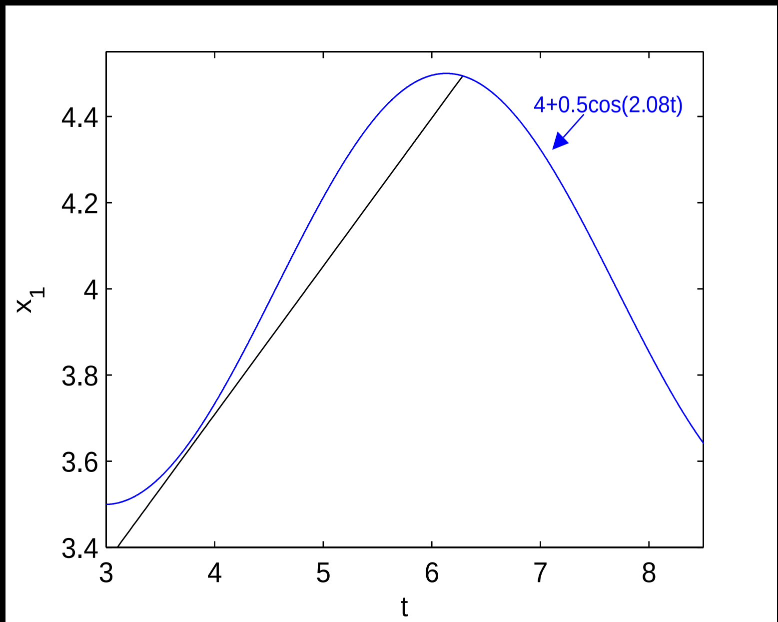
Parameters:

$k = 1,$
$m = 1,$
$g = 10,$
$V_{dr} = 1.$

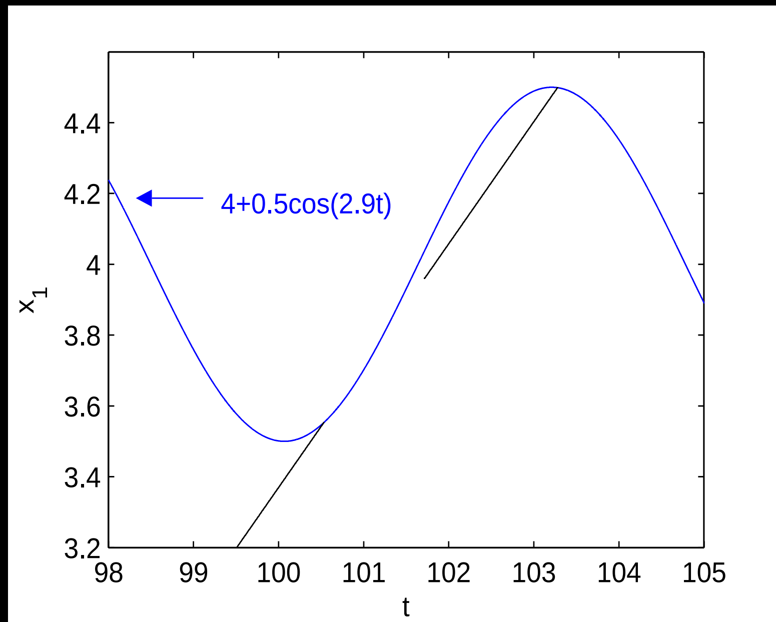
Simulations



Adding-sliding Bifurcation

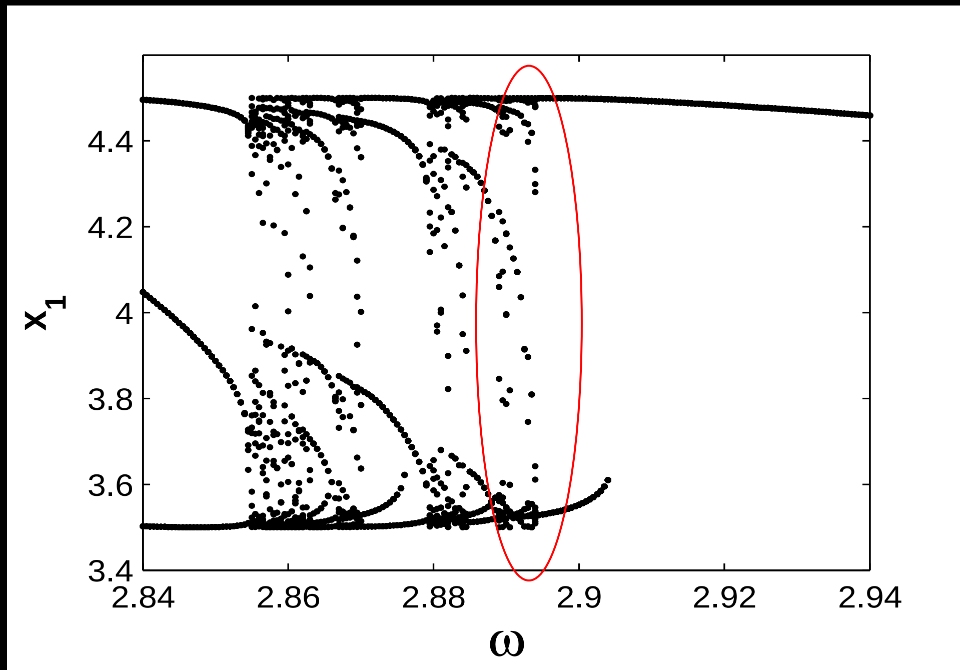


W = 2.908



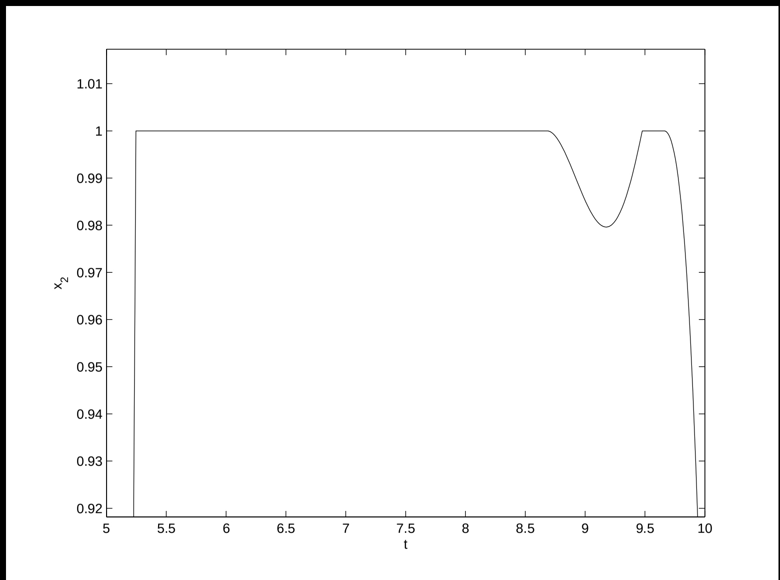
W = 2.9

Simulations

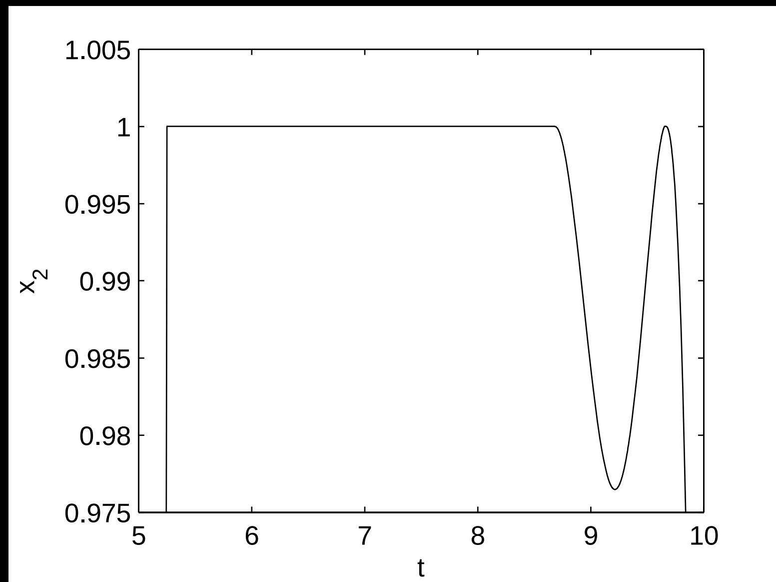


Grazing-sliding Bifurcation

Bifurcation at (A, w) = (0.5, 2.8942)



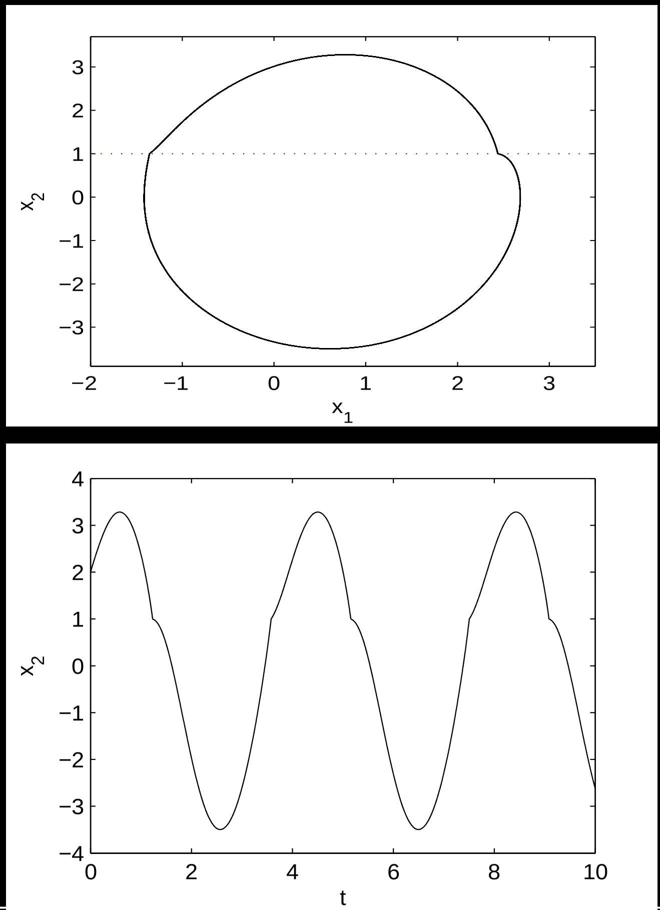
W = 2.895



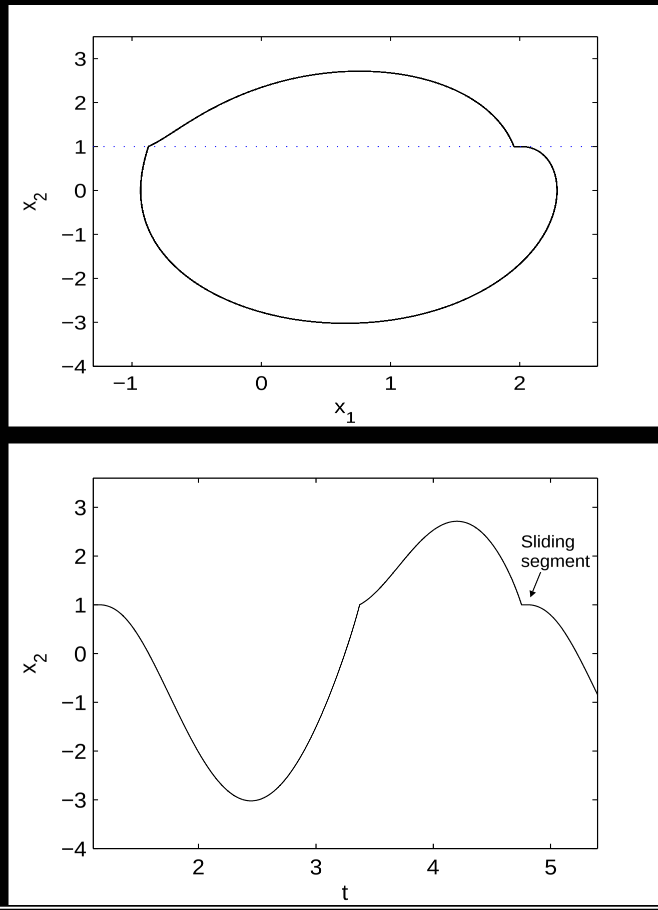
W = 2.893

Crossing-sliding Bifurcation

W = 1.6



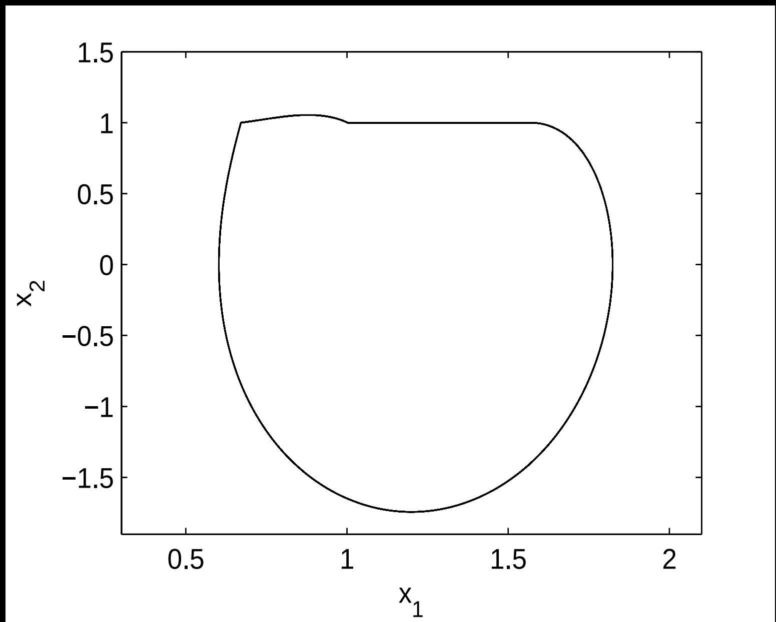
Bifurcation at (A, w) = (5, 1.65768)



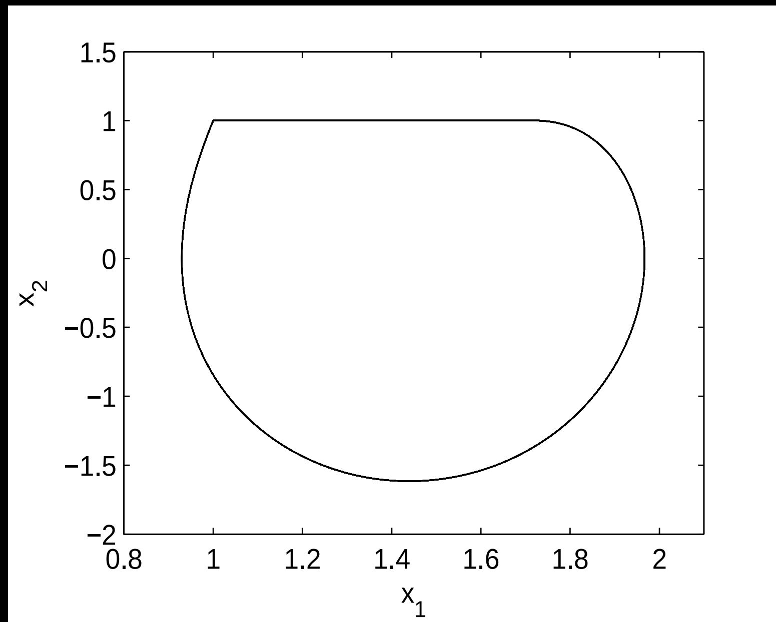
W = 1.72

Switching-sliding Bifurcation

Bifurcation at $(A, w) = (5, 2.7992)$

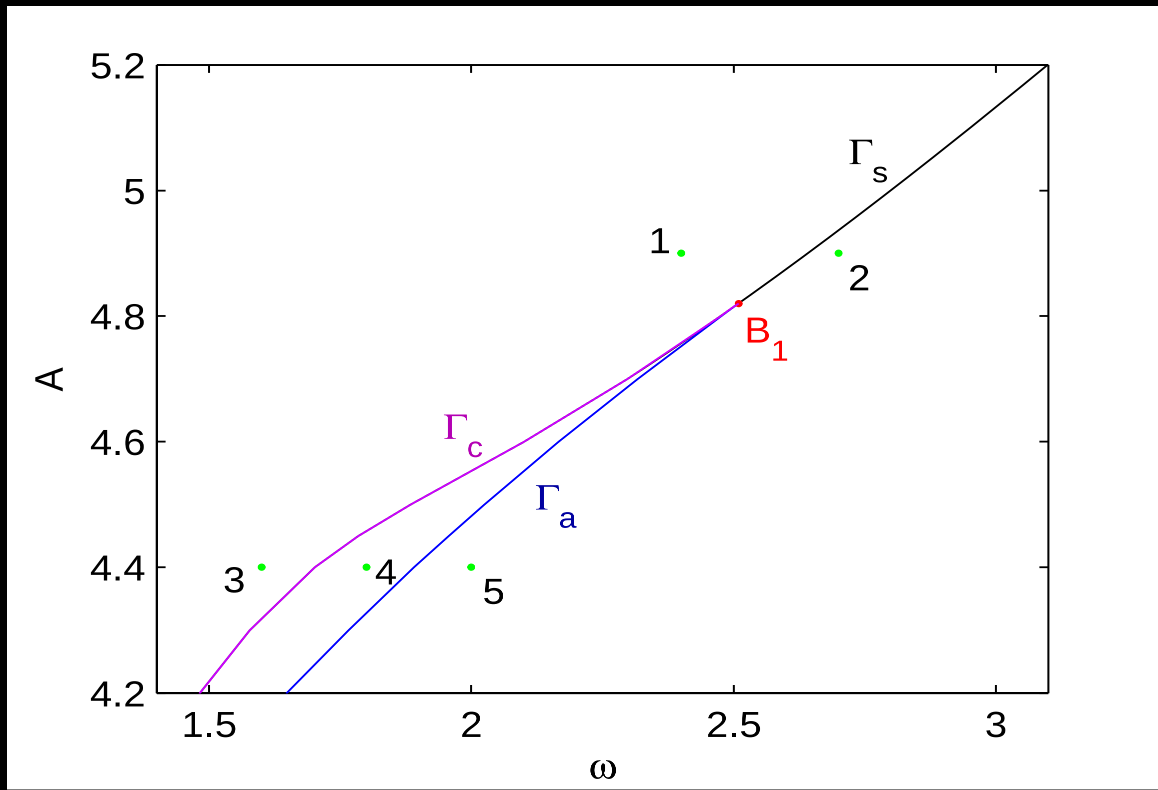


W = 2.5



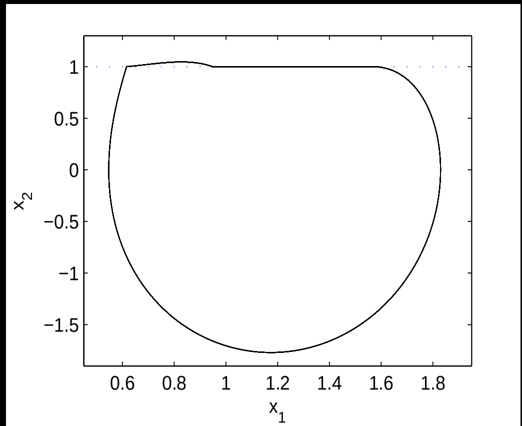
W = 2.8

Degenerate Switching-sliding Bifurcation

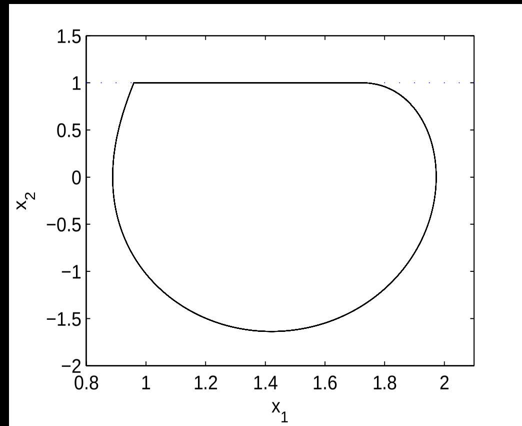


Degenerate Switching-sliding Bifurcation

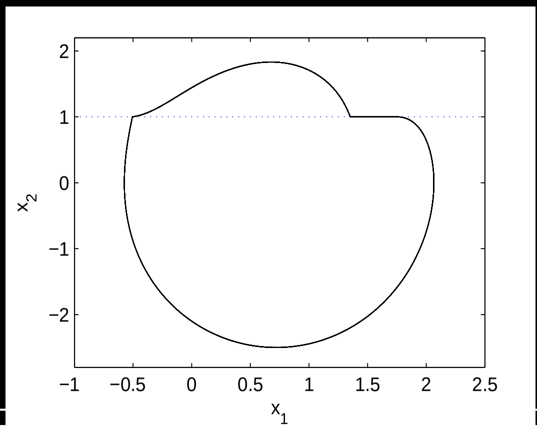
Point 1



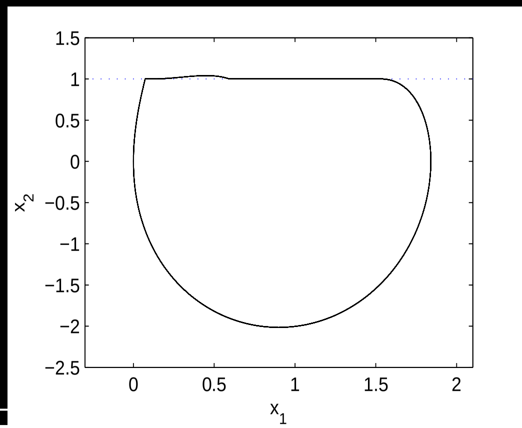
Point 2



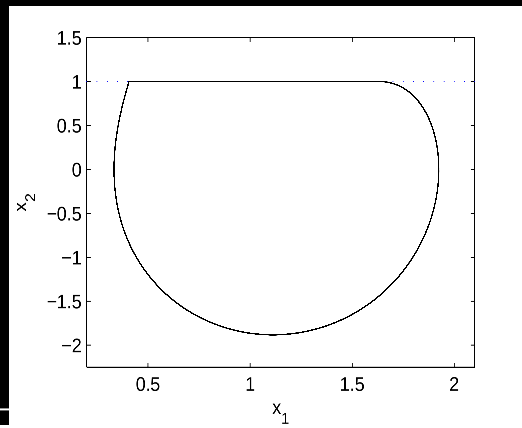
Point 3



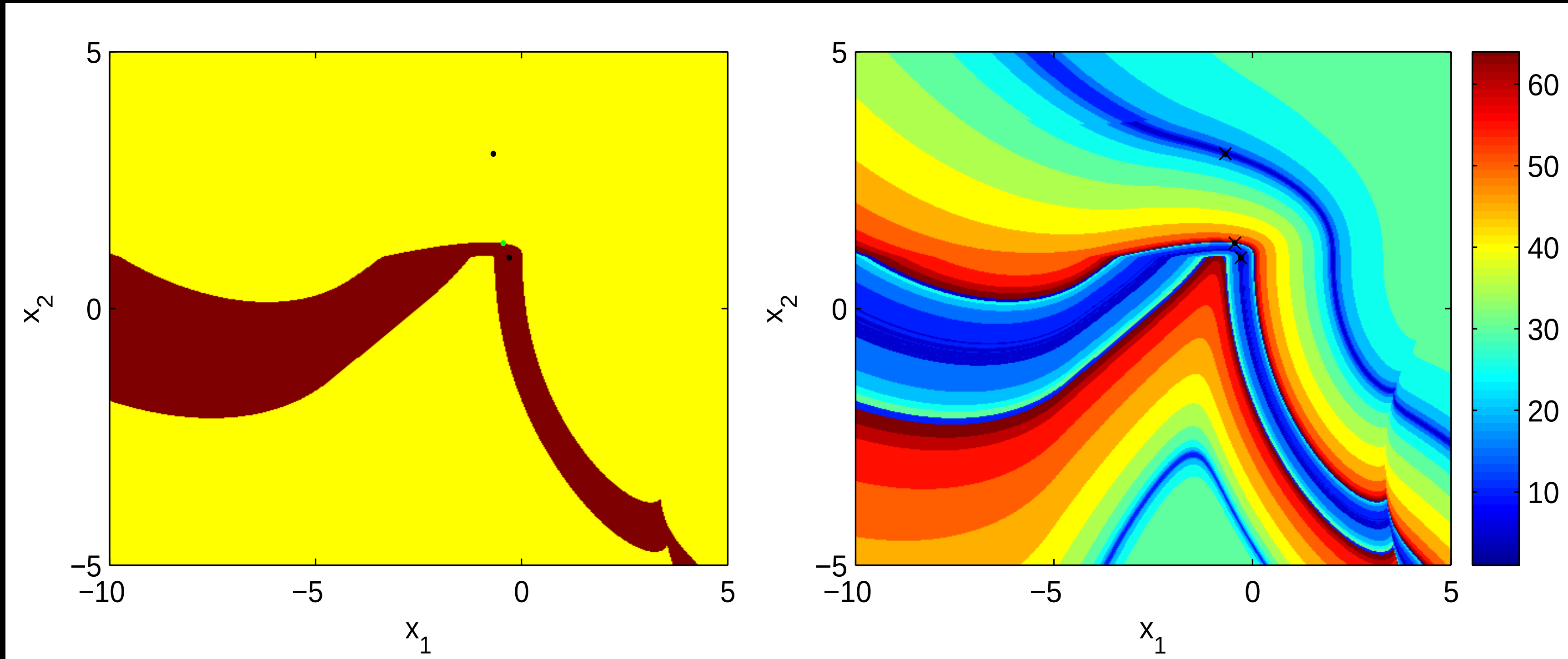
Point 4



Point 5

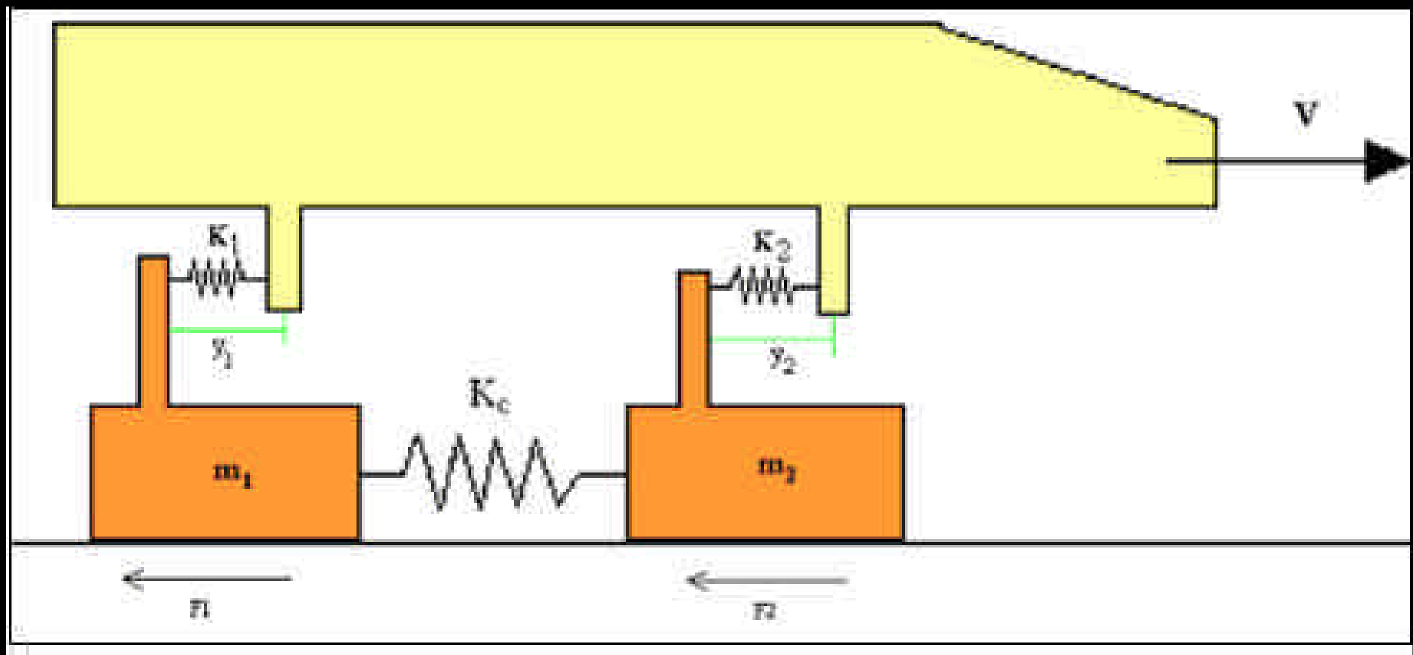


Domain of Attraction



A = 3.6　　W = 1.067

Bifurcations in a two-block stick-slip system



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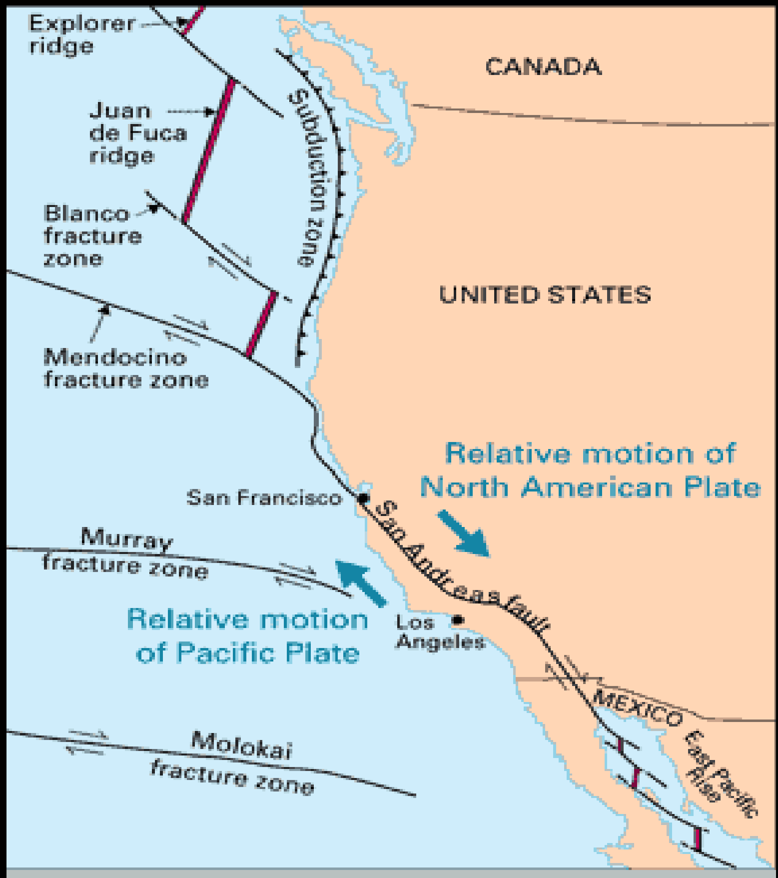

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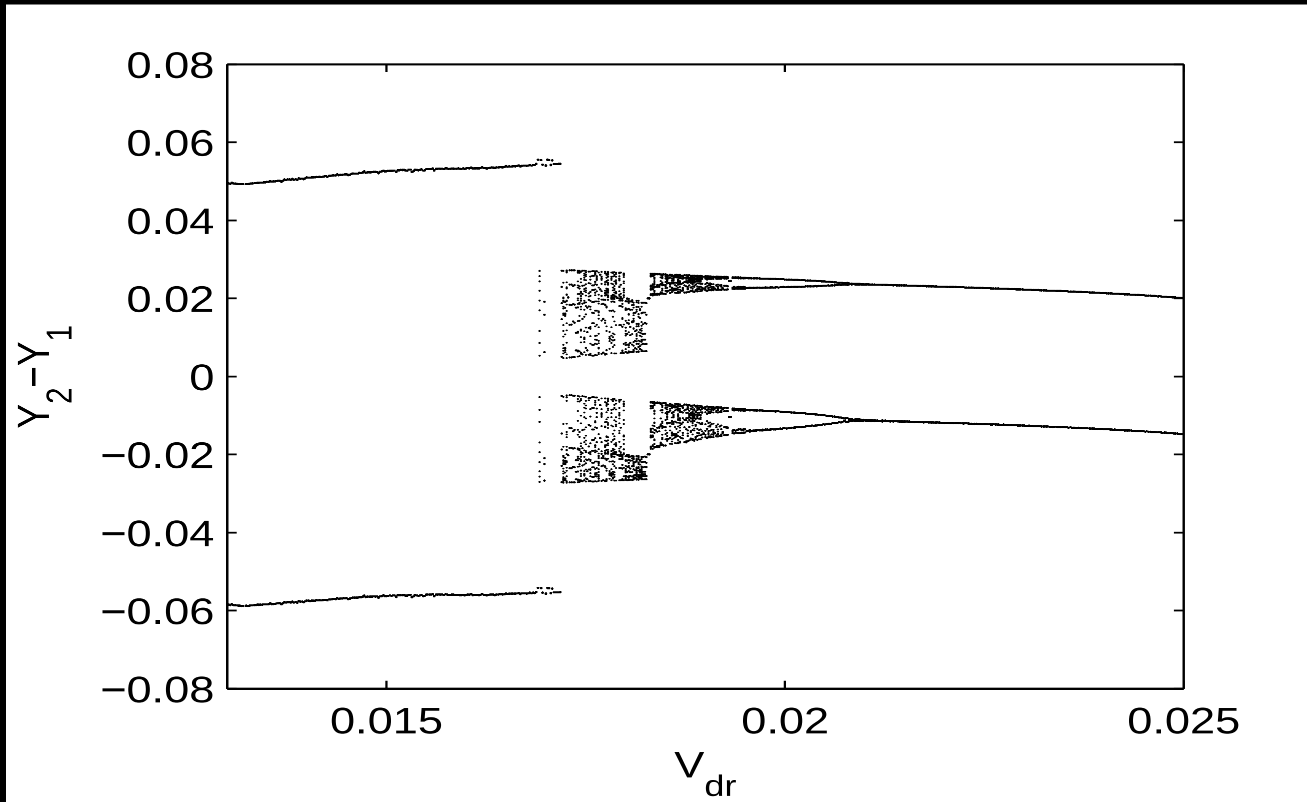

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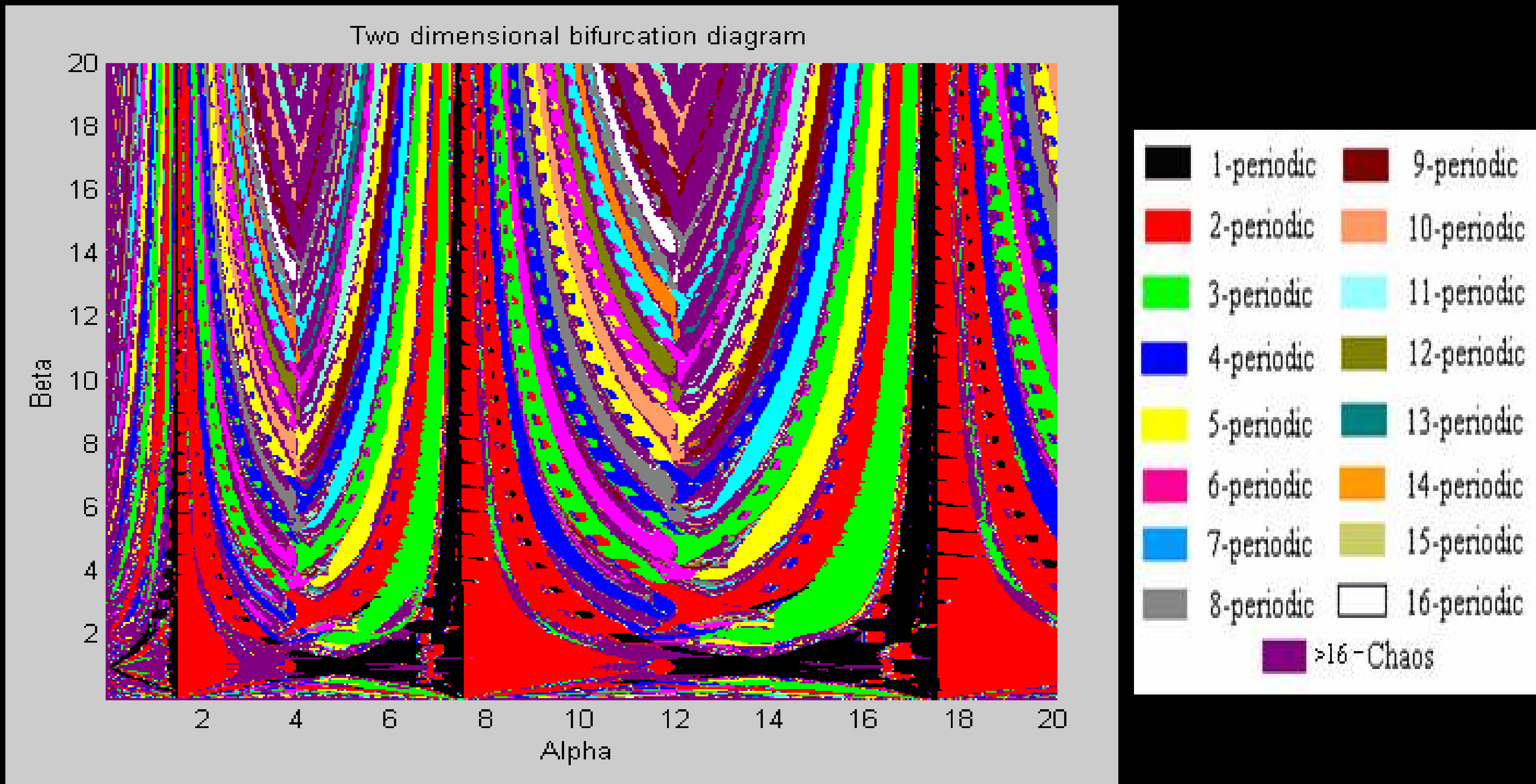

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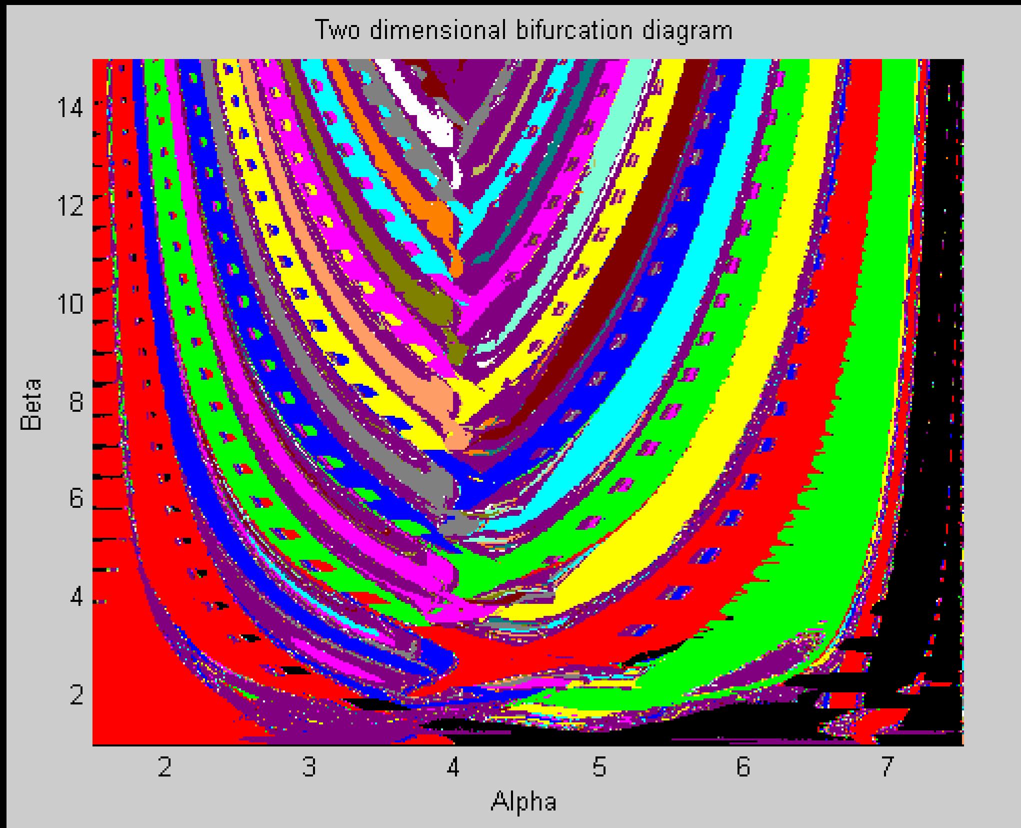

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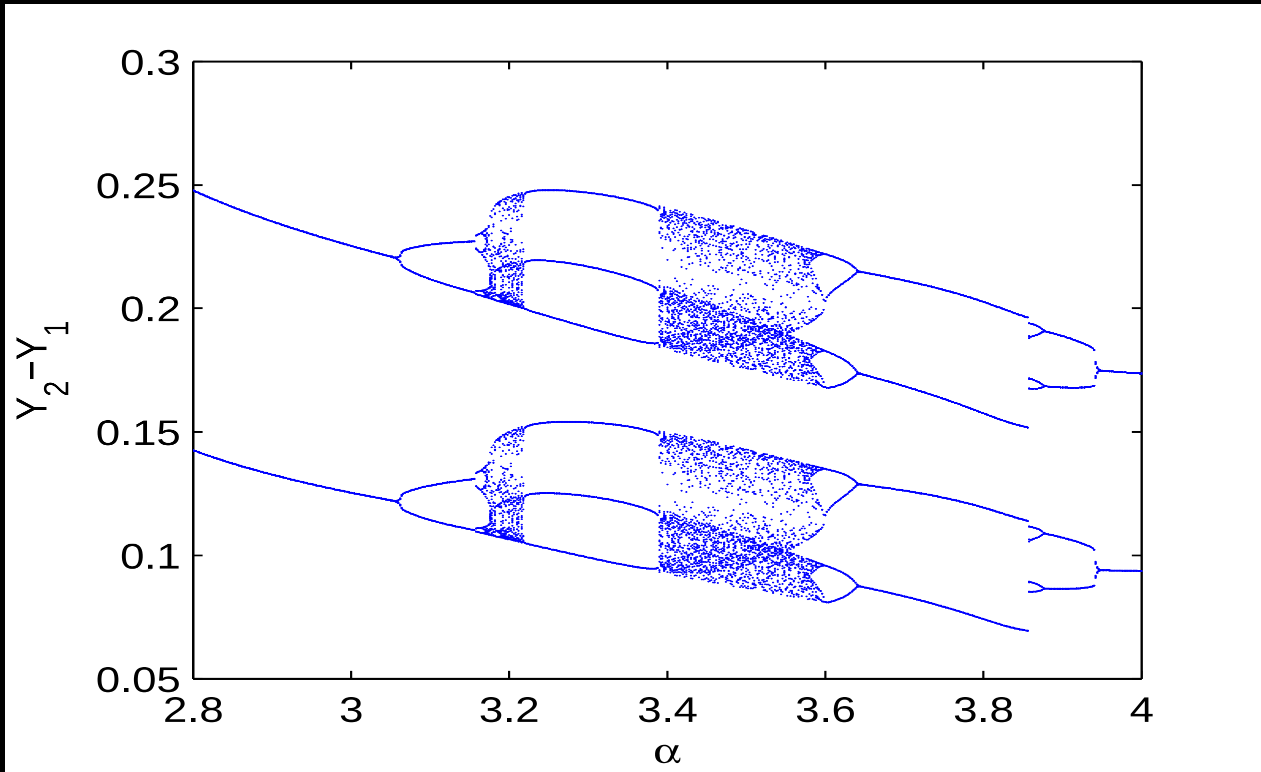
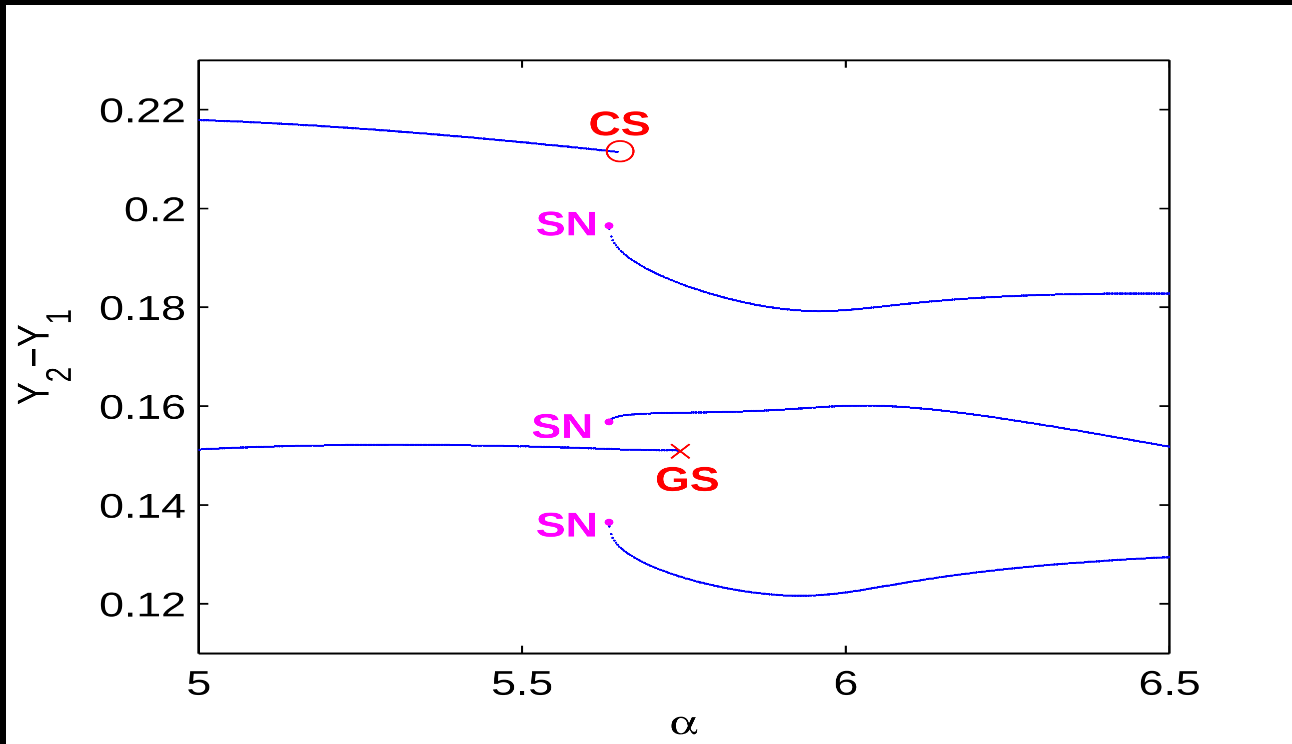
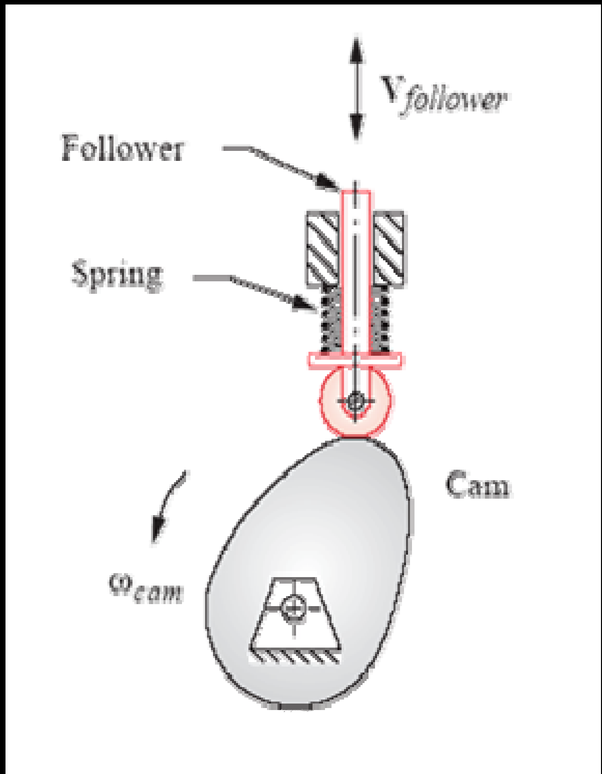
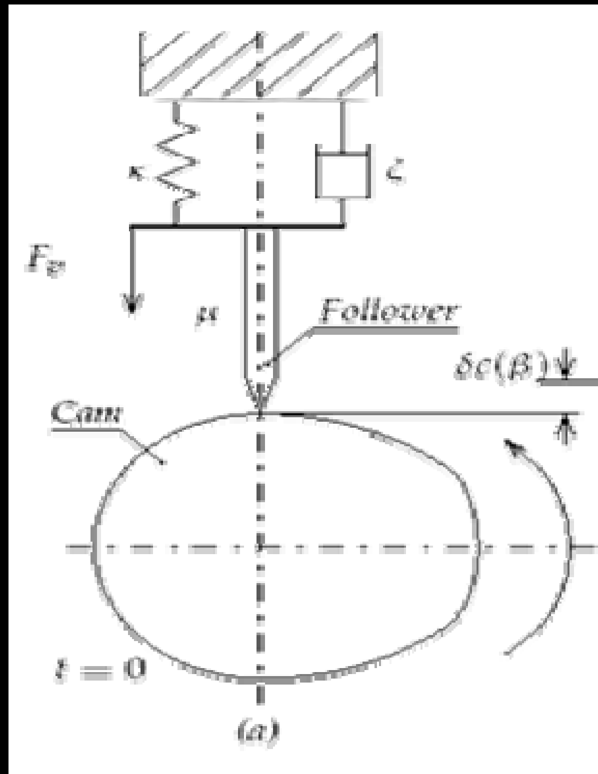


Bifurcations in a two-block stick-slip system



Bifurcations in a Cam Follower System

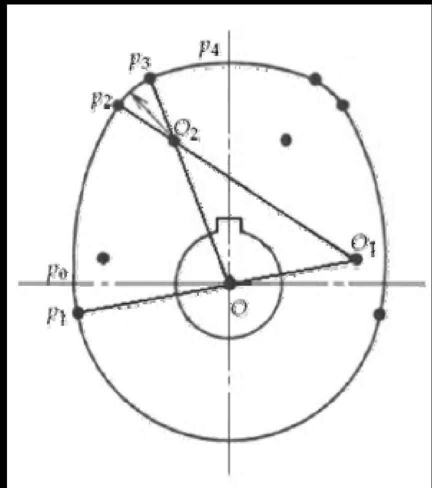




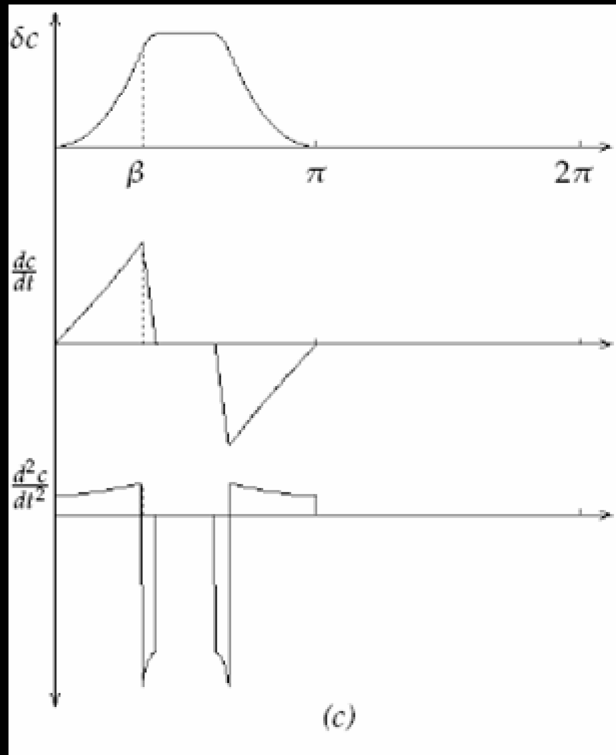
Cam Profile

The cam is PWS model because:

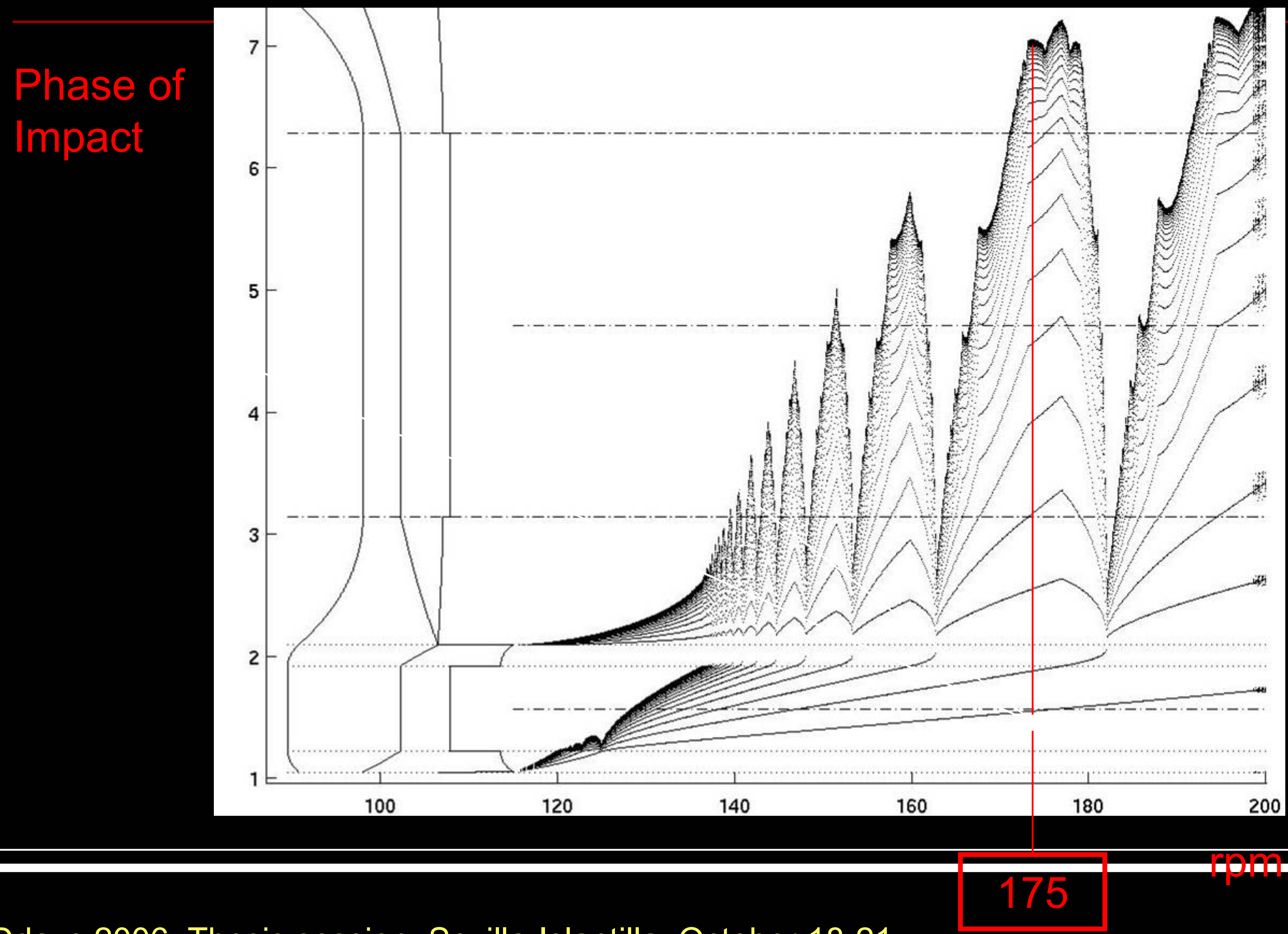
- There are impacts
- The discontinuous nature on the cam profile second derivative



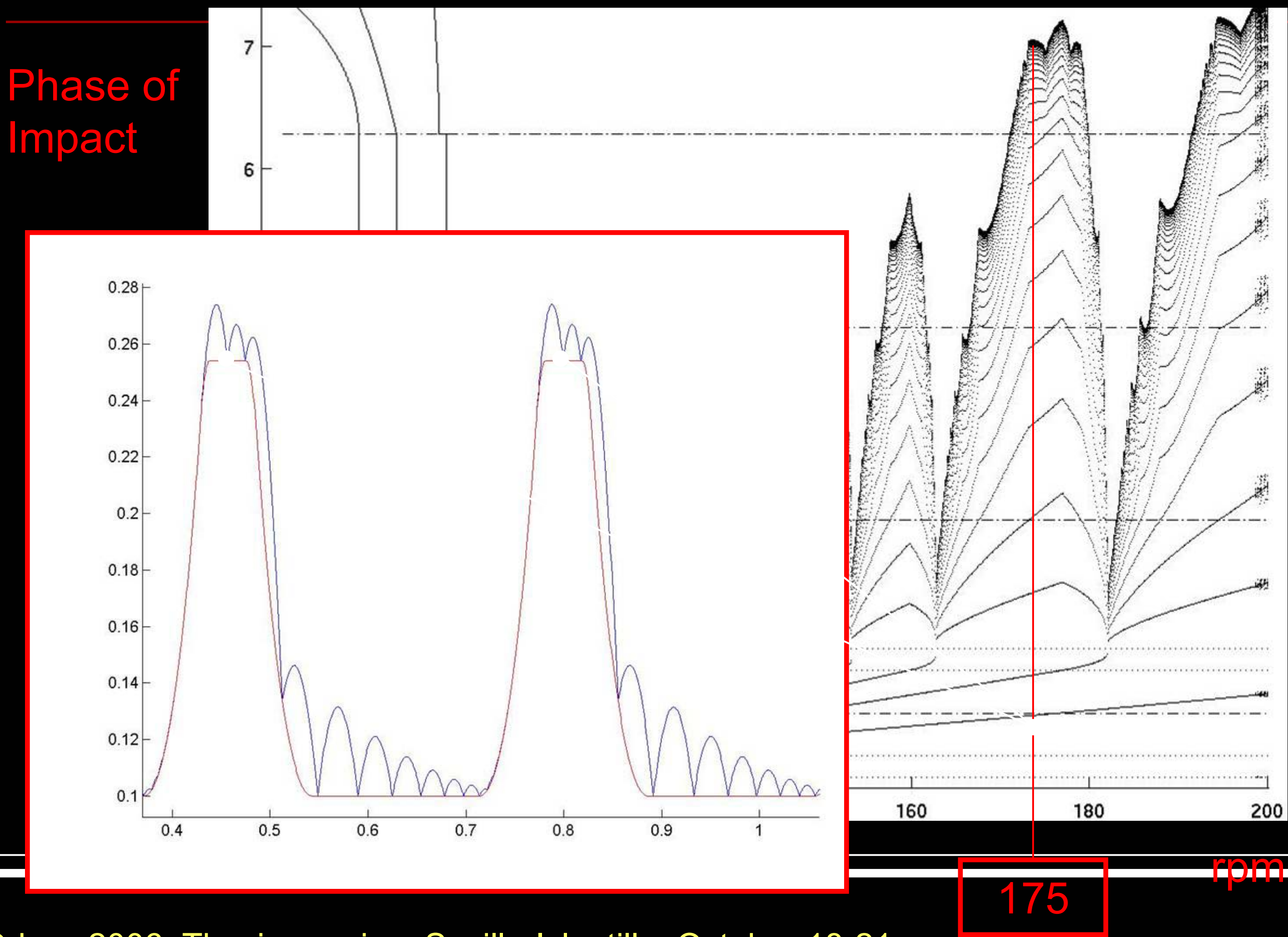
$$c(\beta) = \begin{cases} R_b & \text{If } 0 < \beta \leq \beta_1 \\ -k_1 s_{\beta\beta_1} + \rho_1\sqrt{1 - \frac{k_1^2}{\rho_1^2}c^2_{\beta\beta_1}} & \text{If } \beta_1 < \beta \leq \beta_2 \\ k_2 s_{\beta\beta_3} + \rho_3\sqrt{1 - \frac{k_2^2}{\rho_3^2}c^2_{\beta\beta_3}} & \text{If } \beta_2 < \beta \leq \beta_3 \\ \rho_2 & \text{If } \beta 3 < \beta \leq \pi \end{cases}$$



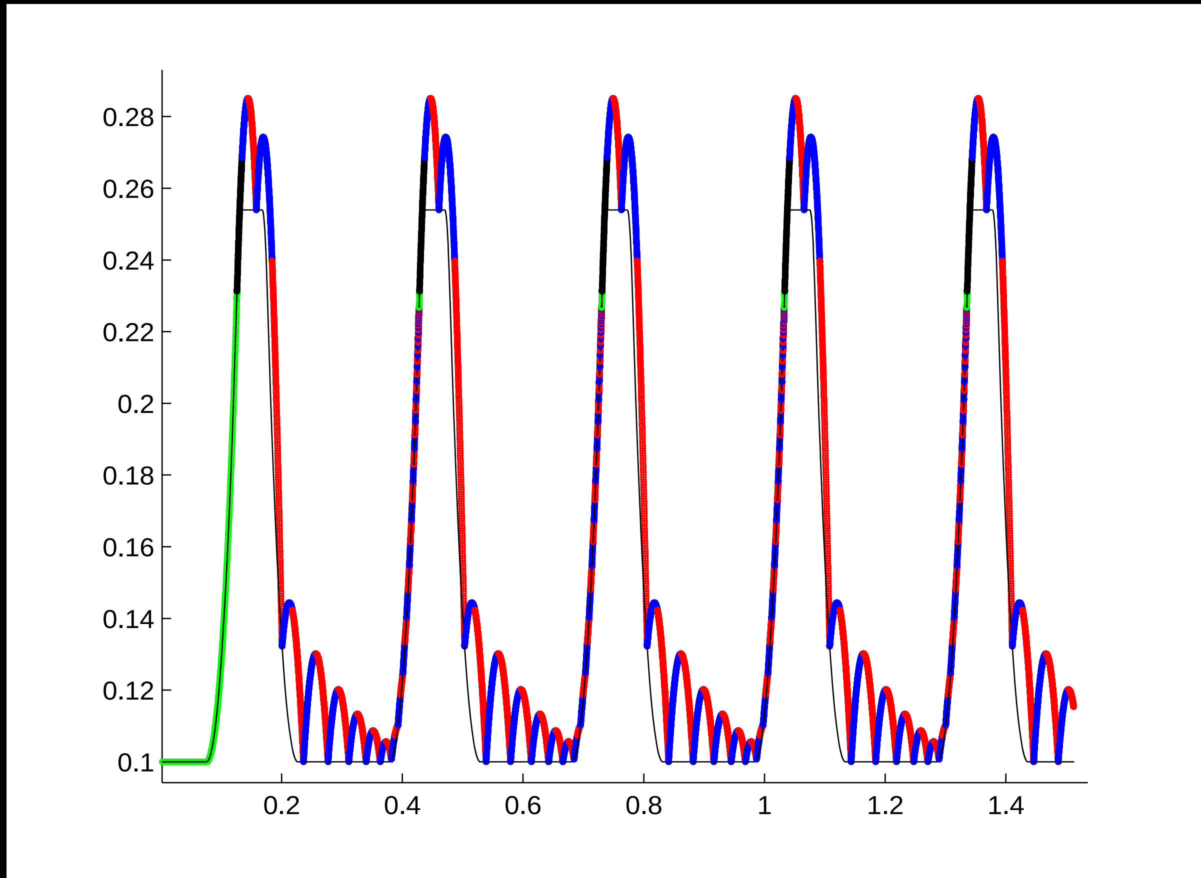
Bifurcation Analysis - Chattering orbits



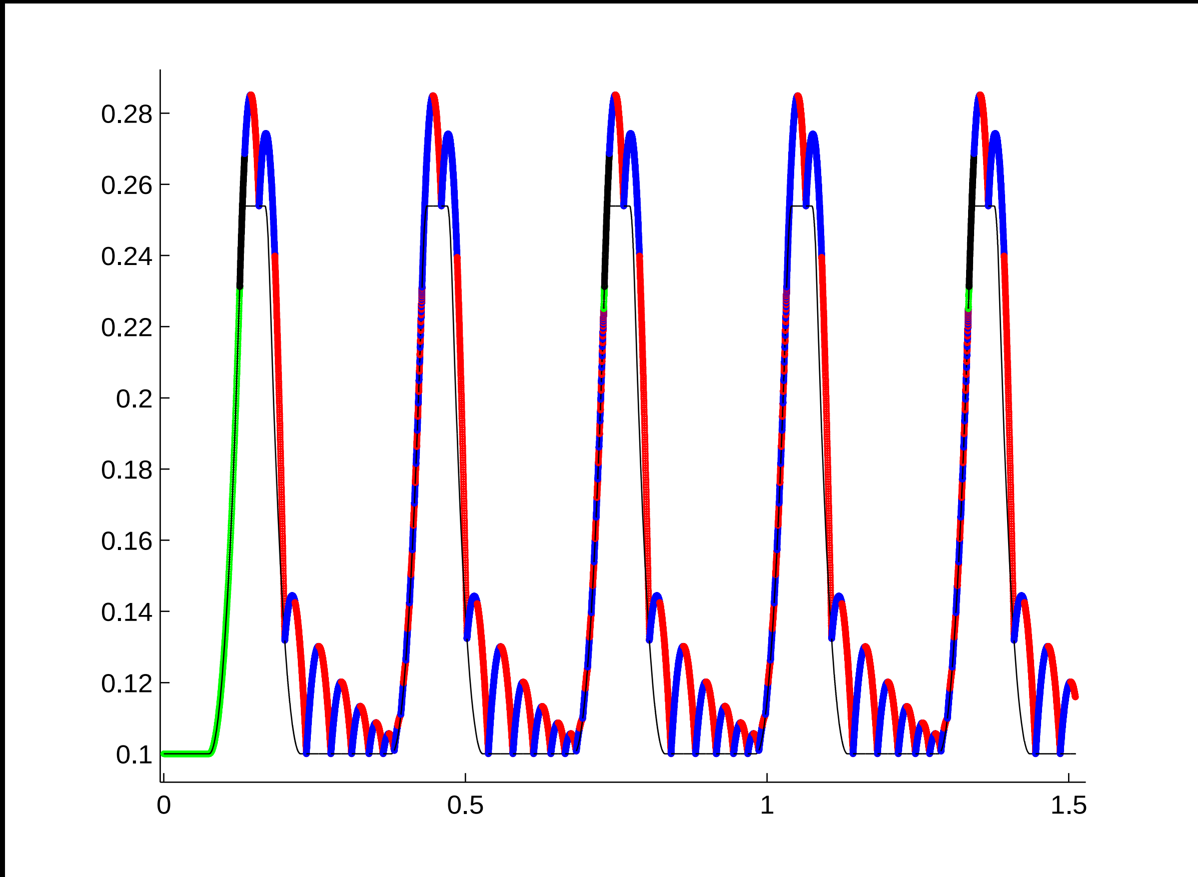
Bifurcation Analysis - Chattering orbits



Transition from complete chattering to no-complete chattering

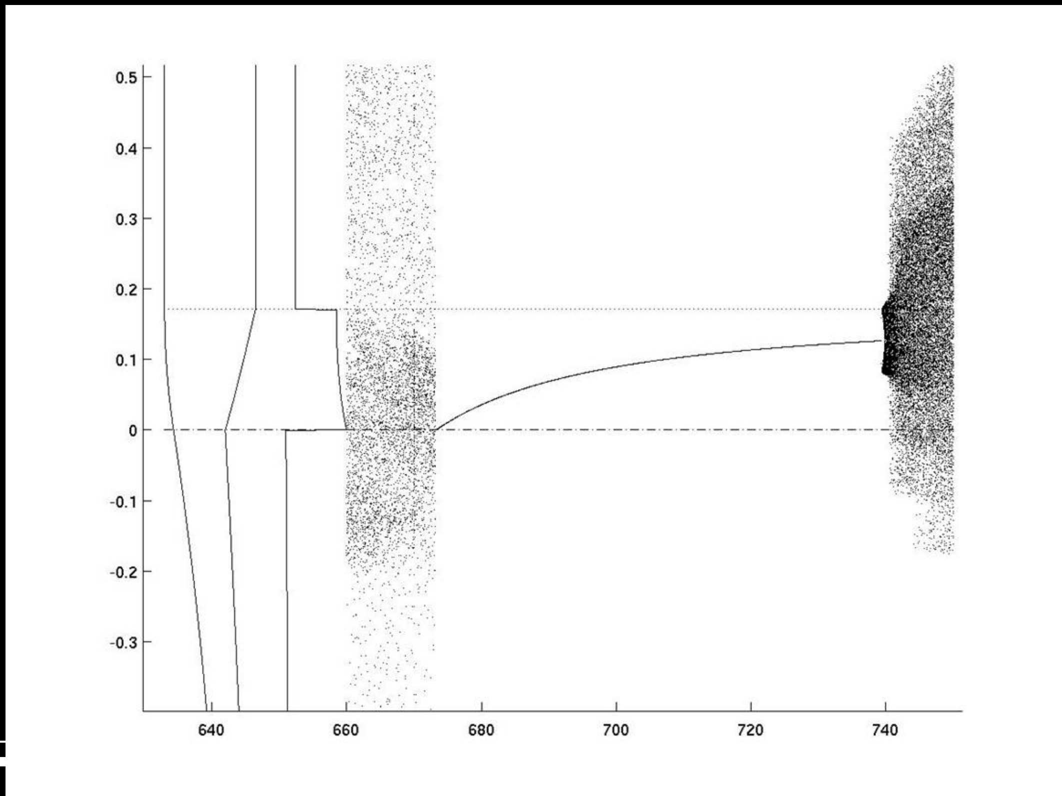


Transition from complete chattering to no-complete chattering



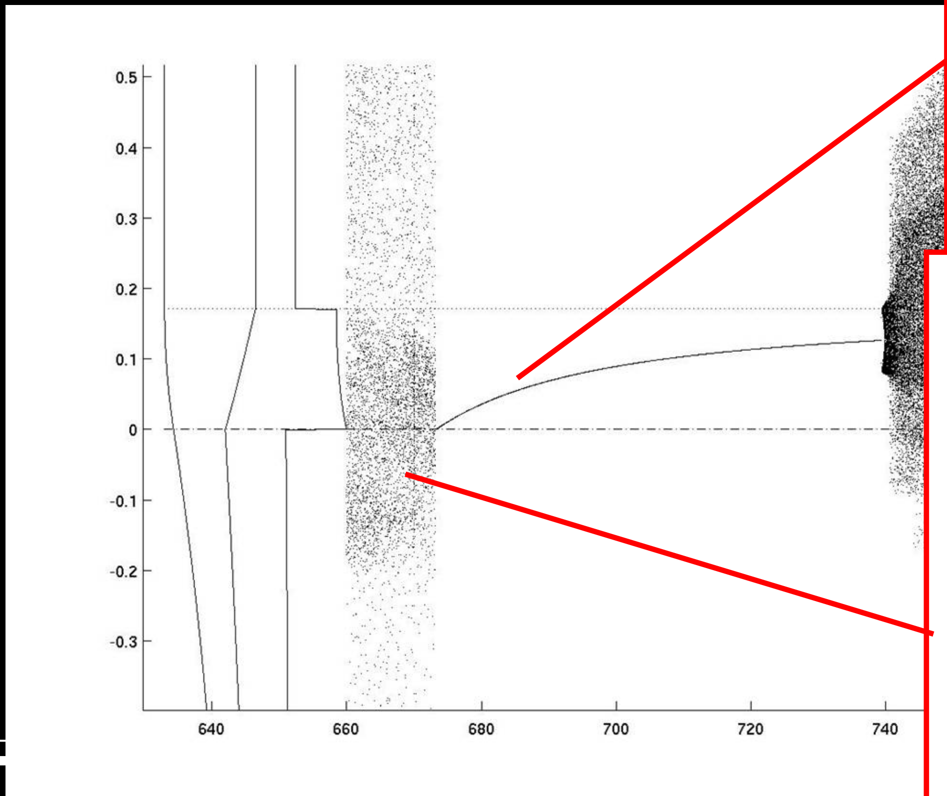
NonSmooth Bifurcation Scenario

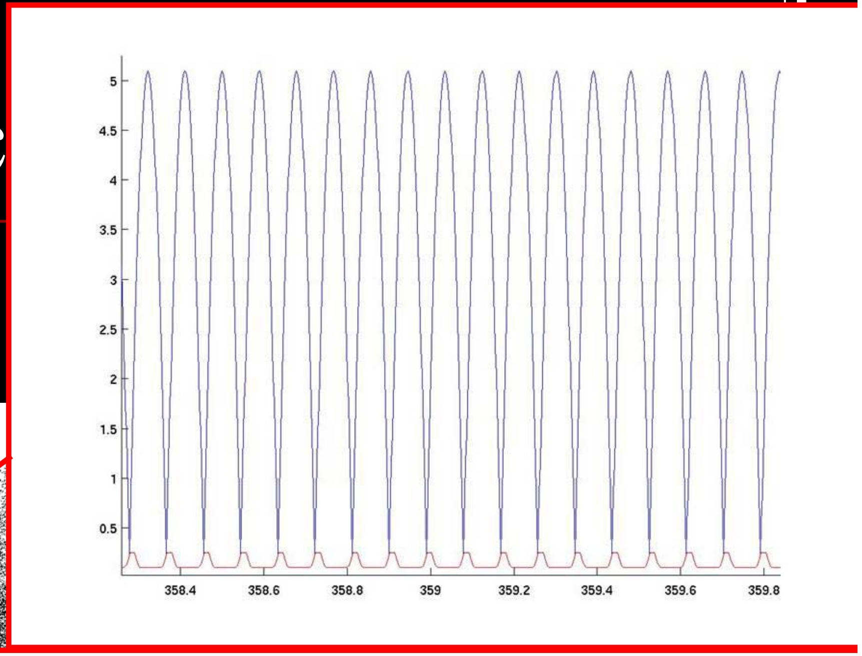
Corner Impact Bifurcation

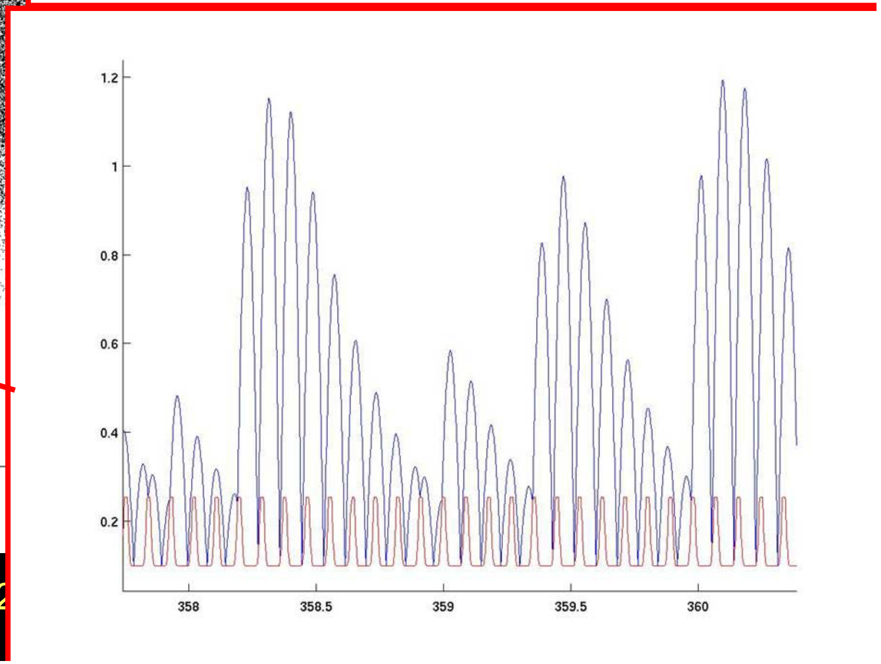


NonSmooth Bifurc

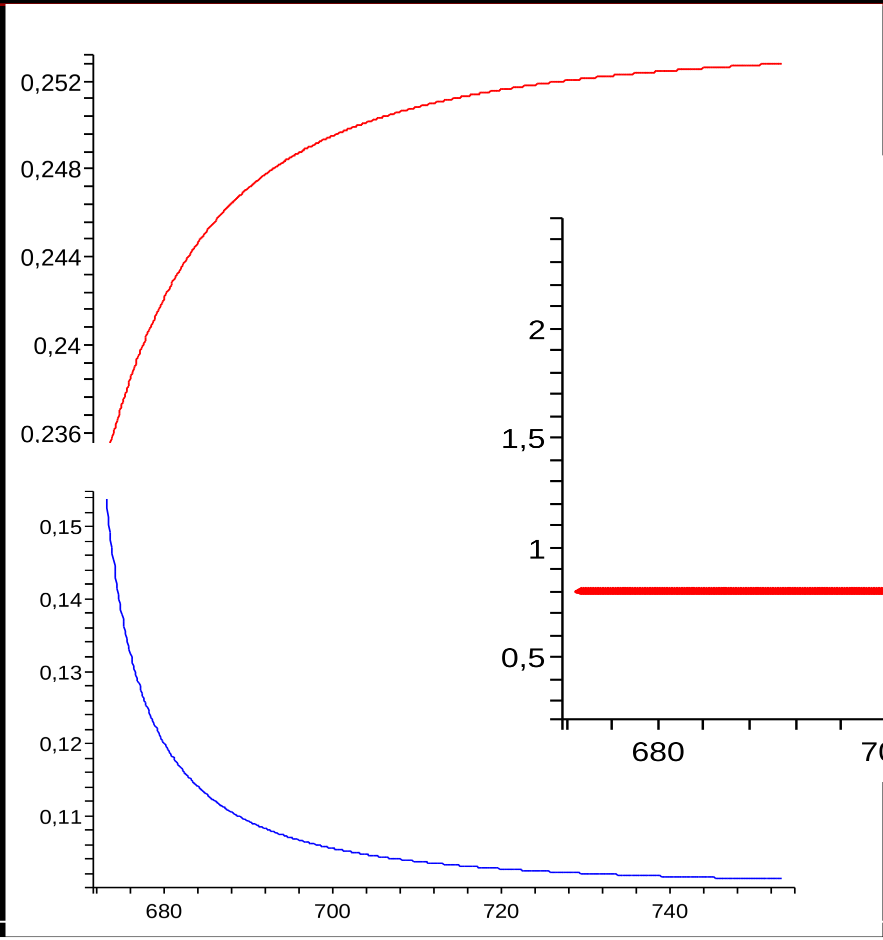
Corner Impact Bifurcation

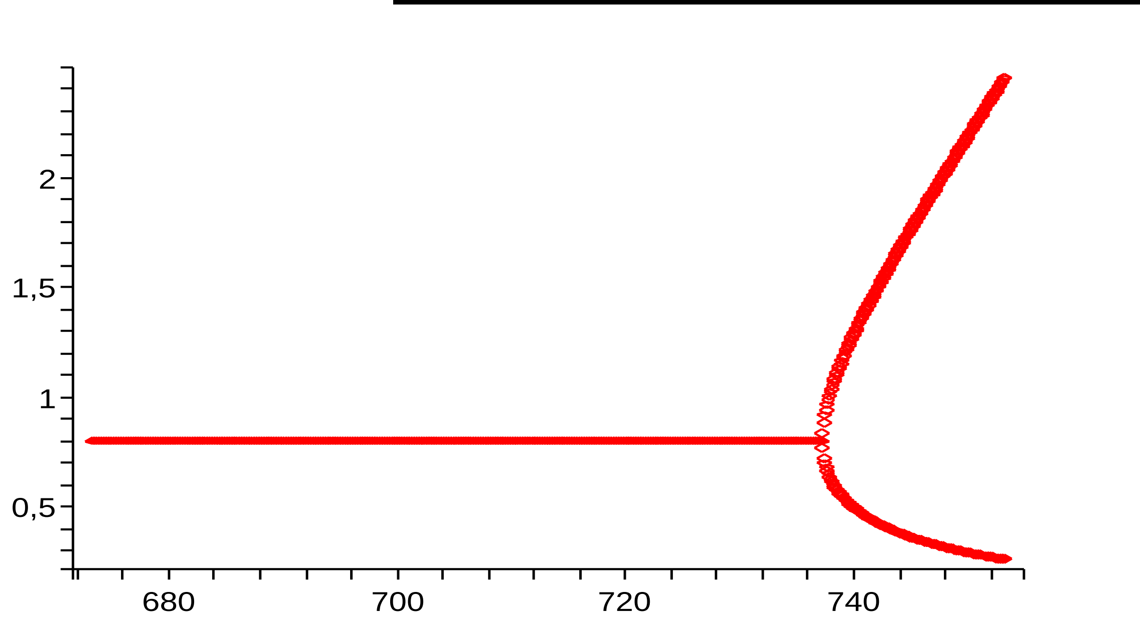




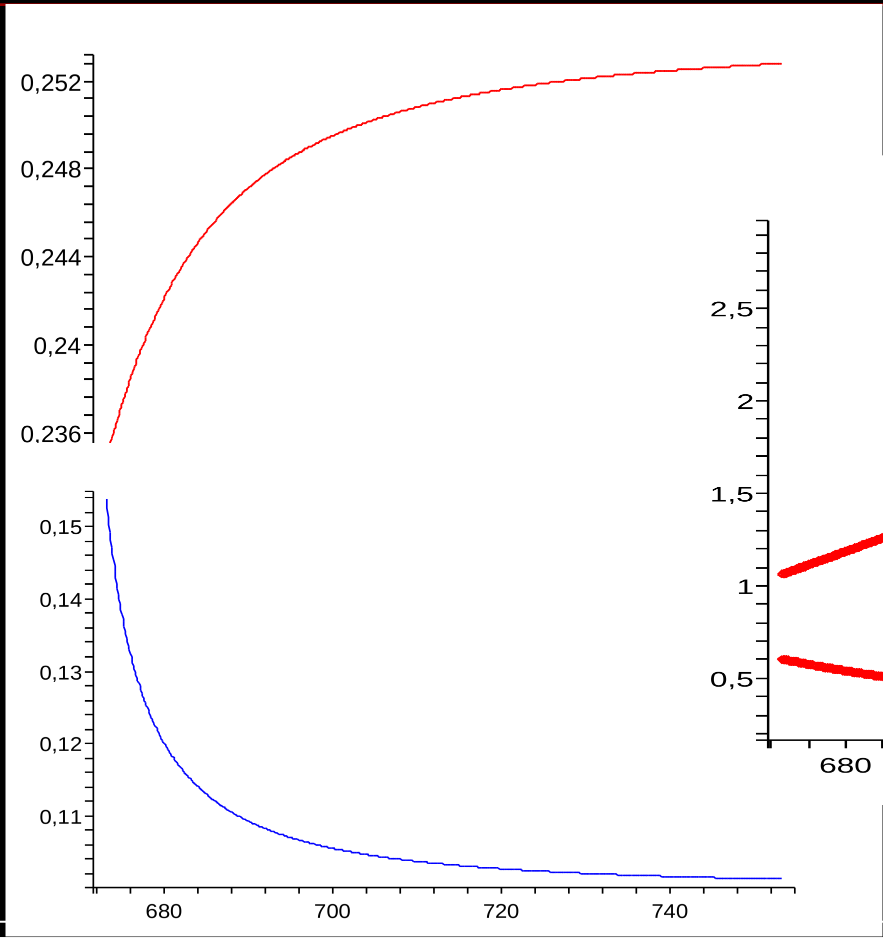


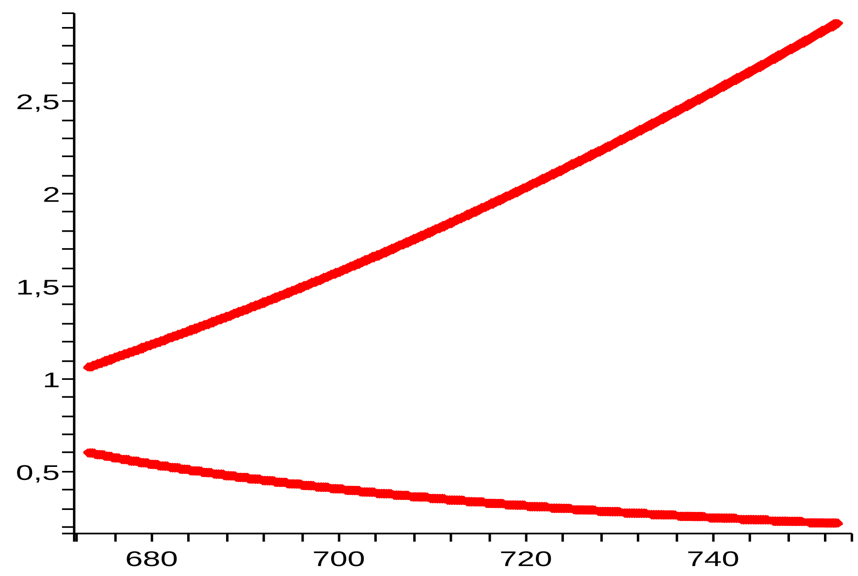
Continuation of periodic orbits



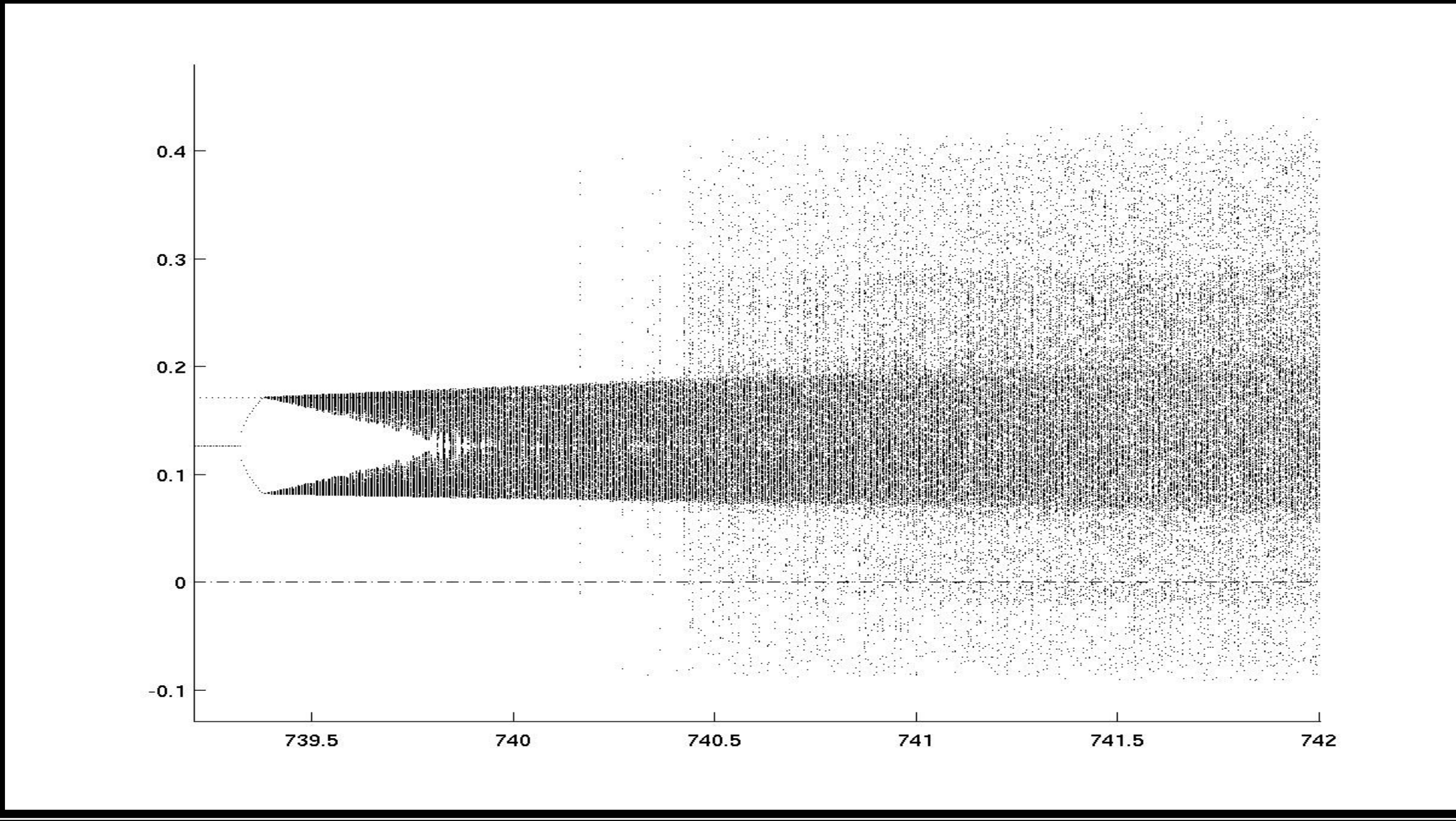


Continuation of periodic orbits





Corner-impact bifurcation in a P(2,1:1)



GRACIAS POR SU ATENCIÓN