Escisión de una órbita heteroclínica y sistema “inner” en despliegues de la singularidad “Hopf-zero”

Inmaculada Baldomá Barraca [1]

Tere M. Seara [2]

[1] Dpt. Enginyeria Informàtica i Matemàtiques. **Universitat Roviri i Virgili**
inma.baldoma@urv.net

[2] Dpt. Matemàtica Aplicada I. **Universitat Politècnica de Catalunya**

Tere.M-Seara@upc.edu

Motivation. Hopf-zero singularity and Shilnikov bifurcation (I)

The Hopf-zero singularity

Assume that a family of vector field $X_\eta \colon \mathbb{R}^3 \to \mathbb{R}^3$, $\eta \in \mathbb{R}^k$, satisfies

- $X_0(0,0,0) = 0$.
-
$$X^* := DX_0(0,0,0) = \begin{pmatrix} 0 & -\alpha^* & 0 \\ \alpha^* & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Motivation. Hopf-zero singularity and Shilnikov bifurcation (II)

Taking into account only the linear part, X^* is generic into a linear family X^*_η if

- Dissipative case: $k = 2$ (codimension two). Eigenvalues of X^*_η are of the form λ, $\mu \pm \mathrm{i}\alpha^*$.
- Conservative case: $k = 1$ (codimension one). Since $\mathrm{tr}(X^*_\eta) = 0$, $\lambda = -2\mu$.

Motivation. Hopf-zero singularity and Shilnikov bifurcation (III)

Normal form

A theorem due to Broer (1980).

Let X_η: $\mathbb{R}^3 \to \mathbb{R}^3$ be a C^∞ family of vector fields having a Hopf zero singularity at $\xi = 0$ and $\eta = 0$, ($\eta \in \mathbb{R}^k$).

Then there exists a C^∞ change of coordinates such that X_η can be expressed as $X_\eta = \overline{X}_\eta + F_\eta$ with F_η a flat functions in $(0,0) \in \mathbb{R}^3 \times \mathbb{R}^k$ and $\overline{X}_\eta$ in cylindrical coordinates:

$$\begin{aligned} \dot\theta &= f(r^2, z, \eta) \\ \dot r &= r g(r^2, z, \eta) \\ \dot z &= h(r^2, z, \eta) \end{aligned} \tag{1}$$

$$f(0,0,0) = \alpha^*$$

$$g(0,0,0) = h(0,0,0) = \partial_z h(0,0,0) = 0.$$

Keys: Borel-Ritt theorem and formal normal form. This argument is not true in the analytic case.

Motivation. Hopf-zero singularity and Shilnikov bifurcation (IV)

The dynamics of the normal form, $\overline{X}_\eta$

Up to generic conditions about the terms of order two of the normal form (**??**):

- It has two fixed points $S_\pm(\eta)$ of saddle-focus type.
- The axis $x = y = 0$ contains a heteroclinic connection between $S_\pm(\eta) = (x_\pm, y_\pm, z_\pm)$:

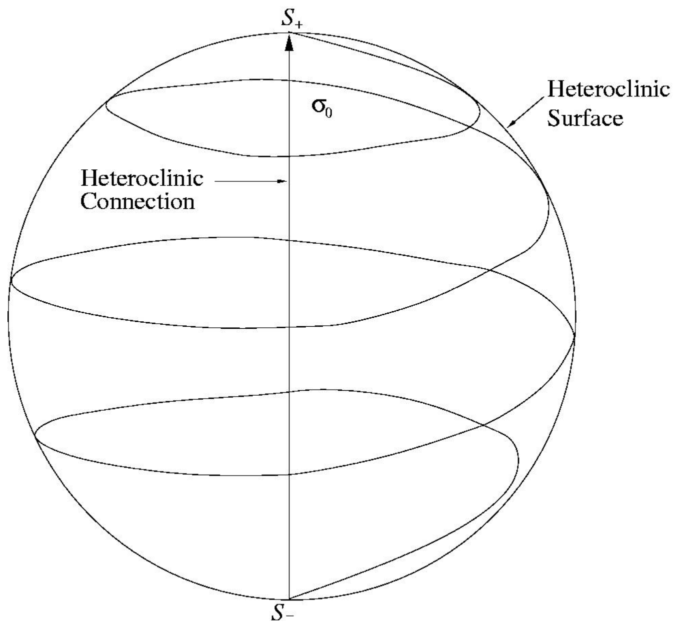
$$W^{\mathrm{s}}(S_-(\eta)) = W^{\mathrm{u}}(S_+(\eta)) = \{(0, 0, z),\ z_- < z < z_+\}$$

- There exists a curve Γ in the η-plane such that $W^{\mathrm{u}}(S_-(\eta)) = W^{\mathrm{s}}(S_+(\eta))$ is a heteroclinic surface for all $\eta \in \mathrm{graph}\Gamma$.

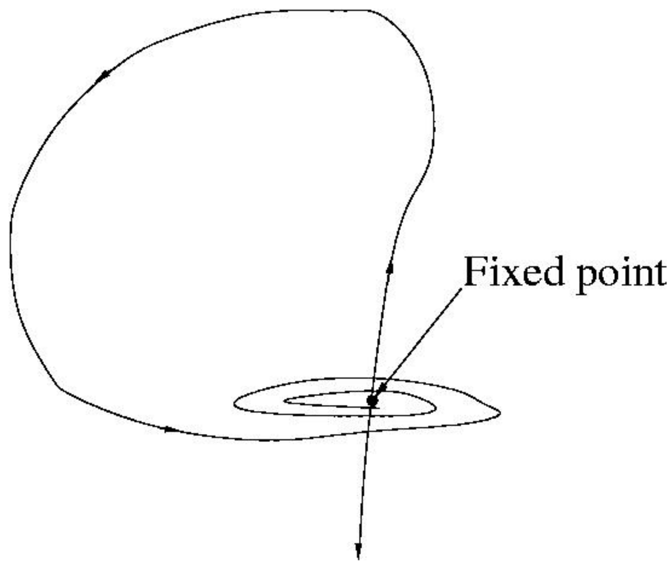
Motivation. Hopf-zero singularity and Shilnikov bifurcation (V)

Shilnikov bifurcation

The Silnikov bifurcation takes place when a critical point of saddle-focus type exists and its stable and unstable manifolds intersect given rise to a homoclinic orbit.



Dynamics of $\overline{X}_\eta$.



Shilnikov bifurcation

Motivation. Hopf-zero singularity and Shilnikov bifurcation (VI)

Trivially, the vector field $\overline{X}_\eta$ have no homoclinic connection.

Theorem (Broer-Vegter, 1984)

Given $\overline{X}_\eta$, there exist flat perturbation F_η such that the full system X_η= $\overline{X}_\eta$ $+F_\eta$ possesses a sequence of Silnikov's bifurcations taking place at a sequence of $\eta_n \in \Gamma$ accumulating at $\eta = 0$.

Motivation. Hopf-zero singularity and Shilnikov bifurcation (VII)

Keys of the proof:

To prove such result is necessary to check that

- The **heteroclinic connection disappears.**
- The two dimensional **stable and unstable manifolds intersect transversally.**

Our work is a very first step towards to prove that a similar result is valid in the analytic case.

Our problem (I)

For simplicity, let us assume that we are in the conservative case.

Let X_μ be a smooth vector field passing through the singularity X^* at $\mu = 0$, $\xi = 0$.

Performing the normal form procedure up to order two and after some scalings and changing the parameter if necessary, we get a system of the form

$$\begin{aligned}
\frac{dx}{dt} &= -xz - y\left(\frac{\alpha}{\delta} + cz\right) + \varepsilon\delta^{-2} f(\delta x, \delta y, \delta z, \delta) \\
\frac{dy}{dt} &= -yz + x\left(\frac{\alpha}{\delta} + cz\right) + \varepsilon\delta^{-2} g(\delta x, \delta y, \delta z, \delta) \\
\frac{dz}{dt} &= -1 + b(x^2 + y^2) + z^2 + \varepsilon\delta^{-2} h(\delta x, \delta y, \delta z, \delta),
\end{aligned}$$

with $\varepsilon = 1$ and f, g, h real analytic functions in all their variables, whose Taylor series begin at least with terms of degree three.

Our problem (II)

When $\varepsilon = 0$, the system has a heteroclinic connection between the critical points $(0, 0, \pm 1)$ parameterized by

$$\{(0, 0, -\tanh t); \quad -\infty < t < +\infty\}$$

which has singularities at $\pm \mathrm{i}\pi/2 + k\pi \mathrm{i}$.

The full system has critical points $S^{\pm}(\delta)$ of saddle-focus type. The fixed points $S^{\pm}(\delta)$ have unidimensional invariant manifolds $\sigma^{\mathrm{s,u}} = (x^{\mathrm{s,u}}, y^{\mathrm{s,u}}, z^{\mathrm{s,u}})$.

Our problem (III). The regular case

When $\varepsilon = \delta^{p+2}$, $p > -2$, it has been proved that

$$x^{s,u}(t), y^{s,u}(t) \sim C\delta^{p+4} \left| t \mp \mathrm{i}\frac{\pi}{2} \right|^{-3}, \quad z^{s,u}(t) \sim C \left| t \mp \mathrm{i}\frac{\pi}{2} \right|^{-1} + C\delta^{p+3} \log \delta \left| t \mp \mathrm{i}\frac{\pi}{2} \right|^{-2}$$

These estimates indicate that when $|t \pm \mathrm{i}\pi/2| = O(\delta)$, then

$$x^{s,u}(t), y^{s,u}(t), z^{s,u}(t) \sim O(\delta^{p+1})$$

Our problem (IV). The singular case

When ε is not small (for instance $\varepsilon = 1$), the previous results do not work.

Even this, they seems to indicate that if $|t \pm \mathrm{i}\pi/2| = O(\delta)$, we will have that

$$x^{\mathrm{s,u}}(t), y^{\mathrm{s,u}}(t), z^{\mathrm{s,u}}(t) \sim O(\delta^{-1})$$

To study the manifolds $\sigma^{\mathrm{s,u}}$ around the singularity $\mathrm{i}\pi/2$, we perform the change

$$\psi = \delta(x + \mathrm{i}y),\ \ \varphi = \delta(x - \mathrm{i}y),\ \ \eta = \delta z,\ \ \tau = \frac{t - \mathrm{i}\pi/2}{\delta}.$$

We obtain a new system

$$\begin{aligned}
\frac{d\psi}{d\tau} &= \big(-(\alpha + c\eta)i - \eta\big)\psi + \varepsilon \tilde{F}_1(\psi, \varphi, \eta, \delta) \\
\frac{d\varphi}{d\tau} &= \big((\alpha + c\eta)i - \eta\big)\varphi + \varepsilon \tilde{F}_2(\psi, \varphi, \eta, \delta) \\
\frac{d\eta}{d\tau} &= -\delta^2 + b\psi\varphi + \eta^2 + \varepsilon \tilde{H}(\psi, \varphi, \eta, \delta)
\end{aligned}$$

Our problem (V). The inner equation

Taking $\delta = 0$ in the previous system we obtain

$$\begin{aligned}
\frac{d\psi}{d\tau} &= \big(-(\alpha + c\eta)i - \eta\big)\psi + \varepsilon F_1(\psi, \varphi, \eta) \\
\frac{d\varphi}{d\tau} &= \big((\alpha + c\eta)i - \eta\big)\varphi + \varepsilon F_2(\psi, \varphi, \eta) \\
\frac{d\eta}{d\tau} &= b\psi\varphi + \eta^2 + \varepsilon H(\psi, \varphi, \eta).
\end{aligned} \tag{2}$$

Which is the system under consideration.

In addition, one expects that $\sigma^{\mathrm{s,u}}$ will behave as

$$x^{\mathrm{s,u}}(\tau), y^{\mathrm{s,u}}(\tau) \sim \frac{1}{\delta\tau^3}, \quad z^{\mathrm{s,u}}(\tau) \sim \frac{1}{\delta\tau},$$

hence we look for solutions of (**??**) satisfying

$$\lim_{\mathrm{Re}\tau \to \pm\infty} \Phi^{\pm}(\tau) = 0, \quad \mathrm{Im}\, \tau < 0.$$

Our result (I)

System (**??**) has two analytic solutions $\Psi^{\pm}$ satisfying

$$\lim_{\mathrm{Re}\tau \to \pm\infty} \Psi^{\pm}(\tau) = 0, \quad \mathrm{Im}\,\tau < 0.$$

Let $\Delta\Psi = \Psi^{-} - \Psi^{+}$.

Then

$$\begin{pmatrix} \pi^{1,2}\Delta\Psi(\tau) \\ \tau^2\pi^3\Delta\Psi(\tau) \end{pmatrix} = \tau \mathrm{e}^{-\mathrm{i}(|\alpha|\tau - c\log\tau)}\varepsilon\big(C(\varepsilon) + \xi(\tau,\varepsilon)\big)$$

where $\xi(\tau, \varepsilon) \to 0$ as $\mathrm{Im}\tau \to -\infty$.

Moreover

$$\pi^{1,2}C(0) = (2\pi\mathrm{i}\hat{m}(\mathrm{i}\alpha), 0)$$

where $\hat{m}$ is the Borel transform of $z^{1+\mathrm{i}c}F_1(0, 0, -z, 0)$.

Our result (II)

Some remarks

- When $\varepsilon = \delta^{p+2}$, the previous result agree with the one obtained in the regular case which is:

$$\pi^{1,2}\Delta\Psi(t) \sim \begin{pmatrix} \mathrm{e}^{-\mathrm{i}\alpha t/\delta}\mathrm{e}^{\mathrm{i}c\log(\cosh t)}\cosh t\ c_1^0 \\ \mathrm{e}^{\mathrm{i}\alpha t/\delta}\mathrm{e}^{-\mathrm{i}c\log(\cosh t)}\cosh t\ c_2^0 \end{pmatrix}$$

with $c_2^0 = \overline{c_1^0}$ and

$$c_1^0 = 2\pi\mathrm{e}^{c\pi/2}\hat{m}(\mathrm{i}\alpha)\delta^p\mathrm{e}^{-\pi|\alpha|/(2\delta)}\mathrm{e}^{-\mathrm{i}c\log\delta} + O(\delta^{p+1})\mathrm{e}^{-\pi|\alpha|/(2\delta)}$$

- The dominant term depend on the full jet of f, g and h.
- Matching complex techniques will be required to prove that, when $\varepsilon = 1$, the distance between the invariant manifolds $\sigma^{\mathrm{s,u}}$ is dominated by $\Delta\Psi$.