Lo que he aprendido sobre ANCES desde los DDays2003

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Un poco de historia del “baile”

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Outline

- A revision of the definition.
- Some open problems on non autonomous quasiperiodically forced dynamical systems.

The start of the story

The term *Strange Non-chaotic attractor (SNA)* was introduced and coined in

[GOPY] C. Grebogi, E. Ott, S. Pelikan, and J. A. Yorke.
Strange attractors that are not chaotic.
Phys. D, 13(1-2):261–268, 1984.

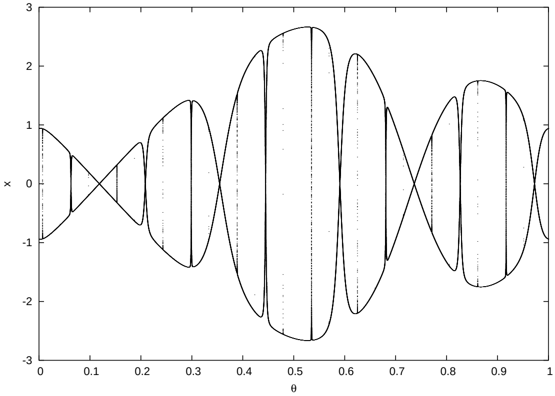
After this paper the study of these objects became rapidly popular and a number a papers studying different related models appeared.

A more complete version of this part of the talk can be found at
`http://mat.uab.es/~alseda/talks/`

The [GOPY] model (ω equals the golden mean)

(1) $$\begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= 2\sigma \tanh(x_n)\cos(2\pi\theta_n) \end{cases}$$

where $x \in \mathbb{R}, \theta \in \mathbb{S}^1$, $\omega \in \mathbb{R} \setminus \mathbb{Q}$ and $\sigma > 1$.



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- it is *non-chaotic* because *the Lyapunov exponents are non positive* (computed numerically).

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B– Since the line $x = 0$ intersects the attractor and it is invariant (a repellor), the basin of attraction of the attractor does not contain an open set.

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- The notion of SNA is neither unified nor precisely formulated
- The existence of SNA, usually, is not proved rigorously. Some authors just give very rough/rude numerical evidences of their existence that easily can turn out to be wrong.
- The theoretical tools to study these objects and derive these consequences, are often used in a wrong way.

On the positive side there are few works where the existence of an SNA is rigorously proved. Mainly:

[BO] Z. I. Bezhaeva and V. I. Oseledets.
On an example of a “strange nonchaotic attractor”.
Funktsional. Anal. i Prilozhen., 30(4):1–9, 95, 1996.

[Kel] G. Keller.
A note on strange nonchaotic attractors.
Fund. Math., 151(2):139–148, 1996.

Aims

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- propose a rigorous definition for the notion of SNA in the topological setting.
- discuss some methodological aspects relative to the non-chaoticity part of the definition.

The notion of attractor

We use the definition of attractor proposed by Milnor in

J. Milnor.
On the concept of attractor.
Comm. Math. Phys., 99(2):177–195, 1985.
(Erratum: Comm. Math. Phys, **102(3)** (1985), 517–519).

The closed set $\mathcal{A}$ is an *attractor* provided its *realm of attraction* $\rho(\mathcal{A}) := \{x \ : \ \omega(x) \subset A\}$ has positive Lebesgue measure and there is no strictly smaller closed set $\mathcal{A}' \subset \mathcal{A}$ so that $\rho(\mathcal{A}')$ coincides with $\rho(\mathcal{A})$ (up to sets of measure zero).

Towards the definition of strangeness

It has been proved by

J. Stark.
Invariant graphs for forced systems.
Phys. D, 109(1-2):163–179, 1997.
Physics and dynamics between chaos, order, and noise (Berlin, 1996).

that the invariant curves of models of type

$$\begin{cases} \theta_{n+1} & = \theta_n + \omega \pmod 1, \\ x_{n+1} & = \psi(\theta_n, x_n) \end{cases}$$

are the graph of a correspondence from $\mathbb{S}^1$ to $\mathbb{R}$.

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The three definitions above are used in articles where two-dimensional systems are studied, while for higher-dimensional system only the first definition is used.

A remark in the two dimensional case

Using elemental dimension theory one can prove that the Hausdorff dimension of the graph of a one-dimensional piecewise differentiable map from $\mathbb{S}^1$ to $\mathbb{R}$ is one.

Therefore, in the two dimensional case the definition (A) above is the most general one.

This justifies the choice of the following

The notion of strangeness

An attractor is called *strange* when it is not a finite set of points neither a piecewise differentiable manifold.

A manifold M is *piecewise differentiable* if there exists a finite set of disjoint differentiable submanifolds $A_1, \ldots, A_k$ such that

$$M \subset \mathrm{Cl}(\cup_{i=1}^{k} A_i).$$

If M has boundary, then it must be piecewise differentiable too.

A usual argument in the literature

Recall that

$$\begin{cases} \theta_{n+1} & = \theta_n + \omega \quad (\text{mod } 1), \\ x_{n+1} & = \psi(\theta_n, x_n) \end{cases}$$

The usual procedure is to compute numerically the vertical Lyapunov exponent:

$$\lambda_v(\theta_0, x_0) = \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right|$$

and call the system *non-chaotic* if $\lambda_v(\theta_0, x_0) < 0$.

Theoretical explanation (in dimension 2 to simplify) — Oseledec's Theorem

Assume that μ is an ergodic measure of the system. Then, according to the Oseledec's Theorem, μ almost every point (θ_0, x_0) is *regular*. A regular point verifies the following properties:

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(R1) the above limit exists and takes a value $\lambda_v = \lambda_v(\theta_0, x_0)$; which it is independent on the choice of the point.

(R2) for all $v \in T_{(\theta_0, x_0)}\mathbb{S}^1 \times \mathbb{R}$, the limit

$$\lim_{n\to\infty} \frac{1}{n} \log \|M(\theta_n, x_n)v\| \quad \text{where} \quad M(\theta_n, x_n) = \begin{pmatrix} \frac{\partial\theta_n}{\partial\theta} & \frac{\partial\theta_n}{\partial x} \\ \frac{\partial x_n}{\partial\theta} & \frac{\partial x_n}{\partial x} \end{pmatrix}$$

takes at most two different values λ_v and another one (which may coincide with λ_v), that we will denote by $\widehat{\lambda}$.

(R3)

$$\begin{aligned}
\lambda_v + \widehat{\lambda} &= \lim_{n\to\infty} \frac{1}{n} \log\det |M(\theta_n, x_n)| \\
&= \lim_{n\to\infty} \frac{1}{n} \log\det \left| \begin{pmatrix} 1 & 0 \\ \frac{\partial x_n}{\partial \theta} & \frac{\partial x_n}{\partial x} \end{pmatrix} \right| \\
&= \lim_{n\to\infty} \frac{1}{n} \log \left| \frac{\partial x_n}{\partial x} \right| = \lambda_v
\end{aligned}$$

Consequently, $\widehat{\lambda} = 0$ and no Lyapunov exponent is positive μ-a.e. if and only if the vertical Lyapunov exponent is not positive μ-a.e..

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(P2) λ_v need not be constant, and

(P3) the formula in (R3) need not hold. Consequently, it well may happen that $\widehat{\lambda}(\theta, x) > 0$.

A very simple example on the non existence of lim in Lyapunov exponents

Consider

$$\begin{cases} \theta_{n+1} & = \theta_n + \omega \pmod 1, \\ x_{n+1} & = \tau(x_n) + \varepsilon \cos(2\pi\theta_n); \end{cases}$$

where $\tau(x)$ is a tent-like map:

$$\tau(x) = \begin{cases} \alpha(x - \frac{1}{2}) + 1 & \text{if } 0 \le x \le \frac{1}{2} \\ -\beta(x - \frac{1}{2}) + 1 & \text{if } \frac{1}{2} \le x \le 1 \end{cases}$$

with $0 \le \alpha, \beta \le 2$.

Computing the vertical Lyapunov exponent

Then,

$$\frac{1}{n}\log\left|\frac{\partial x_n}{\partial x}\right| = \frac{1}{n}\log\left|\frac{\partial \tau^n(x_0)}{\partial x}\right| = \frac{1}{n}\log\left(\alpha^{n_1}\cdot\beta^{n_2}\right)$$

where $n_1 + n_2 = n$ and n_1 (respectively n_2) is the number of times that the orbit $x_0, x_1, \ldots, x_{n-1}$ visits the the interval $[0, \frac{1}{2})$ (respectively $(\frac{1}{2}, 1]$).

Clearly,

$$\frac{1}{n}\log\left(\alpha^{n_1}\cdot\beta^{n_2}\right) = \frac{n_1\log(\alpha) + n_2\log(\beta)}{n_1 + n_2}.$$

By using elementary symbolic dynamics, its is possible to choose plenty of points x (but in a set of zero Lebesgue measure) so that the above sequence has no limit (even it can have the interval formed with endpoints $\log(\alpha)$ and $\log(\beta)$ as the set of accumulation points).

Jager's approach to the definition of non-chaoticity

Another approach to the definition of non-chaoticity is to consider the dynamical system in dimension one restricted to the attractor.

Then the original system is called *non-chaotic* if the unique Lyapunov exponent of this reduced system is non positive.

This argument can be made rigorous by means of the Birkhoff Ergodic Theorem since the dynamics on the attractor is driven by $\theta_{n+1} = \theta_n + \omega \pmod 1$, which is uniquely ergodic with the unique ergodic measure being the Lebesgue measure.

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- In the first approach, the “non-chaotic” points are the regular ones which “live” in the support of invariant measures which are not absolutely continuous with respect to the Lebesgue one. Thus, being not observable in the above sense.
- In the second approach the “non-chaotic” points are almost all points in the attractor. The situation is similar.

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(P2) is to estimate the Lyapunov exponents for all relevant points; since now no point can be chosen as a representative.

(P3) is to compute the maximal Lyapunov exponent (see the next example).

And finally: the notion of non-chaoticity

An attractor $\mathcal{A}$ is *non-chaotic* if the set of points in its realm of attraction $\rho(\mathcal{A})$, whose maximal upper Lyapunov exponent

$$\lambda_{\max}(x) = \limsup_{n\to\infty} \frac{1}{n} \log \|Df^n(x)\|$$

is positive, has zero Lebesgue measure.

An example from de la Llave

We have seen before that the problems (P1), (P2) pointed out before cannot be avoided. The main question is whether the argument presented in (R3) works for non regular points.

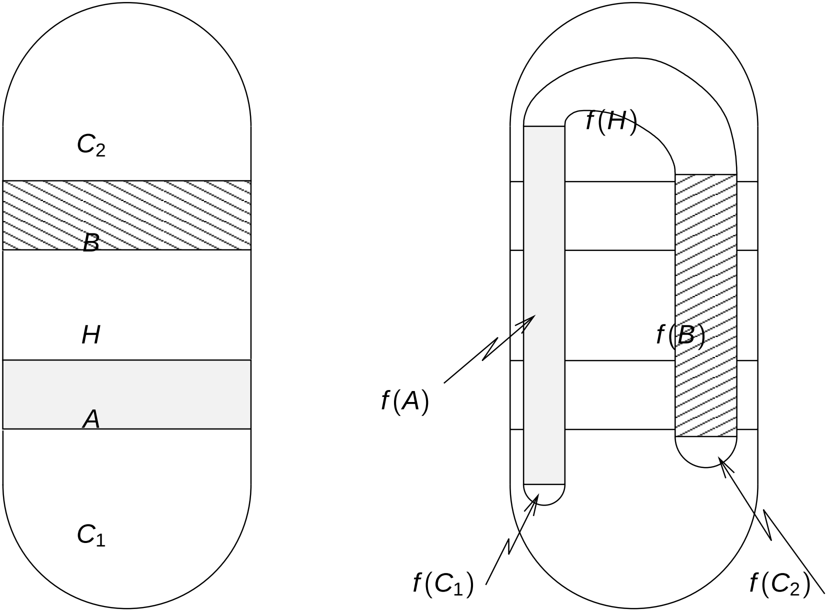
Although this is not known to us in the case of quasiperiodically forced skew products there is the following nice example of de la Llave that suggests that the definition that we gave, in general, cannot be simplified because *the "determinant formula" (R3) does not hold in this case*.

Consider an asymmetric horseshoe $\mathcal{C}^\infty$ diffeomorphism f as the one shown in the next picture ($N = C_1 \cup C_2 \cup A \cup B \cup H$ denotes the whole disc). To fix ideas we assume that

$$Df\big|_A = \begin{pmatrix} \frac{1}{4} & 0 \\ 0 & 24 \end{pmatrix} \qquad \text{and} \qquad Df\big|_B = \begin{pmatrix} -\frac{1}{2} & 0 \\ 0 & -12 \end{pmatrix}.$$

Observe that both matrices have determinant 6.

The asymmetric horseshoe



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2. is *strange*: it is not a finite set of points neither a piecewise differentiable manifold.

3. is *non-chaotic*: the set of points in its realm of attraction $\rho(\mathcal{A})$ whose maximal upper Lyapunov exponent $\lambda_{\max}(x)$ is positive, has zero Lebesgue measure.

Open problems

- Fractalization as a route to SNA
- Combinatorial Dynamics in non autonomous quasiperiodically forced dynamical systems.
- May SNA coexist in models with more complicate base maps?

In previous sessions we have seen that fractalization is not understood, and some models need to be deeply revised (see also)

[HS] A. Haro, C. Simó.
To be or not to be a SNA: That is the question.
preprint, 2005.

Open problems: Fractalization as a route to SNA – I

We aim at elucidating whether the fractalization route to chaos really happens, as in the model studied by of Heagy and Hammel or Prasad, Negi and Ramaswamy presented before. Also, we would like to understand better the dynamics of this model: reducibility, ...

$$\begin{cases} \theta_{n+1} &= \theta_n + \omega \pmod 1, \\ x_{n+1} &= \alpha x_n (1 - x_n) + \varepsilon \cos(2\pi\theta_n) \end{cases}$$

[NK] T. Nishikawa, K. Kaneko.
Fractalization of torus as a strange nonchaotic attractor.
Phys. Rev. E, **56(6)** (1997), 6114–6124.

Open problems: Fractalization as a route to SNA — II

The idea behind this model is that since it relies in the tent map, instead of in the logistic one, the computations can be made explicit with the help of the symbolic dynamics (I do not say that they are easy!). So, this is perhaps/probably a toy model to study and understand *analytically* the the fractalization route to chaos (drawback: the model is not differentiable!).

$$\begin{cases} \theta_{n+1} & = \theta_n + \omega \quad (\text{mod } 1), \\ x_{n+1} & = \tau(x_n) + \varepsilon \cos(2\pi\theta_n); \end{cases}$$

where $\tau(x)$ is a tent-like map:

$$\tau(x) = \begin{cases} x + \frac{1}{2} & \text{if } 0 \leq x \leq \frac{1}{2} \\ \frac{3}{2} - x & \text{if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Open problems: Combinatorial Dynamics in non autonomous quasiperiodically forced dynamical systems

In [FJJK] it is studied the coexistence of periodic *pinched strips* as a generalisation of the Sharkovskii Theorem that studies the coexistence of periodic orbits for interval maps.

[FJJK] R. Fabbri, T. Jäger, R. Johnson, and G. Keller.
A Sharkovskii-type theorem for minimally forced interval maps.
Topological Methods in Nonlinear Analysis, **26** (2005) 163–188.

Open problems: Combinatorial Dynamics in non autonomous quasiperiodically forced dynamical systems

The aim of this problem is to look deeply to the dynamical structure of these objects. Instead of just looking at the period we want to look at the combinatorial structure (“permutation”) of the whole orbit of the periodic pinched strip and derive dynamical consequences from it. Since we are using more information than just the period we will definitely obtain more information on the *forced* dynamics. Namely we aim at

- Study the forcing relation of the orbit of pinched strips,
- Perhaps, construct models with minimal dynamics (fixed a given combinatorial data of an orbit of pinched strips),
- Obtain lower bounds of the topological entropy of the system.

Open problems: May SNA coexist in models with more complicate base maps?

Consider a model of the type

$$(2) \qquad \begin{cases} \theta_{n+1} & = \varphi(\theta_n), \\ x_{n+1} & = \psi(\theta_n, x_n) \end{cases}$$

where φ is a continuous circle map of degree one with *nondegenerate* rotation interval.

The ultimate question is whether may coexist (of course simultaneously) different SNA's associated to the Birkhoff orbits of φ with different irrational rotational number.

Remark

Each of these orbits is semiconjugate — and plays the same role of — a single orbit of the rigid rotation $\theta_{n+1} = \theta_n + \omega$ (mod 1) in the usual models.