Cálculo de una forma normal alrededor de una órbita periódica multicrítica en un problema de Mecánica de Fluidos.

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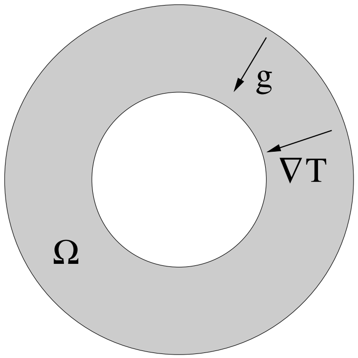
Boussinesq thermal convection in a two-dimensional annulus



Parameters:

$\eta = r_i/r_o$	Radius ratio (near 0.3)
$\sigma = \nu/\kappa$	Prandtl number (0.025)
$Ra = \alpha\Delta T g d^3/\kappa\nu$	Rayleigh number

The conduction state

$$\mathbf{v}_c = 0, \quad T_c = T_i + \ln(r/r_i)/\ln\eta$$

is a solution for any value of Ra.
The velocity field is written in terms of a mean flow f and a streamfunction ψ as

$$\mathbf{v} \quad = \quad f\hat{\mathbf{e}}_\theta + \nabla\times(\psi\hat{\mathbf{e}}_z) = (\partial_\theta\psi/r, f - \partial_r\psi).$$

The non-slip boundary conditions transform into

$$f = \psi = \partial_r\psi = 0 \quad \text{at } r = r_i, r_o.$$

The equations for f, ψ, and the deviation of the temperature from the conduction state, $\Theta = T - T_c$, are:

$$\partial_t \begin{pmatrix} f \\ \Theta \\ \Delta\psi \end{pmatrix} = \begin{pmatrix} \sigma\tilde{\Delta} & 0 & 0 \\ 0 & \Delta & -(r^2 \ln\eta)^{-1}\partial_\theta \\ 0 & \sigma r^{-1} Ra\,\partial_\theta & \sigma\Delta\Delta \end{pmatrix} \begin{pmatrix} f \\ \Theta \\ \psi \end{pmatrix} +$$

$$+ \begin{pmatrix} P_\theta \left[\Delta\psi\partial_\theta\psi\right]/r \\ J(\psi,\Theta) - f\partial_\theta\Theta/r \\ (1-P_\theta)J(\psi,\Delta\psi) + \tilde{\Delta} f\partial_\theta\psi/r - f\partial_\theta\Delta\psi/r \end{pmatrix},$$

with $J(h,g) = (\partial_r h\partial_\theta g - \partial_r g\partial_\theta h)/r$, $\Delta = (\partial_r + 1/r)\partial_r + (1/r^2)\partial^2_{\theta\theta}$, $\tilde{\Delta} = \partial_r(\partial_r + 1/r)$, and boundary conditions

$$f = \psi = \partial_r\psi = \Theta = 0 \qquad \text{on } r = r_i, r_o.$$

The system will be written in the form $\mathcal{L}_0\partial_t u = \mathcal{A}u + \mathcal{B}(u,u)$ with $u = (f,\Theta,\psi)$.

The system

$$\mathcal{L}_0 \partial_t u = \mathcal{A} u + \mathcal{B}(u, u)$$

is $O(2)$-equivariant:

- rotations $(f, \Theta, \psi)(r, \theta) \rightarrow (f, \Theta, \psi)(r, \theta + \theta_0)$
- reflections $(f, \Theta, \psi)(r, \theta) \rightarrow (-f, \Theta, -\psi)(r, 2\theta_0 - \theta)$.

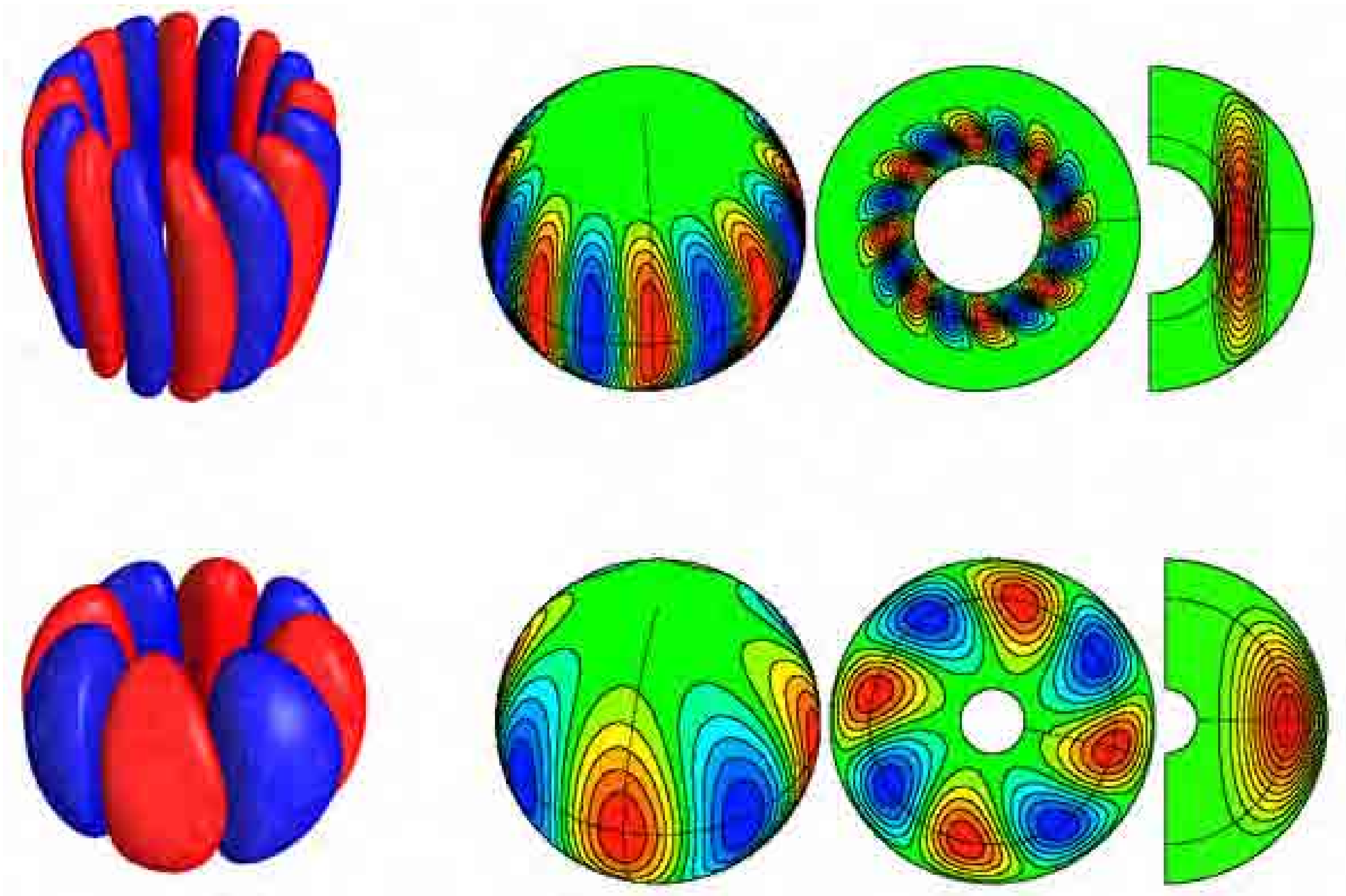
We will also write

$$\mathcal{A} = \mathcal{L}_1 + \varepsilon_1 \mathcal{L}_2,$$

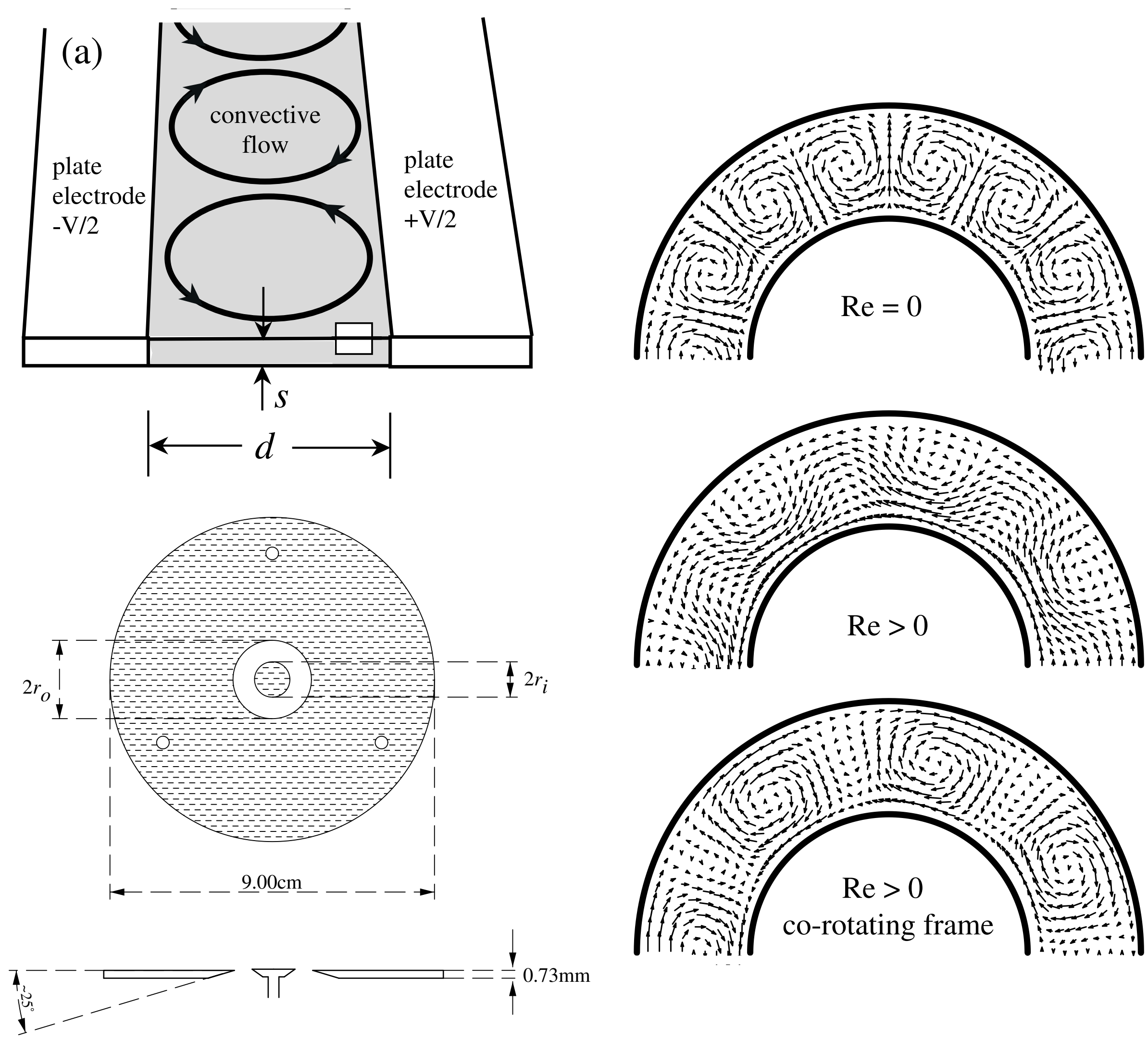
where

$$\varepsilon_1 = (Ra - Ra_c)/Ra_c \quad \text{or} \quad Ra = Ra_c(1 + \varepsilon_1).$$

Thermal convection in a spherical shell



Electroconvection in suspended fluid films



From *Electroconvection in sheared annular fluids films*, Z.A. Daya. Ph. D. Thesis.

Discretization

The functions ψ and Θ are approximated by a pseudo-spectral Fourier method in θ:

$$\psi = \psi(t,r,\theta) = i\sum_{\substack{n=-N/2 \\ n\neq 0}}^{N/2} \psi_n(t,r)e^{in\theta}, \quad \Theta = \Theta(t,r,\theta) = \sum_{n=-N/2}^{N/2} \Theta_n(t,r)e^{in\theta},$$

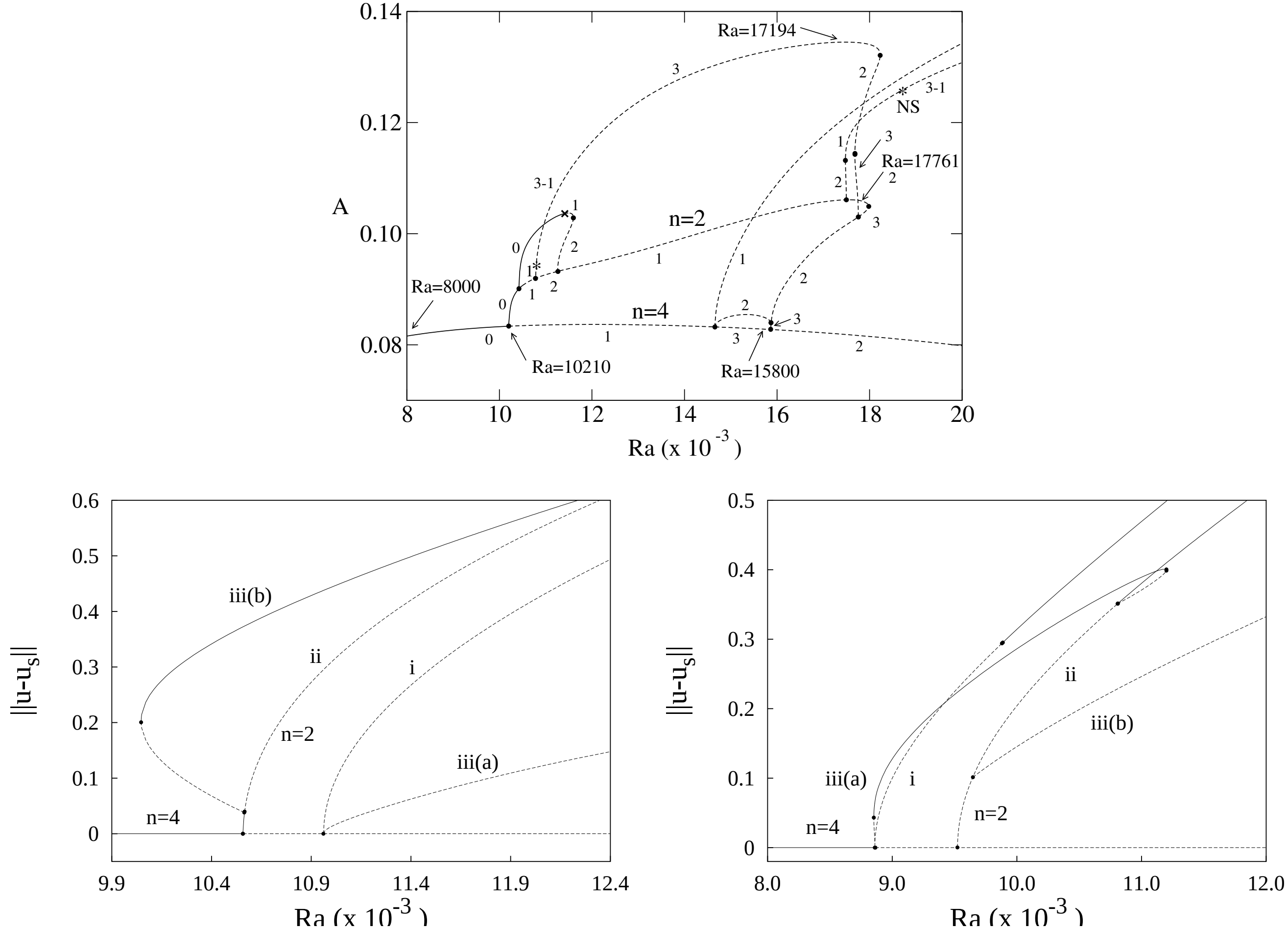
with $\psi_n(t,r)$ and $\Theta_n(t,r)$ polynomials of degree L in r that verify the boundary conditions.
The unknowns of the problem are the values of $f(t,r)$, $\psi_n(t,r)$ and $\Theta_n(t,r)$ on a Gauss-Lobatto mesh (collocation in r).
In most calculations shown $L = 48$, $N = 192$, and the dimension of the system of ODEs obtained is $n = 17758$. Tests have been performed up to $L = 64$, $N = 256$. In this case $n = 31870$.
The stiff system of ODEs obtained

$$\mathcal{L}_0 \partial_t u = \mathcal{A}u + \mathcal{B}(u,u)$$

is integrated by using fourth order BDF-extrapolation formulas:

$$\frac{1}{\Delta t}\mathcal{L}_0 \left(\gamma_0 u^{n+1} - \sum_{i=0}^{k-1} \alpha_i u^{n-i} \right) = \sum_{i=0}^{k-1} \beta_i \mathcal{B}(u^{n-i}, u^{n-i}) + \mathcal{A}u^{n+1}.$$

The codimension-two point



Bifurcation diagrams of symmetric periodic orbits for $\eta = 0.3$, $\eta = 0.32$ and $\eta = 0.35$. In the last two the horizontal line corresponds to the basic periodic solution u_s.
The codimension-two point is located at $Ra_c = 10385$, $\eta_c = 0.3255$.

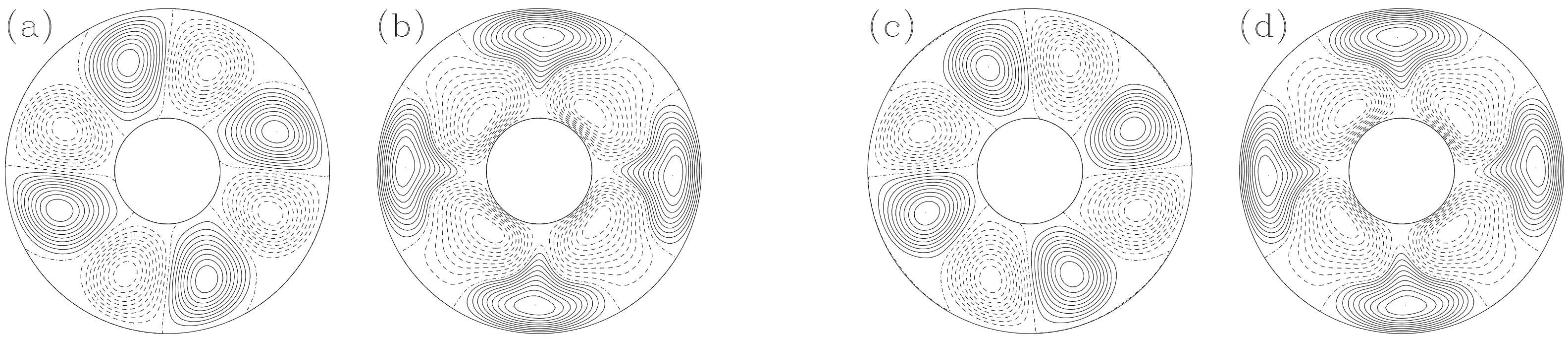
The basic solution at the codimension-two point

A periodic orbit $u(t)$ of period T is said to be a symmetric cycle (S-cycle) if there is a linear transformation R such that $R^2 = I$, and

$$u(t + T/2) = Ru(t).$$

All the periodic orbits we have found are S-cycles. The transformations being the *reflections*

$$\sigma_{\theta_1} : (f, \Theta, \psi)(r, \theta) \longrightarrow \zeta(f, \Theta, \psi)(r, 2\theta_1 - \theta), \quad \text{with} \quad \zeta(f, \Theta, \psi) = (-f, \Theta, -\psi).$$



Instantaneous streamlines and temperature perturbation for $u_s(0)$, and $u_s(T/2)$.

The isotropy group of spatio-temporal symmetries of u_s is the dihedral group D_4 generated, for instance, by

$$\begin{array}{llll} \rho_{\pi/2} : & & \theta \to \theta + \pi/2, & u_s \to u_s, \\ \sigma_{\pi/4} : & t \to t + T/2, & \theta \to \pi/2 - \theta, & u_s \to \zeta u_s. \end{array}$$

The normal form

The center manifold of the basic periodic orbit, u_s, is

$$u(t,r,\theta) = u_s(\tau,r,\tilde{\theta}) + \sum_{j=1}^{3} A_j(t)U_j(\tau,r,\tilde{\theta}) + \cdots ,$$

with $\tau = t - \phi(t)$, and $\tilde{\theta} = \theta - \varphi(t)$.
The normal form is choosen to be equivariant by the transformations of the isotropy group of u_s, D_4.
The action of three of its elements on the eigenfunctions is

$$\left.\begin{array}{l} \tau \to \tau + T/2, \\ \tilde{\theta} \to -\tilde{\theta}, \quad \zeta \end{array}\right\} \Longrightarrow U_1 \to -U_1, \quad U_2 \to U_2, \quad U_3 \to -U_3,$$

$$\left.\begin{array}{l} \tau \to \tau + T/2, \\ \tilde{\theta} \to \pi - \tilde{\theta}, \quad \zeta \end{array}\right\} \Longrightarrow U_1 \to U_1, \quad U_2 \to -U_2, \quad U_3 \to -U_3,$$

$$\tilde{\theta} \to \tilde{\theta} + \pi/2 \Longrightarrow U_1 \to -U_2, \quad U_2 \to U_1, \quad U_3 \to -U_3.$$

In terms of the amplitudes A_j, the phase shift φ, and the time shift ϕ, the equations must be invariant under the transformations

$$
\begin{array}{llll}
\varphi \to -\varphi, & A_1 \to -A_1, & A_3 \to -A_3, & \\
\varphi \to \pi - \varphi, & A_2 \to -A_2, & A_3 \to -A_3, & \\
\varphi \to \varphi + \pi/2, & A_1 \to -A_2, & A_2 \to A_1, & A_3 \to -A_3.
\end{array}
$$

Up to third order, the most general equations that are invariant under the above symmetries are of the form

$$
\begin{aligned}
\dot{A}_1 &= (\alpha_1\varepsilon_1 + \alpha_3\varepsilon_2)A_1 + \beta_1 A_2 A_3 + (\gamma_1 A_1^2 + \gamma_2 A_2^2 + \gamma_3 A_3^2)A_1, \\
\dot{A}_2 &= (\alpha_1\varepsilon_1 + \alpha_3\varepsilon_2)A_2 + \beta_1 A_1 A_3 + (\gamma_1 A_2^2 + \gamma_2 A_1^2 + \gamma_3 A_3^2)A_2, \\
\dot{A}_3 &= (\alpha_2\varepsilon_1 + \alpha_4\varepsilon_2)A_3 + \beta_2 A_1 A_2 + (\gamma_4 (A_1^2 + A_2^2) + \gamma_5 A_3^2)A_3, \\
\dot{\varphi} &= \delta(A_1^2 - A_2^2)A_3, \\
\dot{\phi} &= -\nu_1\varepsilon_1 - \nu_2\varepsilon_2 + \xi_1(A_1^2 + A_2^2) + \xi_2 A_3^2,
\end{aligned}
$$

where the unfolding parameters are, $\varepsilon_1 = (Ra - Ra_c)/Ra_c$, and $\varepsilon_2 = \eta - \eta_c$.

The perturbation method

$$\mathcal{L}_0 \partial_t u = \mathcal{L}_1 u + \varepsilon_1 \mathcal{L}_2 u + \mathcal{B}(u, u).$$

The basic solution u_s and the eigenfunctions U_j are expanded in powers of ε_1, as

$$u_s = u_s^0 + \varepsilon_1 u_s^1 + \cdots , \qquad U_j = U_j^0 + \varepsilon_1 U_j^1 + \cdots \quad \text{for } j = 1, \ldots, 3.$$

The center manifold is rewritten to include higher order terms, as

$$u = u_s^0 + \varepsilon_1 u_s^1 + \sum_{j=1}^{3} A_j (U_j^0 + \varepsilon_1 U_j^1) + \sum_{\substack{k,l=1 \\ k \le l}}^{3} A_k A_l U_{kl}^1 + \sum_{j=1}^{3} A_j \sum_{k=1}^{3} A_k^2 U_{jk}^2 + \ldots ,$$

where all the functions u_s^m, U_j^m, and U_{jk}^m depend on $(\tau, r, \tilde{\theta})$, and A_j, φ and ϕ depend on t. We are neglecting ε_1-perturbations of U_{kl}^1 and U_{jk}^2, which are not needed below, and only those third order terms that contribute to the amplitude equations are displayed.
As $\tau = t - \phi(t)$, and $\tilde{\theta} = \theta - \varphi(t)$, the time derivative becomes

$$\partial_t \to (1 - \dot{\phi}) \partial_\tau - \dot{\varphi} \partial_{\tilde{\theta}} + \partial_t .$$

Non-homogeneous linear equations

$$\mathcal{O}(1): \quad \mathcal{L}_0\partial_\tau u_s^0 = \mathcal{L}_1 u_s^0 + \mathcal{B}(u_s^0, u_s^0)$$

$$\mathcal{O}(A_j): \quad \mathcal{L}_0\partial_\tau U_j^0 = \mathcal{L}_1 U_j^0 + \mathcal{B}(u_s^0, U_j^0) + \mathcal{B}(U_j^0, u_s^0)$$

$$\begin{aligned}\mathcal{O}(\varepsilon_1): \quad \mathcal{L}_0\partial_\tau u_s^1 = \; & \mathcal{L}_1 u_s^1 + \mathcal{B}(u_s^0, u_s^1) + \mathcal{B}(u_s^1, u_s^0)\\ & +\mathcal{L}_2 u_s^0 - \tilde{\nu}_1 \mathcal{L}_0 \partial_\tau u_s^0\end{aligned}$$

$$\begin{aligned}\mathcal{O}(\varepsilon_1 A_j): \quad \mathcal{L}_0\partial_\tau U_j^1 = \; & \mathcal{L}_1 U_j^1 + \mathcal{B}(u_s^0, U_j^1) + \mathcal{B}(U_j^1, u_s^0)\\ & +\mathcal{L}_2 U_j^0 + \mathcal{B}(u_s^1, U_j^0) + \mathcal{B}(U_j^0, u_s^1) - \tilde{\nu}_1 \mathcal{L}_0 \partial_\tau U_j^0 - \tilde{\alpha}_j \mathcal{L}_0 U_j^0\end{aligned}$$

$$\begin{aligned}\mathcal{O}(A_k A_l):\; k \neq l \quad \mathcal{L}_0\partial_\tau U_{kl}^1 = \; & \mathcal{L}_1 U_{kl}^1 + \mathcal{B}(u_s^0, U_{kl}^1) + \mathcal{B}(U_{kl}^1, u_s^0)\\ & +\mathcal{B}(U_k^0, U_l^0) + \mathcal{B}(U_l^0, U_k^0) - \textstyle\sum_{j=1}^3 \tilde{\beta}_{jkl} \mathcal{L}_0 U_j^0\end{aligned}$$

$$\begin{aligned}\mathcal{O}(A_k A_l):\; k = l \quad \mathcal{L}_0\partial_\tau U_{kk}^1 = \; & \mathcal{L}_1 U_{kk}^1 + \mathcal{B}(u_s^0, U_{kk}^1) + \mathcal{B}(U_{kk}^1, u_s^0)\\ & +\mathcal{B}(U_k^0, U_k^0) + \tilde{\xi}_k \mathcal{L}_0 \partial_\tau u_s^0\end{aligned}$$

$$
\begin{array}{lll}
\mathcal{O}(A_j A_k^2): & \mathcal{L}_0 \partial_\tau U^2_{jk} = & \mathcal{L}_1 U^2_{jk} + \mathcal{B}(u^0_s, U^2_{jk}) + \mathcal{B}(U^2_{jk}, u^0_s) \\
j \neq k & & +\mathcal{B}(U^1_{kk}, U^0_j) + \mathcal{B}(U^0_j, U^1_{kk}) + \mathcal{B}(U^1_{jk}, U^0_k) + \mathcal{B}(U^0_k, U^1_{jk}) \\
 & & -2\sum_{l=1}^{3} \hat{\beta}_{ljk} \mathcal{L}_0 U^1_{kl} - \tilde{\gamma}_{jk} \mathcal{L}_0 U^0_j \\
 & & +\tilde{\delta}_{jk} \mathcal{L}_0 \partial_\theta u^0_s + \tilde{\xi}_k \mathcal{L}_0 \partial_\tau U^0_j
\end{array}
$$

$$
\begin{array}{lll}
\mathcal{O}(A_j A_k^2): & \mathcal{L}_0 \partial_\tau U^2_{jj} = & \mathcal{L}_1 U^2_{jj} + \mathcal{B}(u^0_s, U^2_{jj}) + \mathcal{B}(U^2_{jj}, u^0_s) \\
j = k & & +\mathcal{B}(U^1_{jj}, U^0_j) + \mathcal{B}(U^0_j, U^1_{jj}) - \tilde{\gamma}_{jj} \mathcal{L}_0 U^0_j
\end{array}
$$

Solvability conditions

The previous non-homogeneous linear equations can be written as

$$\mathcal{L}_0 \partial_\tau U = \mathcal{L} U + H - \sum_{j=1}^{5} b_j \mathcal{L}_0 U_j^0,$$

where $\mathcal{L} = \mathcal{L}_1 + \mathcal{B}(.,u_s^0) + \mathcal{B}(u_s^0,.)$, $H(\tau) = H(\tau + T)$, and U_j^0 are five linearly independent T-periodic orbits of $\mathcal{L}_0 \partial_\tau U = \mathcal{L} U$.

Lemma 1 *The equation $\mathcal{L}_0 \partial_\tau U = \mathcal{L} U + \tilde{H}$ exhibits T-periodic solutions if and only if*

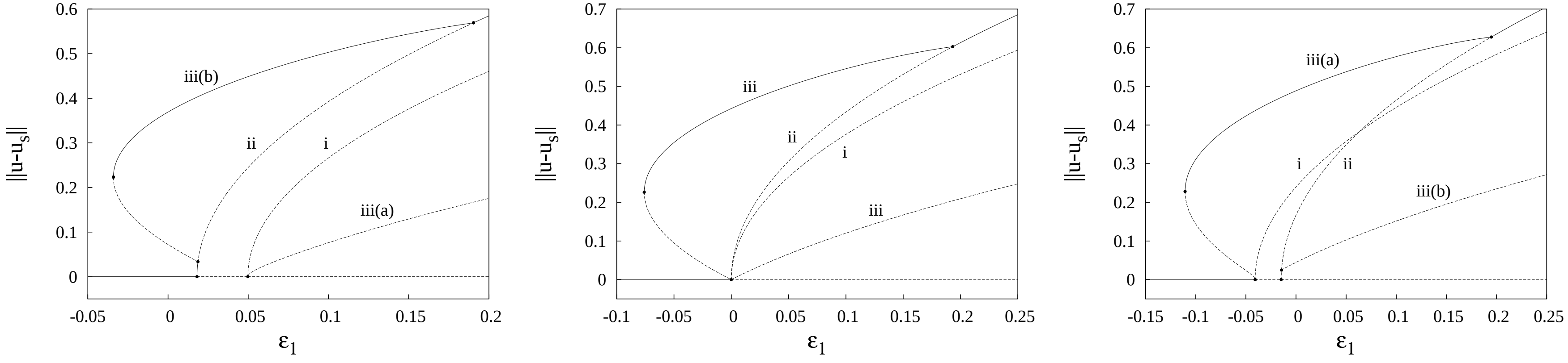
$$\int_0^T \langle \tilde{H}, U_j^* \rangle \, d\tau = 0 \quad \textit{for } j = 1, \ldots, 5,$$

where U_j^* are five linearly independent, T-periodic eigenfunctions of the adjoint problem

$$-\mathcal{L}_0^\top \partial_\tau U^* = \mathcal{L}^\top U^*,$$

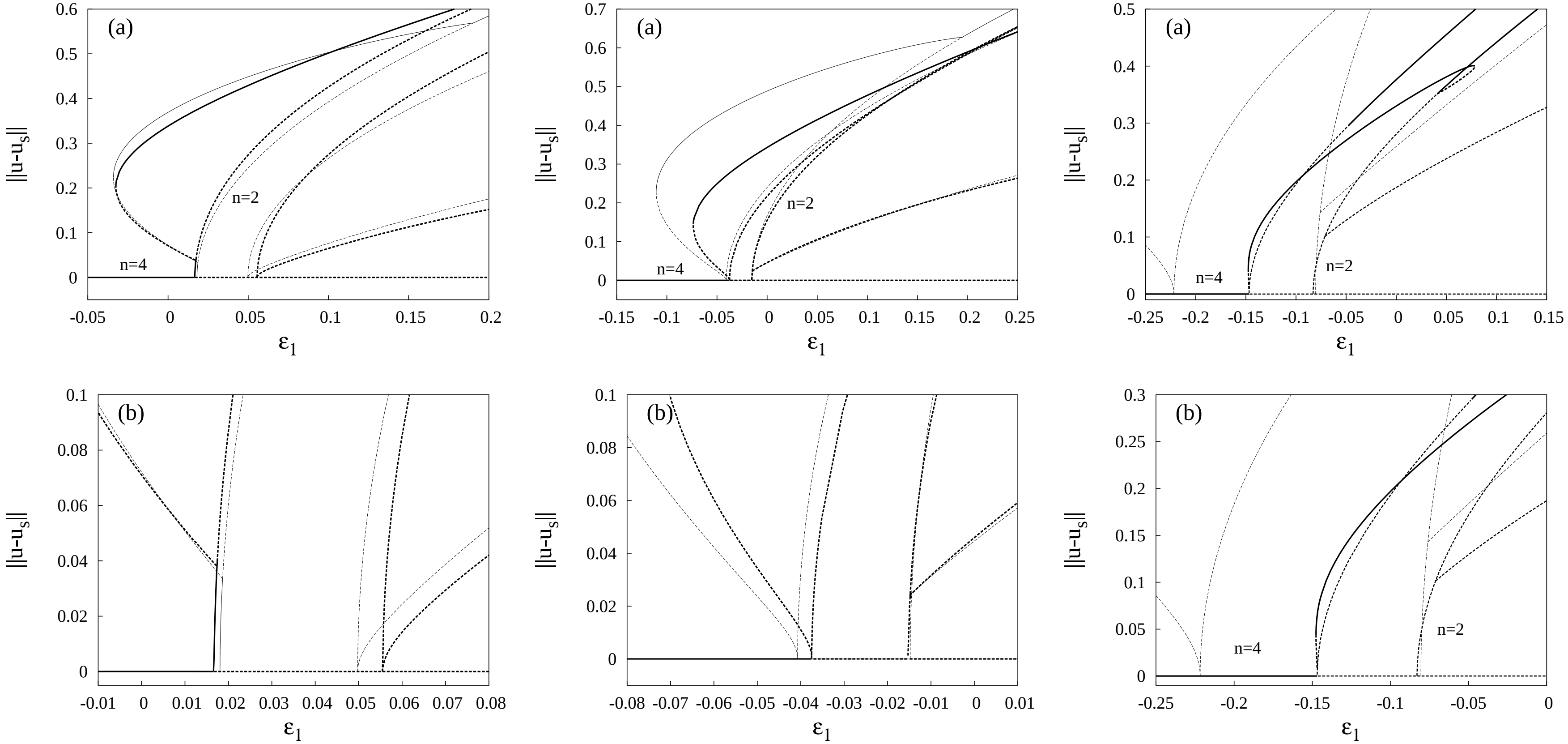
which also exhibits the Floquet multiplier $\mu = +1$ with multiplicity five. If these conditions hold, then $\mathcal{L}_0 \partial_\tau U = \mathcal{L} U + \tilde{H}$ possesses a five-dimensional, linear manifold of periodic solutions.

Bifurcation diagrams for the normal form



Bifurcation diagrams for the normal form at $\varepsilon_2 = -0.0055$, $\varepsilon_2 = 0$, and $\varepsilon_2 = +0.0045$.

Comparison with the original EDP



Superposition for $\varepsilon_2 = -0.0055$, $\varepsilon_2 = +0.0045$, and $\varepsilon_2 = +0.0245$.

The thick and thin lines correspond to the PDE and the ODE systems, respectively.

Conclusions

- We have developed a perturbation technique to determine the coefficients of the amplitude equations near a bifurcation of spatio-temporal symmetric periodic orbits. The only requirement is the availability of accurate time evolution codes.
- The method is relatively easy to apply in the case of quadratic non-linear terms.
- The presence of the degeneration induced by the $O(2)$ symmetry group of the equations has been included and has helped to understand the lack of drifting solutions, which can only appear after all spatio-temporal symmetries have been broken. This information is contained in the equation for the azimuthal phase φ:

$$\dot{\varphi} = \delta(A_1^2 - A_2^2)A_3.$$

- The use of the adjoint problem can be avoided by minimizing the linear growth with respect to the parameters to be determined in each linear non-homogeneous equation.

The variational equations and the adjoint problem

Let u_s be the basic T-periodic solution of $\mathcal{L}_0 \partial_t u = \mathcal{L}_1 u + \mathcal{B}(u,u)$. The linearized equations about u_s (first variationals) are

$$\mathcal{L}_0 \partial_t U = \mathcal{L} U \quad (1)$$

where $\mathcal{L} = \mathcal{L}_1 + \mathcal{B}(\cdot, u_s) + \mathcal{B}(u_s, \cdot)$.
If $u_i = (f_i, \Theta_i, \psi_i)$, $i = 1, 2$ we define the inner product

$$\langle u_1, u_2 \rangle = \frac{1}{2\pi} \int_0^{2\pi} \int_{R_i}^{R_o} (f_1 f_2 + \Theta_1 \Theta_2 + \psi_1 \psi_2)\, r dr d\theta,$$

Then the adjoint problem of (1) with respect to $\langle \cdot, \cdot \rangle$ is $\mathcal{L}_0^\top \partial_t U^* = -\mathcal{L}^\top U^*$.
The latter is integrated backwards in time due to the parabolic nature of (1). Moreover, $\mathcal{L}_0$ is self-adjoint and then the equation to consider is

$$\mathcal{L}_0 \partial_t U^* = \mathcal{L}^\top U^*. \quad (2)$$

The eigenfunctions of the monodromy matrices of (1) and (2) can be obtained by a power-like method (subspace iteration or Arnoldi method). Their periodic orbits correspond to the multiplier $+1$.

The adjoint of $\mathcal{L}$

$$\mathcal{L}^\top u = \begin{pmatrix} \sigma\tilde{\Delta} & 0 & 0 \\ 0 & \Delta & -\sigma r^{-1} Ra\, \partial_\theta \\ 0 & (r^2 \ln \eta)^{-1} \partial_\theta & \sigma \Delta\Delta \end{pmatrix} \begin{pmatrix} f \\ \Theta \\ \psi \end{pmatrix} + \begin{pmatrix} A \\ B \\ C \end{pmatrix}$$

with

$$\begin{aligned} A =& P_\theta \Big[\tilde{\Delta} \left(\psi \partial_\theta \psi_s / r \right) - \Theta \partial_\theta \Theta_s / r - \psi \partial_\theta \Delta \psi_s / r \Big] \\ B =& -J(\psi_s, \Theta) + f_s \partial_\theta \Theta / r \\ C =& (1 - P_\theta) \Big[\Delta J(\psi, \psi_s) - J(\psi, \Delta\psi_s) - J(\Theta, \Theta_s) \Big] \\ & + \Delta \left(\left(f \partial_\theta \psi_s + f_s \partial_\theta \psi \right) / r \right) \\ & - \left(f \partial_\theta \Delta \psi_s + \tilde{\Delta} f_s \partial_\theta \psi \right) / r \end{aligned}$$

where $u_s = (f_s, \Theta_s, \psi_s)$, $\Delta = (\partial_r + 1/r)\partial_r + (1/r^2)\partial^2_{\theta\theta}$, $\tilde{\Delta} = \partial_r(\partial_r + 1/r)$, and $J(h, g) = (\partial_r h \partial_\theta g - \partial_r g \partial_\theta h)/r$.

Lemma 2 *The solutions of the equation* $\mathcal{L}_0 \partial_\tau U = \mathcal{L} U + \tilde{H}$ *have the form*

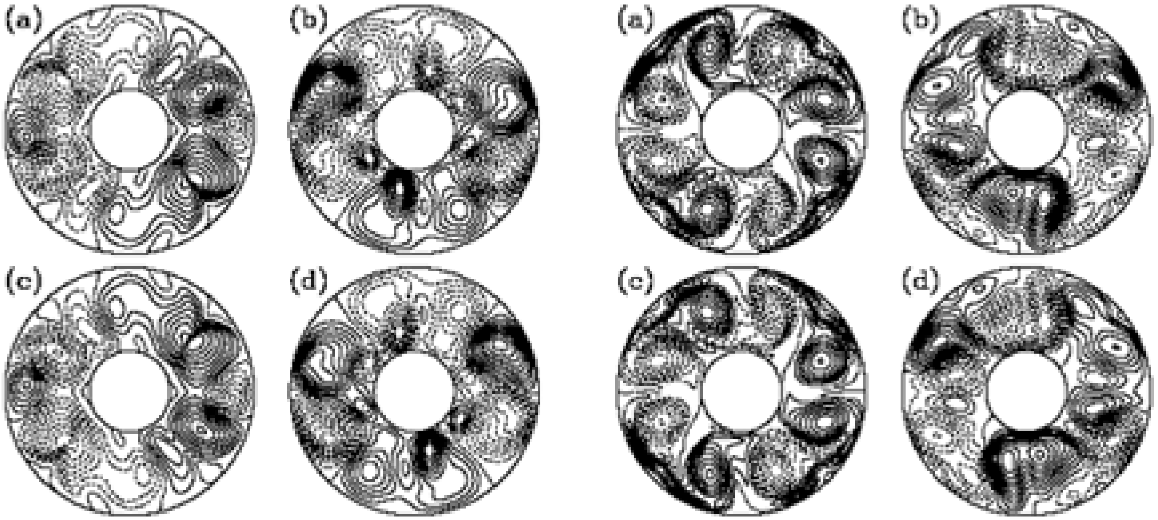
$$U = \sum_{j=1}^{5} a_j \tau U_j^0 + V + \textit{E.S.T}, \qquad (3)$$

where $U_1^0, \ldots, U_5^0$ *are five linearly independent periodic solutions of* $\mathcal{L}_0 \partial_\tau U = \mathcal{L} U$ *associated with the Floquet multiplier* $\mu = +1$*,* V *is* T*-periodic, and E.S.T denote exponentially small terms as* $\tau \to \infty$*. Thus, this system exhibits periodic solutions if and only if* $a_j = 0$*,* $j = 1, \cdots, 5$*.*

Eigenfunctions at the multicritical point

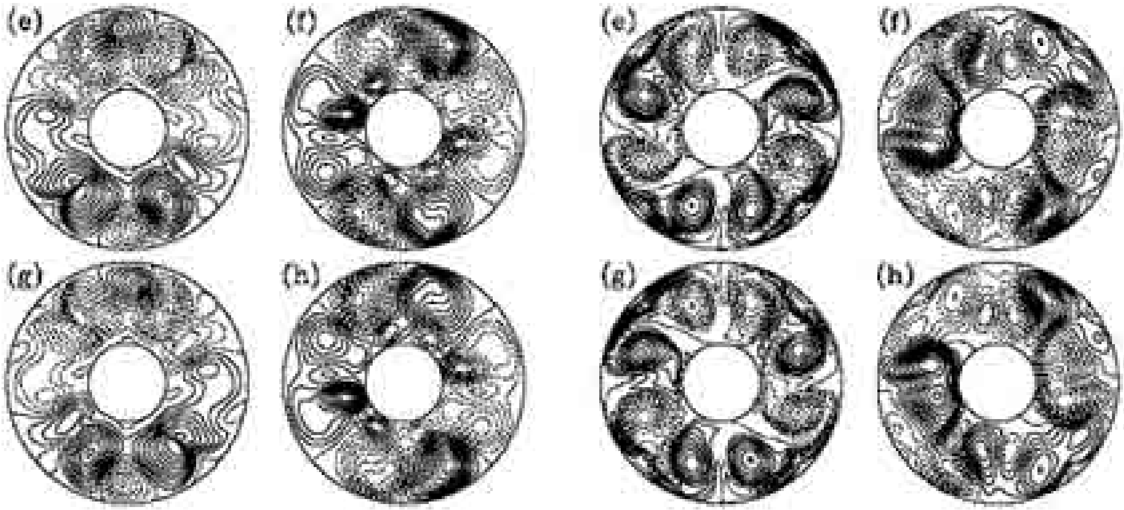
Instantaneous streamlines and temperature perturbation for the critical eigenfunctions, and eigenfunctions of the adjoint problem.

U_1



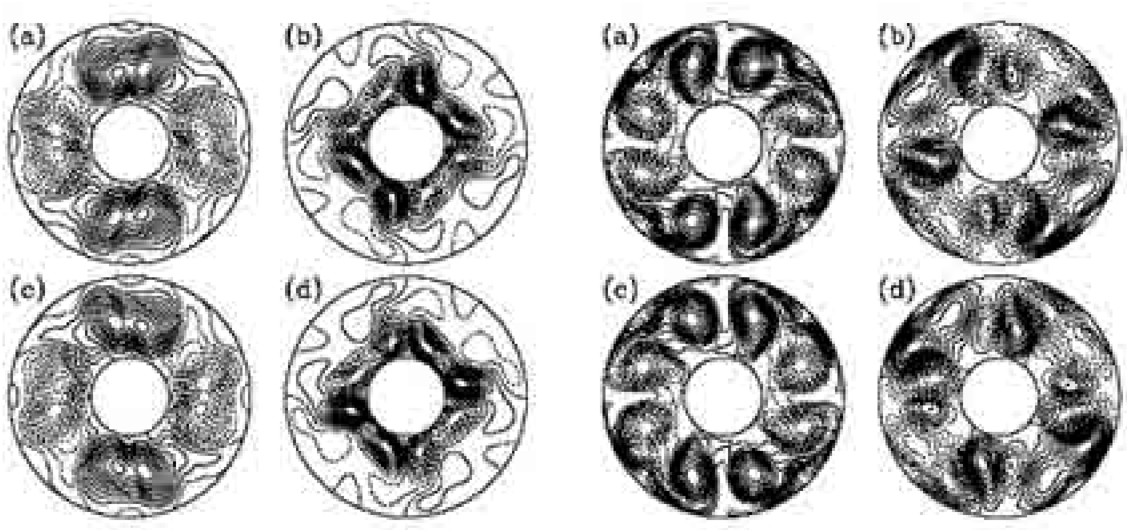
Left: $U_1(0)$, and $U_1(T/2)$. Right: $U_1^*(0)$, and $U_1^*(T/2)$.

U_2



Left: $U_2(0)$, and $U_2(T/2)$. Right: $U_2^*(0)$, and $U_2^*(T/2)$.

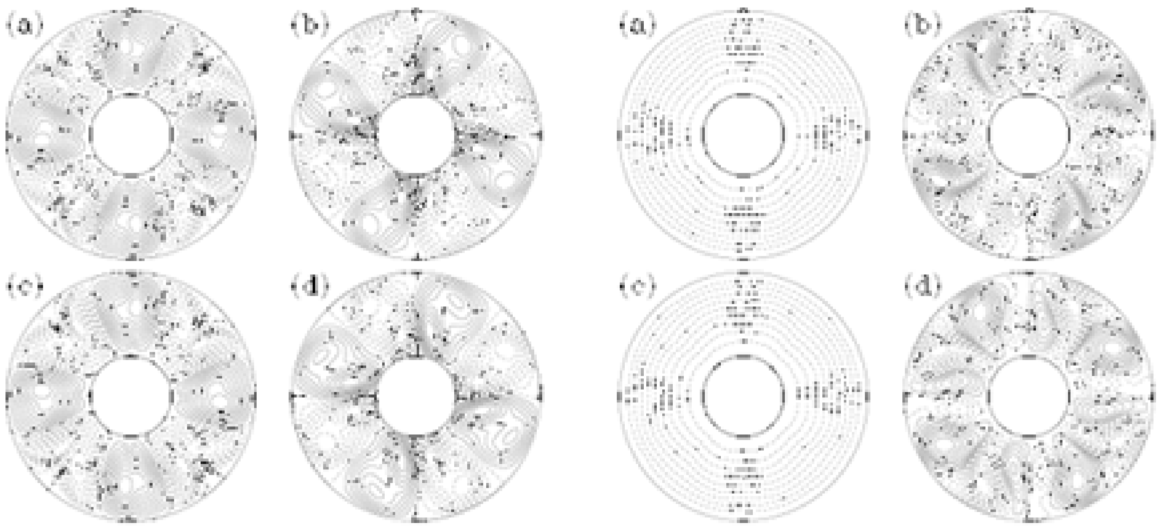
U_3



Left: $U_3(0)$, and $U_3(T/2)$. Right: $U_3^*(0)$, and $U_3^*(T/2)$.

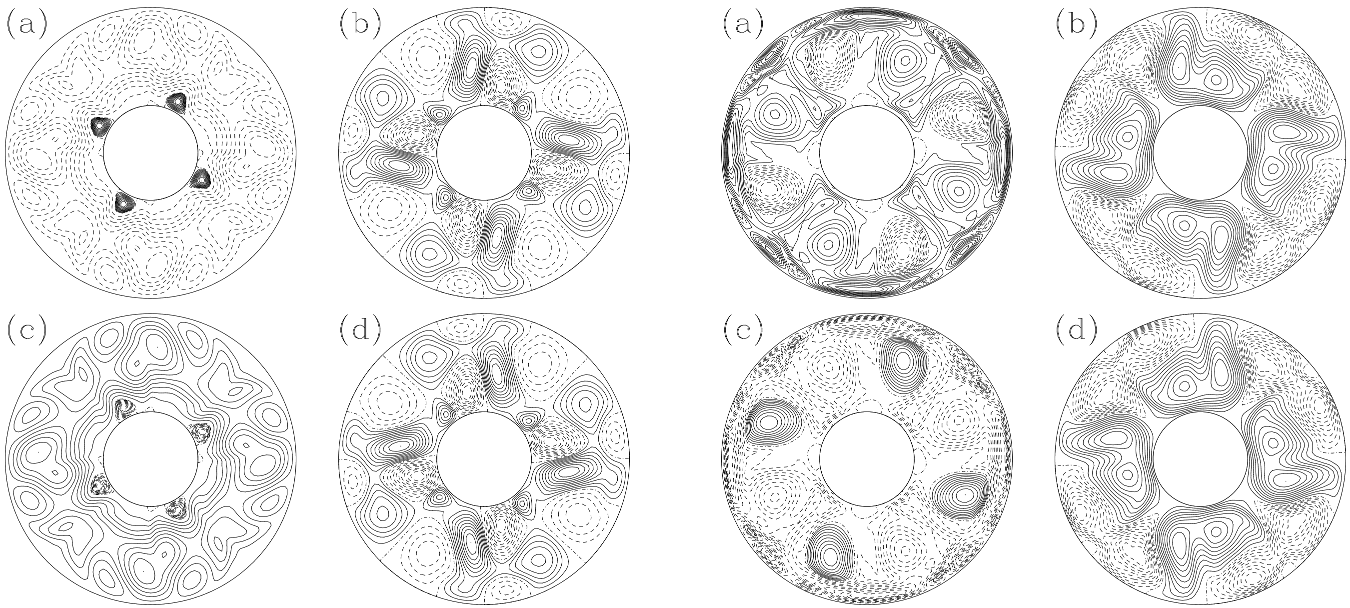
Neutral eigenfunctions.

$U_4 = \partial_\theta u_s$



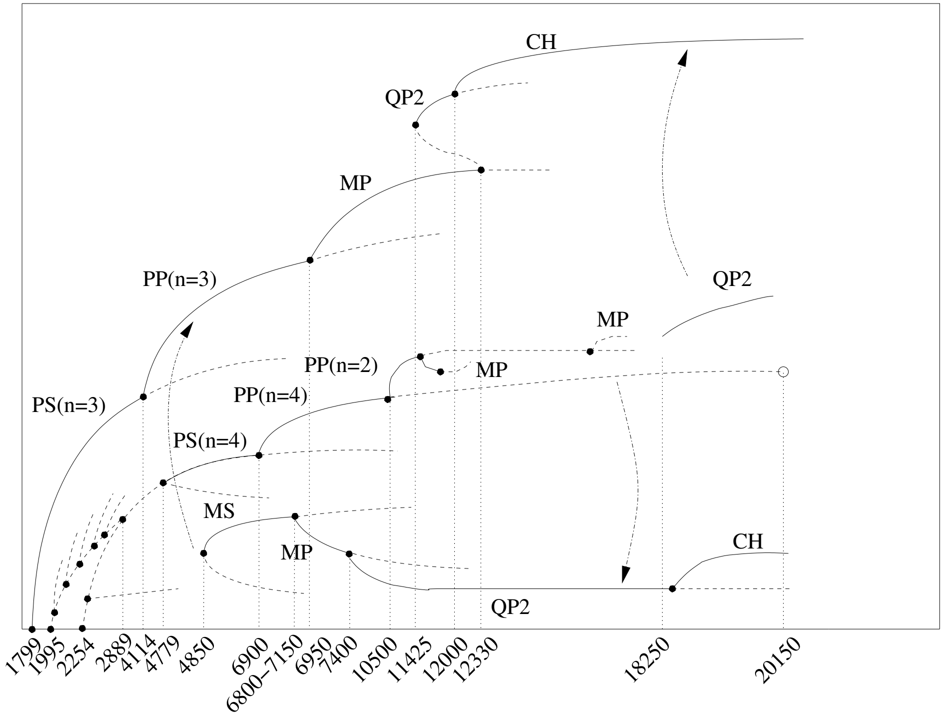
Left: $U_4(0)$, and $U_4(T/2)$. Right: $U_4^*(0)$, and $U_4^*(T/2)$.

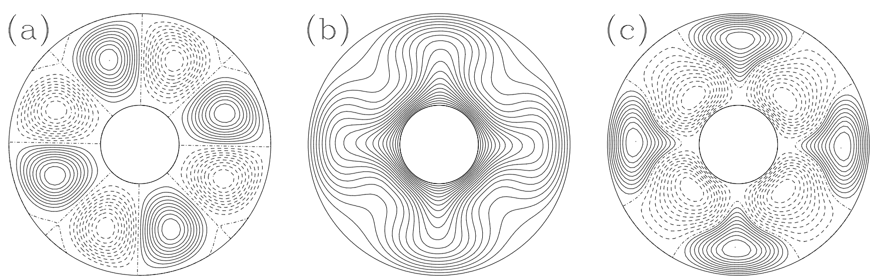
$U_5 = \partial_t u_s$



Left: $U_5(0)$, and $U_5(T/2)$. Right: $U_5^*(0)$, and $U_5^*(T/2)$.

Some previous results





Steady solution at $Ra = 6300$, $\eta = 0.3$.