El límite de viscosidad pequeña en el estudio de ondas de Faraday

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Summary

- Faraday waves
- Nearly marginal modes in water waves at low viscosity: surface modes, hydrodynamic modes
- Inviscid mean flow, viscous mean flow, Stokes drift
- Coupled amplitude-mean flow equations
- Spatially modulated Faraday waves of the standing type
- Concluding remarks

Faraday waves

- Faraday (*Phil. Trans. Roy. Soc. London* 121(1831)319-340) waves are water waves produced in the free surface of the liquid upon vertical vibration of the container.
- A paradigm of pattern forming system.
- Viscous effects are usually weak (except for quite viscous liquids in quite small containers). *Small viscous effects are quite subtle*, but allow to go further analytically.
- Except under quite special conditions (quite small containers, quite small vibrating frequency), these waves *are multi-mode waves* (spatially modulated), and *are dynamically coupled to a mean flow.*
- Even if they are nearly-inviscid (dispersive), they show a *fully dissipative* (diffusive) *behavior near threshold*, and *lead to quintic nonlinearities.*

Nondimensionalized with ω^{-1} and ℓ, defined as $\omega^2 = g/\ell + \sigma/(\rho\ell^3)$
$\boldsymbol{v}$ = velocity, p = pressure, f= free surface elevation

$$\boldsymbol{\nabla} \cdot \boldsymbol{v} = 0, \quad \frac{\partial \boldsymbol{v}}{\partial t} + \boldsymbol{\nabla}\frac{|\boldsymbol{v}|^2}{2} - \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) = -\boldsymbol{\nabla} p + \varepsilon \Delta \boldsymbol{v}$$

in $\Omega \times (-d, f)$, with boundary conditions

$$\begin{aligned}
&\boldsymbol{v} = \mathbf{0} \quad \text{at } z = -d, \quad \boldsymbol{v} = 0, \quad f = 0 \text{ if } (x, y) \in \partial\Omega \\
&\boldsymbol{v} \cdot \boldsymbol{n} = (\partial f/\partial t)(\boldsymbol{e}_z \cdot \boldsymbol{n}), \quad [(\boldsymbol{\nabla} \boldsymbol{v} + \boldsymbol{\nabla} \boldsymbol{v}^\top) \cdot \boldsymbol{n}] \times \boldsymbol{n} = 0, \\
&p - |\boldsymbol{v}|^2/2 - (1 - S)f + S\boldsymbol{\nabla} \cdot (\boldsymbol{\nabla} f/\sqrt{1 + |\boldsymbol{\nabla} f|^2}) = \\
&\qquad 4af \cos 2t + \varepsilon[(\boldsymbol{\nabla} \boldsymbol{v} + \boldsymbol{\nabla} \boldsymbol{v}^\top) \cdot \boldsymbol{n}] \cdot \boldsymbol{n} \quad \text{if } z = f.
\end{aligned}$$

$\varepsilon = \nu/(\omega\ell^2) \ll 1, \quad d = d^*/\ell \quad \exp(-2d) \ll 1, \quad a = a^*/\ell \ll 1,$
S =capillary gravity balance $= \sigma/(\sigma + \rho g d^2) \in [0, 1]$.
Quiescent solution (moving axes): $\boldsymbol{v} = \mathbf{0}$, $p = 0$, $f = 0$, which is destabilized (*parametric excitation* in a Floquet problem).

Linear stability: $(\boldsymbol{v}, p, f) = (\boldsymbol{V}, P, F)\mathrm{e}^{\lambda t}$:

- Surface modes ($\lambda = \pm\mathrm{i} + O(\sqrt{\varepsilon})$, with $\lambda \boldsymbol{V} \simeq -\boldsymbol{\nabla} P$, except in boundary layers); *surface waves*
- Hydrodynamic modes ($\lambda \sim \varepsilon$), with $\lambda \boldsymbol{V} \simeq -\boldsymbol{\nabla} P + \varepsilon \Delta \boldsymbol{V}$ everywhere: *viscous mean flow.* Excited by time-averaged Reynolds stresses in boundary layers; produce a tangential velocity near solid walls (Rayleigh 1883, Schlichting 1932) and a tangential stress near the free surface (Longuet-Higgings 1954); both quadratic in the surface wave steepness, and nonzero as $\varepsilon \to 0$!!.

2D weakly nonlinear theory at large aspect ratio
(Vega-Knobloch-Martel *Physica D*154(2001)313-336): Seek

$$\begin{aligned}
(\boldsymbol{v}, p, f) = {} & A^+ \mathrm{e}^{\mathrm{i}(t-x)}(\boldsymbol{V}, P, F) + A^- \mathrm{e}^{\mathrm{i}(t+x)}(\boldsymbol{V}^S, P, F) + \text{c.c.} \\
& + (\boldsymbol{v}^m, p^m, f^m) + NRT,
\end{aligned}$$

$A^\pm$ and $(\boldsymbol{v}^m, p^m, f^m)$ slowly varying in x and t.

Mean flow (mass transport due to global circulation produced by surface waves through nonlinearity). Sources:

- *Mass conservation* (Davey-Stewartson, *Proc. Roy. Soc. London* A338(1974)101-110)
- *Time-averaged Reynolds stresses* in oscillatory boundary layer (Lord Rayleigh *Phil. Trans. Roy. Soc. London* A175(1883)1-21 1873 , Schlichting *Phys Z* 33(1932)327-335, Longuet-Higgins *Phil. Trans. Roy. Soc. London* A245(1954)535-581).
- *Stokes drift* (Stokes *Trans. Cam. Phil. Soc.* 8(1847)441-455), a purely kinematic effect; kinematics produce surprising effects, like chaotic advection (Aref *J. Fluid Mech.* 212(1984)337-356).

$$\frac{d\boldsymbol{x}}{dt} = \mu \boldsymbol{V}(\boldsymbol{x})\mathrm{e}^{\imath t} + \text{c.c.},$$

with $\mu \ll 1$. Set $\boldsymbol{x} = \mu\boldsymbol{x}_1 + \mu^2\boldsymbol{x}_2 + \ldots$. Then

$$\frac{d\boldsymbol{x}_1}{dt} = \boldsymbol{V}(\boldsymbol{x}_1)\mathrm{e}^{\imath t} + \text{c.c.}, \quad \frac{d\boldsymbol{x}_2}{dt} = (\boldsymbol{V}\cdot\boldsymbol{\nabla})\boldsymbol{V}\mathrm{e}^{2\imath t} + (\bar{\boldsymbol{V}}\cdot\boldsymbol{\nabla})\boldsymbol{V} + \text{c.c.},$$

which yields a mean flow velocity $\boldsymbol{v}^m = \bar{\boldsymbol{V}}\cdot\boldsymbol{\nabla}\boldsymbol{V} + \text{c.c.}$. In the incompressible case, the Stokes drift is nonzero only if the waves are progressive, namely exhibit a nonconstant spatial phase.

$$
\begin{aligned}
A_t^{\pm} \mp v_g A_x^{\pm} + \mathrm{i}\alpha_0 A_{xx}^{\pm} &= -(2\varepsilon + \mathrm{i}\delta)A^{\pm} + \mathrm{i}(\alpha_1|A^{\pm}|^2 - \alpha_2|A^{\mp}|^2)A^{\pm} \\
&\quad \pm 2\mathrm{i}\int_{-1}^{0} \mathrm{e}^{2y}\varphi_y^m \, dy \, A^{\pm} + a\bar{A}^{\mp}, \\
\varphi_{yyt}^m &= \varphi_{yyyy}^m, \quad \text{in } -d < y < 0,
\end{aligned}
$$

with boundary conditions

$$
\begin{gathered}
\varphi_x^m - f_t^m = 2(|A^-|^2 - |A^+|^2)_x, \quad \varphi_{yy}^m = 8(|A^+|^2 - |A^-|^2), \\
(1-S)f_x^m + \varepsilon\varphi_{yyy}^m = 0 \quad \text{at } y = 0; \\
\varphi^m = \varphi_y^m = 0 \quad \text{at } y = -d.
\end{gathered}
$$

φ^m= *streamfunction* ($\boldsymbol{v}^m = (-\varphi_y^m, \varphi_x^m)$). $L \gg 1$, $\delta \ll 1$, and

$$
\begin{aligned}
v_g &= \frac{1+2S}{2}, \quad \alpha_0 = 3S - \frac{3(1+2S)^2}{2}, \\
\alpha_1 &= \frac{3S}{1-3S} + \frac{8-3S}{4}, \quad \alpha_2 = -\frac{2}{1+3S} - \frac{4+3S}{2}.
\end{aligned}
$$

- 'Generic' under the following assumptions: *Weak dissipation and weak forcing* $\varepsilon \ll 1$, $a \ll 1$; *long wave* $|\partial/\partial x| \ll 1$; *slowly varying* $|\partial/\partial t| \ll 1$; and *weakly nonlinear* $|A^{\pm}| \ll 1$.
- *Invariant under* $O(2)$:

$$
A^+ \leftrightarrow A^-, \quad (x, \varphi^m) \to -(x, \varphi^m); \qquad x \to x + c.
$$

- *Resonances avoided*: $1-S$ and $1-3S$ not too small.
- *Nonlocal effect* of the mean flow, which is parallel in this limit.
- *No diffusion*. *Broad band instabilities* are avoided by dispersion/nonlinearity, as in KdV.
- Two slow spatial scales (instabilities): *convection at the group velocity* ($x \sim \varepsilon^{-1}$) and *dispersion* ($x \sim \varepsilon^{-1/2}$).
- *'Simplest' (standing-wave) limit*: dissipative, Ginzburg-Landau-like equation.

Set $(A^+, A^-) = (A_0, \bar{A}_0)e^{\lambda t + i\kappa x}$ into the amplitude equations, and linearize. *Dispersion relation:*

$$\lambda = -2\varepsilon \pm \sqrt{a^2 - v_g^2\kappa^2} \simeq a - a_c - v_g^2\kappa^2/(4\varepsilon) + \dots,$$

as $\kappa \ll a_c \equiv 2\varepsilon \ll 1$. This means that the linear part of the amplitude equation ($(A^+, A^-) \simeq (A, \bar{A})$, with $A \in \mathbb{C}$) should be $A_t = v_g^2\kappa^2 A_{xx}/(4\varepsilon)$, and we expect a *Ginzburg-Landau-like equation.* In order to guess nonlinear terms, we seek spatially constant solutions, of the form $A^\pm = Re^{\pm i\kappa x + i\nu}$; replacing this into the amplitude equations, we obtain

$$4\varepsilon^2 + [v_g\kappa + (\alpha_1 + \alpha_2)R^2]^2 = a^2, \quad 2\varepsilon = a\cos 2\nu,$$

which setting $|a - a_c| \ll 1$, yields

$$\varepsilon(a - a_c) \simeq v_g^2\kappa^2/4 + (\alpha_1 + \alpha_2)^2 R^4/4 + v_g(\alpha_1 + \alpha_2)\kappa R^2/2.$$

This suggests the following *quintic equation*

$$A_t = \frac{\beta_1}{\varepsilon}A_{xx} + (a - a_c)A - \frac{\beta_2}{\varepsilon}|A|^4 A - i\frac{\beta_3}{\varepsilon}|A|^2 A_x,$$

which contains higher order terms 'generically'. But this is not the end of the story because the right equation also has a nonlinear term

$$-i\frac{\beta_4}{\varepsilon}(|A|^2)_x A, \quad \text{with } \beta_4 = -\frac{\alpha_1 v_g}{2},$$

which could never have been guessed using simple priori arguments. The additional terms break a *spurious reflection symmetry.* Note:

- We are obtaining diffusion in a non-diffusive problem. In fact, 'diffusion' comes from transport at the group velocity.
- The mean flow plays no role in this.
- We are obtaining a nongeneric equation under 'generic' conditions, which is surprising. The question is: What has happened with cubic nonlinearity?

The answer is: We are obtaining a nongeneric equation because the starting point is a nongeneric system: *viscous effects* have been only considered in the linear damping term. Both nonlinear damping and nonlinear forcing have been neglected in the coupled amplitude-mean flow equations:

$$\begin{aligned} A_t^{\pm} \mp v_g A_x^{\pm} + \mathrm{i}\alpha_0 A_{xx}^{\pm} &= -(2\varepsilon + \mathrm{i}\delta)A^{\pm} + \mathrm{i}(\alpha_1|A^{\pm}|^2 - \alpha_2|A^{\mp}|^2)A^{\pm} \\ &\quad \pm 2\mathrm{i}\int_{-1}^{0} \mathrm{e}^{2y}\varphi_y^m\, dy\, A^{\pm} + a\bar{A}^{\mp}, \\ \varphi_{yyt}^m &= \varphi_{yyyy}^m, \quad \text{in } -d < y < 0, \end{aligned}$$

with boundary conditions

$$\begin{gathered} \varphi_x^m - f_t^m = 2(|A^-|^2 - |A^+|^2)_x, \quad \varphi_{yy}^m = 8(|A^+|^2 - |A^-|^2), \\ (1-S)f_x^m + \varepsilon\varphi_{yyy}^m = 0 \quad \text{at } y = 0, \\ \varphi^m = \varphi_y^m = 0 \quad \text{at } y = -d, \\ A^{\pm}(x+L,t) \equiv A^{\pm}(x,t), \\ \varphi^m(x+L,y,t) \equiv \varphi^m(x,y,t), \quad f^m(x+L,t) \equiv f^m(x,t), \\ \textstyle\int_0^L f^m\, d\xi = \int_0^L \varphi_{yyy}^m\, dx = 0. \end{gathered}$$

When cubic $O(|A^{\pm}|^3)$-corrections are added to both damping and forcing, the following generic equations result (Mancebo-Vega, *Physica D*197(2004)346-363):

$$\begin{aligned} A_t &= \frac{\beta_1}{\varepsilon}A_{xx} + (a - a_c)A - \beta_5\varepsilon|A|^2A \\ &\quad - \frac{\beta_2}{\varepsilon}|A|^4A - \mathrm{i}\frac{\beta_3}{\varepsilon}|A|^2A_x - \mathrm{i}\frac{\beta_4}{\varepsilon}(|A|^2)_xA - \mathrm{i}\frac{\beta_6}{\varepsilon}\phi_x, \\ \phi_t &= \frac{\beta_7}{\varepsilon}\phi_{xx} - \frac{\beta_8}{\varepsilon}(|A|^2)_{xx}, \end{aligned}$$

where ϕ comes from the mean flow. This system includes as particular case its cubic version, which is also obtained in weakly nonlinear dynamics of conservation laws invariant under the $O(2)$ group (Matthews-Cox *Nonlinearity* 13(2000)1293-1320, Vega *Nonlinearity* 18(2005)1425-1441).

- *Water waves* (namely, liquid motions with interfaces at low viscosity) still involve fascinating open problems from both the fluid dynamicist and the applied mathematician points of view.
- Weakly nonlinear dynamics must be analyzed from *first principles*; ad hoc guessing is useful, but dangerous.
- Weakly nonlinear dynamics must include, as determining modes, *all nearly-marginal modes*.
- Non-conservative hyperbolic problems may exhibit parabolic behavior (e.g., the sine-Gordon equation exhibits Ginzburg-Landau dynamics).