Fractalización y ANCE

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Consider

$$\left.\begin{array}{rcl} \bar{x} & = & f_\mu(x,\theta), \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\}$$

where $x \in \mathbb{R}$, $\theta \in \mathbb{T}^1$, $\mu \in \mathbb{R}$ is a parameter, $\omega \in (0, 2\pi) \setminus 2\pi\mathbb{Q}$ and f_μ is smooth enough.

Assume that, for a given μ_0, there is an attracting invariant curve, $x_{\mu_0}(\theta)$ with rotation number ω,

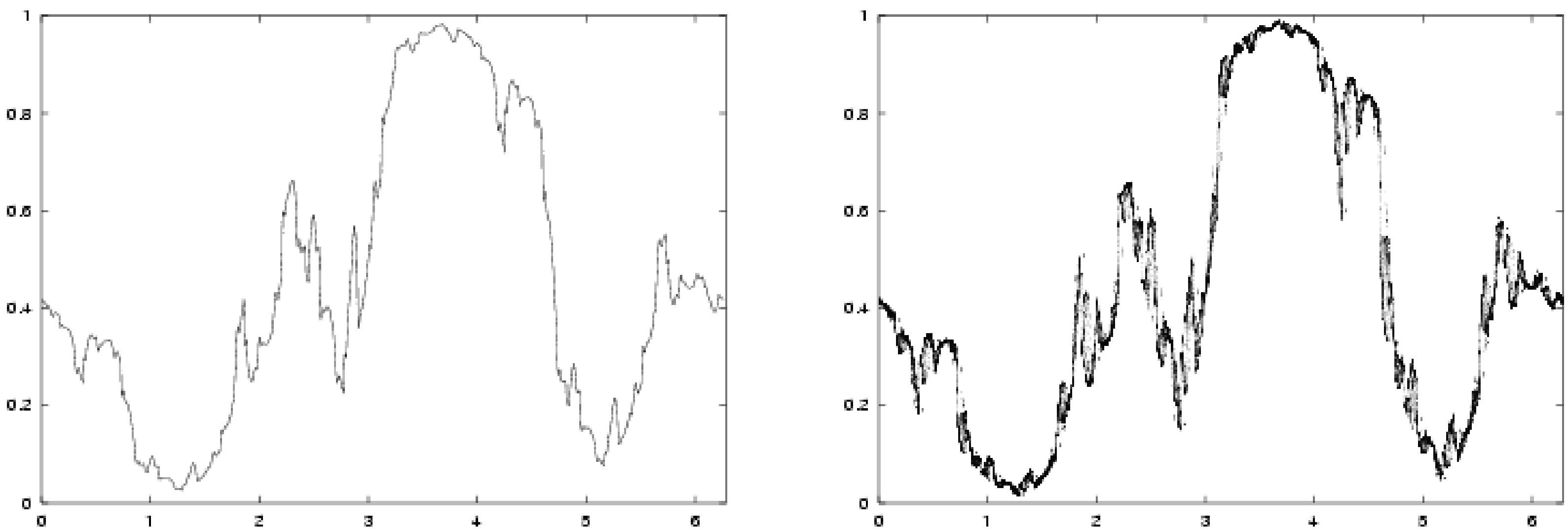
$$f_{\mu_0}(x_{\mu_0}(\theta),\theta) = x_{\mu_0}(\theta+\omega), \qquad \forall\, \theta \in \mathbb{T}^1.$$

We want to study the continuation (and the bifurcations) of x_{μ_0} with respect to the parameter μ.

Example: the quasiperiodically forced logistic map.

$$\left.\begin{array}{rcl} \bar{x} & = & \alpha(1+\varepsilon\cos(\theta))x(1-x), \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\}$$

with $\omega = \pi(\sqrt{5}-1)$ and $\varepsilon = 0.5$.



Left: $\alpha = 2.65$, $\Lambda \approx -0.03884$. Right: $\alpha = 2.665$, $\Lambda \approx -0.00845$.

In this talk we consider the fractalization process as a bifurcation.

Assume that

$$\left.\begin{array}{rcl}\bar{x} & = & f_\mu(x,\theta),\\ \bar{\theta} & = & \theta+\omega,\end{array}\right\}$$

has a C^r $(r \geq 0)$ invariant curve $x = u_0(\theta)$ for $\mu = 0$.

This curve satisfies the functional equation $F(u_0, 0) = 0$, where $F : C^r(\mathbb{T}^1, \mathbb{R}) \times \mathbb{R} \to C^r(\mathbb{T}^1, \mathbb{R})$ and, if $(u, \mu) \in C^r(\mathbb{T}^1, \mathbb{R}) \times \mathbb{R}$,

$$F(u,\mu)(\theta) = f_\mu(u(\theta),\theta) - u(\theta+\omega).$$

To apply the Implicit Function Theorem, $D_uF(u_0, 0)$ needs to be a linear bounded operator with bounded inverse.

The action of $D_uF(u, \mu)$ on an element $v \in C^r(\mathbb{T}^1, \mathbb{R})$ is given by

$$[D_uF(u,\mu)v](\theta) = D_xf_\mu(u(\theta),\theta)v(\theta) - v(\theta+\omega).$$

As $f_0(u_0(\theta)+h,\theta) = f_0(u_0(\theta),\theta) + D_x f_0(u_0(\theta),\theta)h + \cdots$, the linearized dynamics around $u_0(\theta)$ is given by

$$\left.\begin{array}{rcl} \bar{x} & = & a(\theta)x, \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\} \tag{1}$$

where $a(\theta) = D_x f_0(u_0(\theta),\theta)$.

In what follows, we will assume that $a(\theta) \not\equiv 0$.

Definition 1 *(1) is called* reducible *iff there exists a linear change of variables $x = c(\theta)y$ such that (1) becomes*

$$\left.\begin{array}{rcl} \bar{y} & = & by, \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\}$$

where b does not depend on θ.

The bifurcations of reducible curves can be studied by means of normal form techniques.

Proposition 1 *Assume that ω satisfies a Diophantine condition,*

$$|q\omega - 2\pi p| \geq \frac{\gamma}{|q|^\tau}, \qquad \text{for all } (p,q) \in \mathbb{Z} \times (\mathbb{Z} \setminus \{0\}),$$

and that a is C^∞. Then, (1) is reducible iff a has no zeros.

This result also holds if $a \in C^r$, for r big enough but, due to the effect of the small divisors, the reducing transformation does not need to belong to C^r.

The Lyapunov exponent of (1) at θ is

$$\lambda(\theta) = \limsup_{n\to\infty} \frac{1}{n} \ln \left| \prod_{j=0}^{n-1} a(\theta + j\omega) \right| .$$

We define

$$\Lambda = \frac{1}{2\pi} \int_0^{2\pi} \ln |a(\theta)| \, d\theta.$$

If Λ is finite, then the Birkhoff ergodic theorem implies that

$$\lambda(\theta) = \Lambda, \qquad \text{for Lebesgue-a.e. } \theta \in \mathbb{T}^1.$$

The value Λ is usually known as the Lyapunov exponent of the skew product.

Proposition 2 *If $a(\theta)$ is C^0 and the skew product is reducible, then the Lyapunov exponent at θ, $\lambda(\theta)$, does not depend on θ.*

Theorem *Let us consider a one-parametric family of linear skew-products*

$$\left.\begin{aligned} \bar{x} &= a(\theta,\mu)x, \\ \bar{\theta} &= \theta+\omega, \end{aligned}\right\}$$

where ω *is Diophantine and* μ *belongs to an open subset of* $\mathbb{R}$. a *is a* C^∞ *function of* θ *and* μ. *We assume that:*

1. *For each* μ, $a(\cdot,\mu)$ *has finitely many zeros, each of them are simple except maybe one of multiplicity 2.*

   Let us call M *the (open) set of values of* μ *for which all the zeros of* $a(\cdot,\mu)$ *are simple.*

2. *If* $a(\cdot,\mu)$ *has a zero of multiplicity 2 at* $\theta=\theta_0$ *for* $\mu=\mu_0$, *then*

$$\frac{\partial a}{\partial \mu}(\theta_0,\mu_0)\neq 0.$$

Then, the Lyapunov exponent $\Lambda(\mu)$ *depends continuously on* μ*, and*

1. Λ *is* C^∞ *on* M.
2. *If* $\mu_0 \notin M$*, then*
   (a) *if the number of zeros of* $a(\cdot,\mu)$ *increases at* μ_0*, then*

$$\lim_{\mu\to\mu_0^-} \Lambda'(\mu) = -\infty, \textit{ and } \lim_{\mu\to\mu_0^+} \Lambda'(\mu) \textit{ exists and is finite}$$

   (b) *if the number of zeros of* $a(\cdot,\mu)$ *decreases at* μ_0*, then*

$$\lim_{\mu\to\mu_0^-} \Lambda'(\mu) \textit{ exists and is finite, and } \lim_{\mu\to\mu_0^+} \Lambda'(\mu) = +\infty.$$

Moreover, for $\mu \to \mu_0^-$ *in (a) and for* $\mu \to \mu_0^+$ *in (b), we have*

$$\Lambda(\mu) = \Lambda(\mu_0) + A\sqrt{|\mu-\mu_0|} + O(|\mu-\mu_0|). \qquad (A > 0).$$

Definition 2 *If* $a \in C^r(\mathbb{T}^1, \mathbb{R})$*, the transfer operator* $\mathcal{L} : C^r \to C^r$ *is defined as*

$$(\mathcal{L}\psi)(\theta) = a(\theta - \omega)\psi(\theta - \omega) \quad \forall\, \theta \in \mathbb{T}^1. \tag{2}$$

It is easy to check that we can apply the IFT if and only if 1 does not belong to the spectrum of the transfer operator.

The reducibility depends on the existence of eigenfunctions for $\mathcal{L}$.

Regardless of the reducibility, the spectrum of $\mathcal{L}$ is invariant by rotations (Mather, 1968).

Proposition 3 *Let $\mathcal{L} : C^0 \to C^0$ and Λ denote, respectively, the transfer operator and the Lyapunov exponent of (1). Then,*

$$\rho(\mathcal{L}) = \exp(\Lambda).$$

If a is C^r, $\mathcal{L}$ can be defined acting on any C^s, $0 \leq s \leq r$. It can be shown (A. Haro & R. de la Llave, 2005) that its spectrum does not depend on s.

Proposition 4 *If a has zeros (this implies that the skew product is not reducible), then*

$$\mathrm{Spec}\,(\mathcal{L}) = \{z \in \mathbb{C} \textit{ such that } |z| \leq \exp(\Lambda)\}.$$

Affine systems

$$\left.\begin{array}{rcl}\bar{x} & = & \alpha\, a(\theta)x + b(\theta), \\ \bar{\theta} & = & \theta + \omega,\end{array}\right\} \tag{3}$$

where a and b are C^r functions and α is a real positive parameter. It is clear that, for any invariant curve of (3), its linearized normal behaviour is described by

$$\left.\begin{array}{rcl}\bar{x} & = & \alpha\, a(\theta)x, \\ \bar{\theta} & = & \theta + \omega.\end{array}\right\} \tag{4}$$

In what follows, we will assume that (4) is not reducible.
The Lyapunov exponent is given by

$$\Lambda = \ln \alpha + \frac{1}{2\pi}\int_0^{2\pi} \ln |a(\theta)|\, d\theta.$$

If the previous integral exists (and it is finite), then the Lyapunov exponent is negative for sufficiently small values of α, namely,

$$\alpha < \alpha_0 = \exp\left(-\frac{1}{2\pi}\int_0^{2\pi} \ln|a(\theta)|\, d\theta\right).$$

In particular this implies that, for $\alpha < \alpha_0$, *any* invariant curve of

$$\left.\begin{array}{rcl} \bar{x} & = & \alpha\, a(\theta)x + b(\theta), \\ \bar{\theta} & = & \theta + \omega, \end{array}\right\}$$

is attracting and, therefore, it must be unique.

Proposition 5 *If a and b are of class C^r and $\alpha < \alpha_0$, then there exists a unique attracting invariant curve of class C^r of (3).*

Fractalization

As we are dealing with an affine system and the sup norm of a curve does not need to be bounded, we will say that a curve is fractalizing when its C^1 norm –taken on any closed nontrivial interval for θ– goes to infinity much faster than its C^0 norm, that is, when

$$\limsup_{\alpha\to\alpha_0} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty,$$

where $\|\cdot\|_{I,\infty}$ is the sup norm on a nontrivial closed interval I.

Theorem 1 *Assume that* $a, b \in C^1(\mathbb{T}, \mathbb{R})$ *and that (4) is not reducible. Then,*

a) If $\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty < +\infty$,

and $b \in D_1$ *(*D_1 *is a suitable residual set), we have*

$$\limsup_{\alpha \to \alpha_0^-} \|x'_\alpha\|_{I,\infty} = +\infty,$$

for any nontrivial closed interval $I \subset \mathbb{T}$*.*

b) If $\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_\infty = +\infty$,

then, for any nontrivial closed interval $I \subset \mathbb{T}$*, we have*

$$\limsup_{\alpha \to \alpha_0^-} \|x_\alpha\|_{I,\infty} = +\infty, \qquad \textit{and} \qquad \limsup_{\alpha \to \alpha_0^-} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty.$$

On repelling continuous curves

Now we assume that $\alpha > \alpha_0$ which implies that the origin of is a repellor. As before, we are assuming that the skew product is not reducible and we are interested in the existence of a repelling invariant curve.

Proposition 6 *Assume, for all $\theta \in \mathbb{T}^1$, that $a(\theta) \geq 0$. Then the operator*

$$x(\theta) \mapsto x(\theta + \omega) - \alpha a(\theta) x(\theta),$$

defined on $C^0(\mathbb{T}^1, \mathbb{R})$, is not surjective. In particular, there is no $x \in C^0(\mathbb{T}^1, \mathbb{R})$ such that $x(\theta + \omega) = \alpha a(\theta) x(\theta) + 1$.

Proposition 7 *Assume, in the hypothesis of Proposition 6, that $a(\theta)$ is not always positive. Then, there exists $b \in C^0(\mathbb{T}^1, \mathbb{R})$ for which there is no $x \in C^0(\mathbb{T}^1, \mathbb{R})$ such that $x(\theta + \omega) = \alpha a(\theta) x(\theta) + b(\theta)$.*

In this section we focus on the fractalization phenomena for the affine system (3), but assuming that a is a positive function with at least a zero (so that the skew product is not reducible).

Proposition 8 *Assume, in (3), that $a, b \in C^1(\mathbb{T}, \mathbb{R})$, $a(\theta) \geq 0$ for all $\theta \in \mathbb{T}^1$ and there exists a value θ_0 such that $a(\theta_0) = 0$. We also assume that b never vanishes. Then,*

a) If $a, b \in C^r(\mathbb{T}, \mathbb{R})$, $r \geq 1$, then $x_\alpha \in C^r(\mathbb{T}, \mathbb{R})$ for $0 < \alpha < \alpha_0$.

b) For any nontrivial closed interval $I \subset \mathbb{T}$, we have

$$\lim_{\alpha \to \alpha_0^-} \|x_\alpha\|_{I,\infty} = +\infty, \quad \text{and} \quad \lim_{\alpha \to \alpha_0^-} \frac{\|x'_\alpha\|_{I,\infty}}{\|x_\alpha\|_\infty} = +\infty.$$

c) For $\alpha > \alpha_0$, there is no $x \in C^0(\mathbb{T}, \mathbb{R})$ such that $x(\theta + \omega) = \alpha a(\theta) x(\theta) + b(\theta)$.

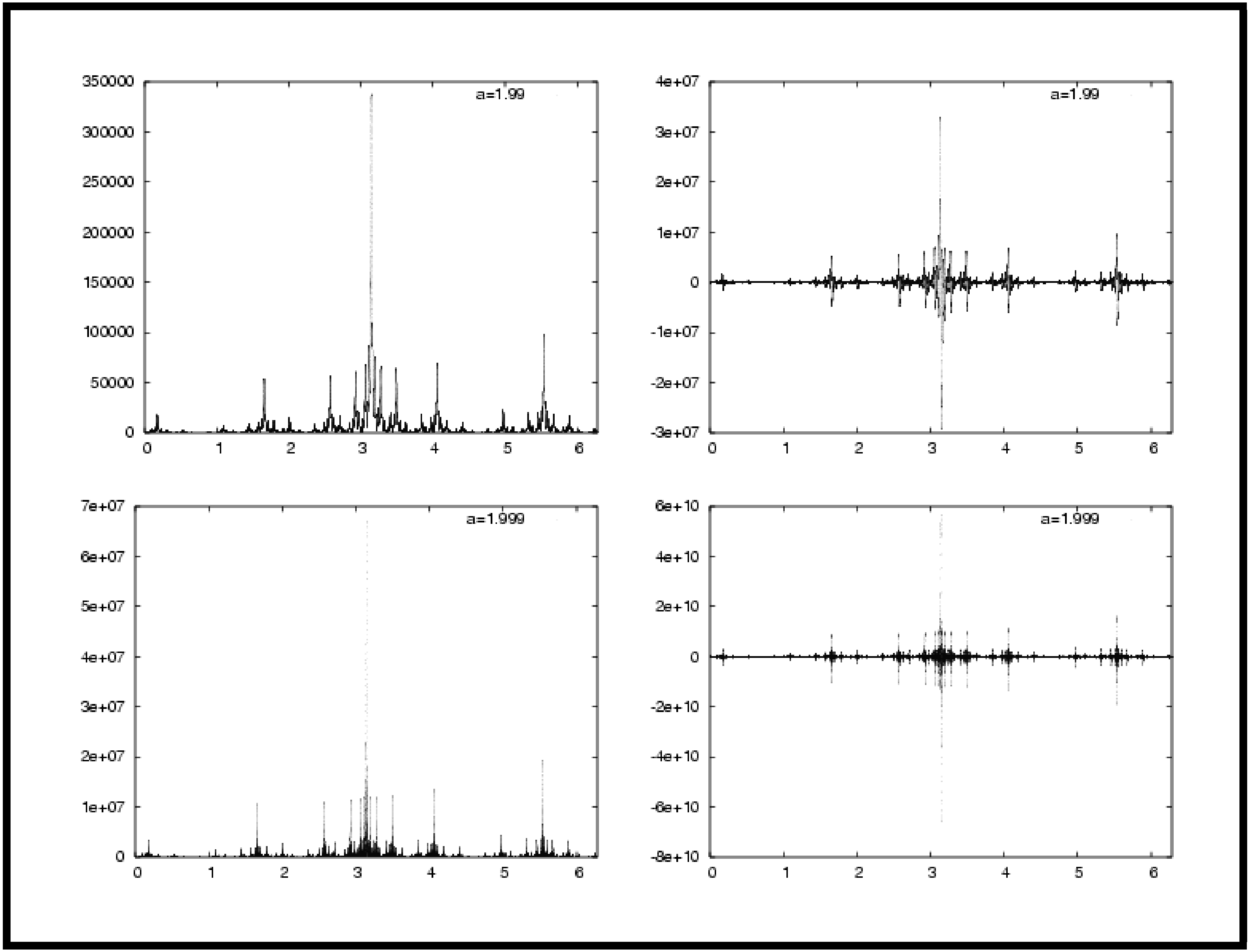
Some numerical examples

$$\left.\begin{array}{rcl}\bar{x} & = & \alpha\,(1+\cos\theta)x+1,\\ \bar{\theta} & = & \theta+\omega,\end{array}\right\}$$

where ω is the golden mean. We note that $1+\cos\theta \geq 0$ so we are in the hypotheses of the last proposition.

The Lyapunov exponent of the linear skew product is $\Lambda = \ln\alpha - \ln 2$ and, therefore, the critical value α_0 is 2.

We have proved that there exists a unique invariant attracting curve for $0 < \alpha < 2$, that undergoes a fractalization process when $\alpha \to 2^-$.


a=1.99
350000
300000
250000
200000
150000
100000
50000
0
0 1 2 3 4 5 6
a=1.99
4e+07
3e+07
2e+07
1e+07
0
-1e+07
-2e+07
-3e+07
0 1 2 3 4 5 6
a=1.999
7e+07
6e+07
5e+07
4e+07
3e+07
2e+07
1e+07
0
0 1 2 3 4 5 6
a=1.999
6e+10
4e+10
2e+10
0
-2e+10
-4e+10
-6e+10
-8e+10
0 1 2 3 4 5 6

Another example.

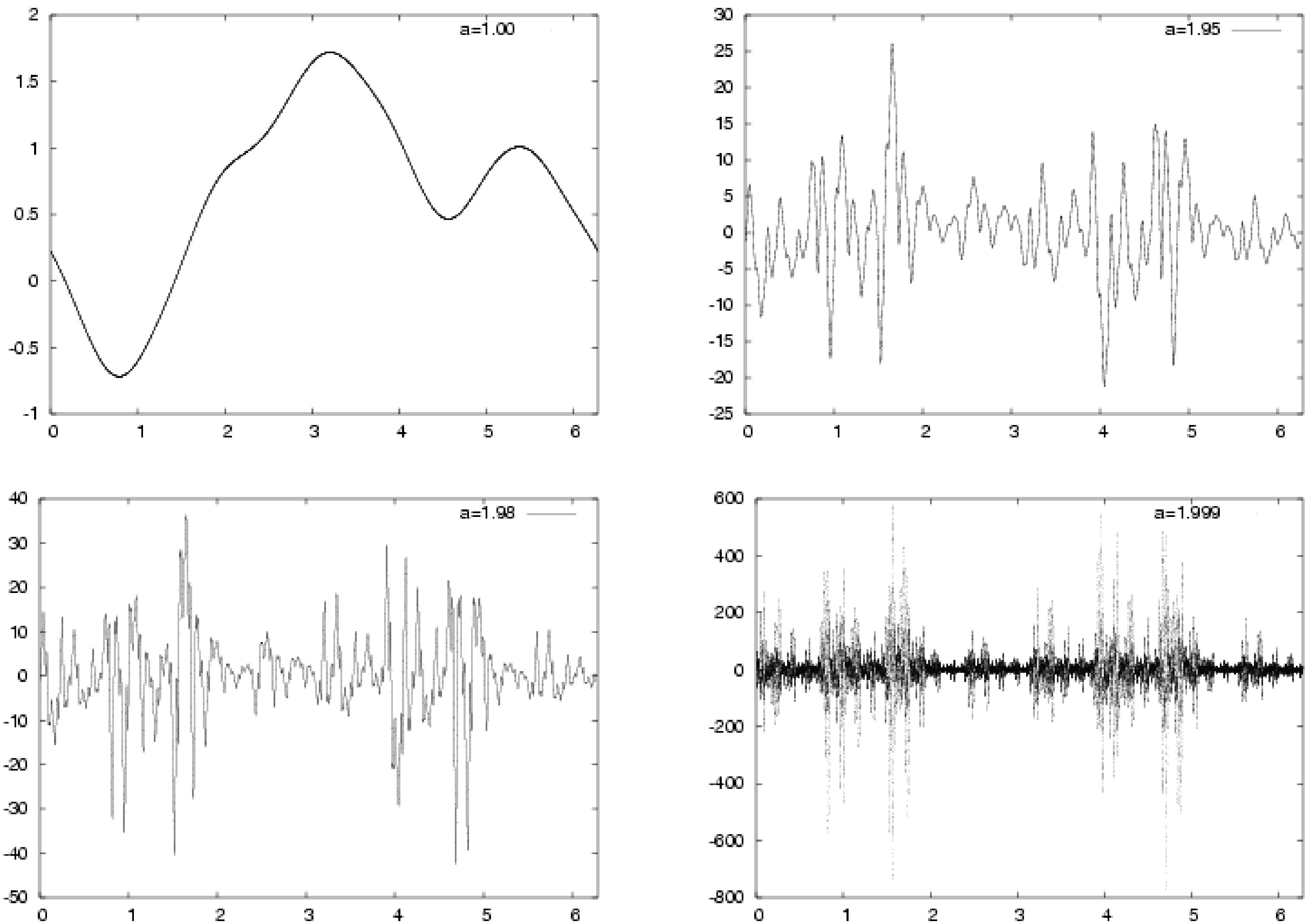
$$\left.\begin{array}{rcl} \bar{x} & = & \alpha\,\cos(\theta)\,x+1, \\ \bar{\theta} & = & \theta+\omega, \end{array}\right\}$$

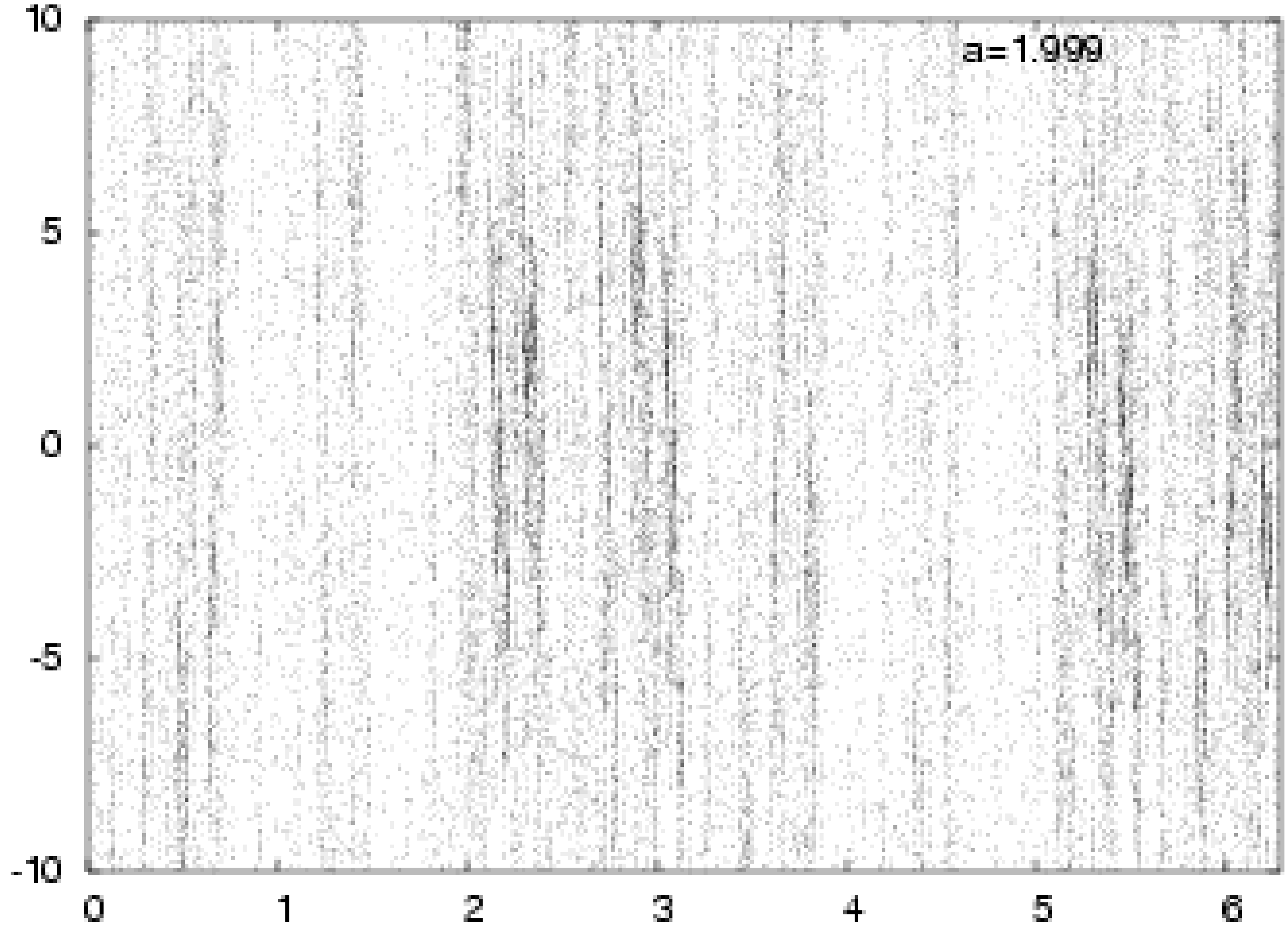
being α a positive parameter.

It is easy to see that its Lyapunov exponent is $\ln\alpha - \ln 2$.

If $\alpha < 2$, the Lyapunov exponent is negative. Therefore, we must have a unique and global attracting set.

Next slides show the attractor for several values $\alpha < 2$.


a=1.00
a=1.95
a=1.98
a=1.999


a=1.999
10
5
0
-5
-10
0
1
2
3
4
5
6

The quasiperiodically forced logistic map.

$$\left.\begin{array}{rcl}\bar{x} & = & \alpha(1+\varepsilon\cos(\theta))x(1-x), \\ \bar{\theta} & = & \theta+\omega,\end{array}\right\}$$

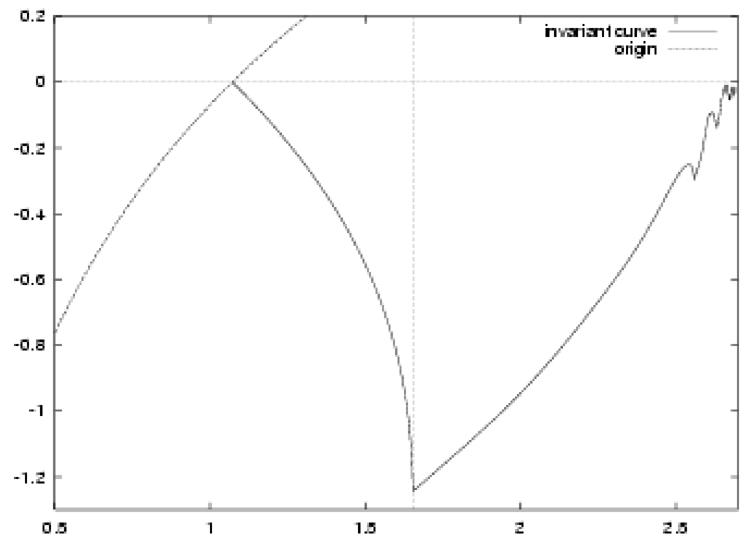
with $\omega = \pi(\sqrt{5}-1)$.

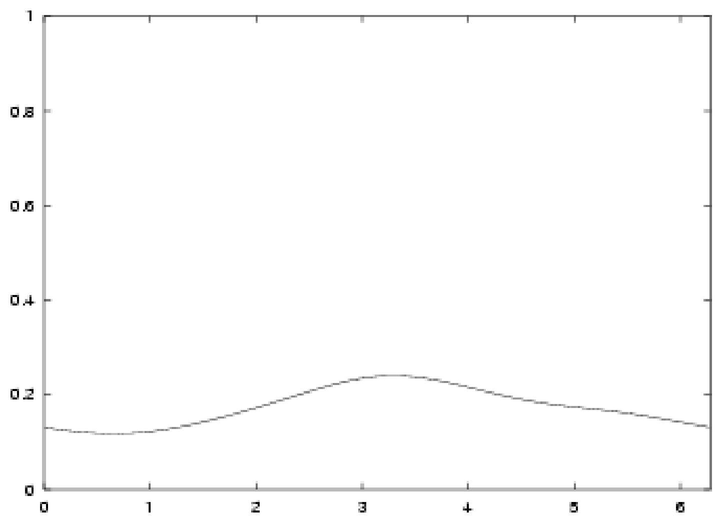
Let $x(\theta)$ be a continuous invariant curve of this map; if h denotes an infinitesimal displacement w.r.t. the curve then

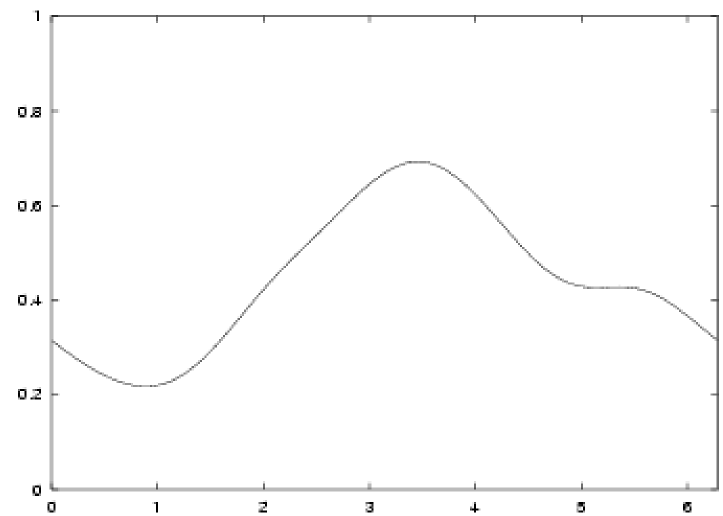
$$\left.\begin{array}{rcl}\bar{h} & = & D_x f_{\alpha,\varepsilon}(x,\theta)h = \alpha(1+\varepsilon\cos\theta)(1-2x(\theta))h, \\ \bar{\theta} & = & \theta+\omega.\end{array}\right\}$$

It is clear that $|\varepsilon| \geq 1$ or $x(\theta_0) = \frac{1}{2}$ for some θ_0 imply non-reducibility. On the other hand, if $|\varepsilon| < 1$, $x(\theta) \neq \frac{1}{2}$ (for all θ) and $x(\theta)$ is smooth, the curve is reducible.

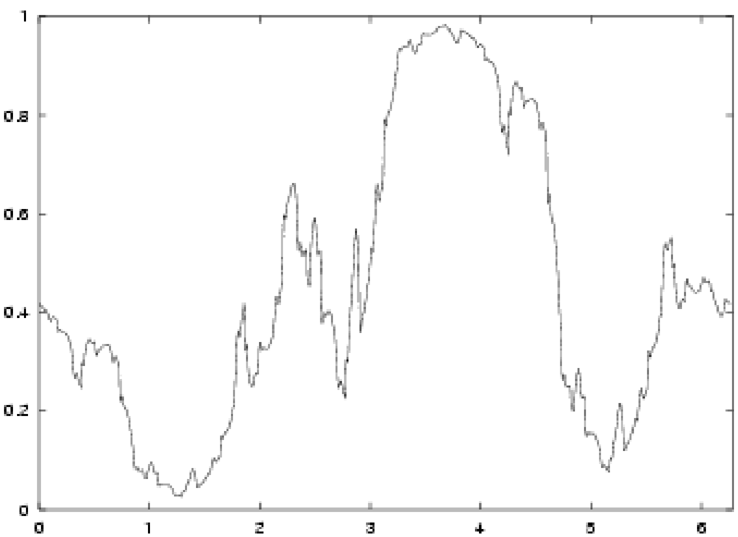
Let us select $\varepsilon = 0.5$.

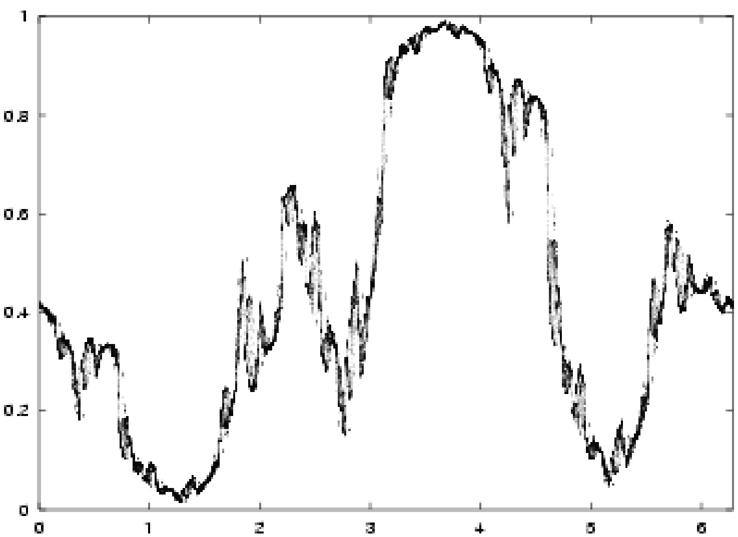




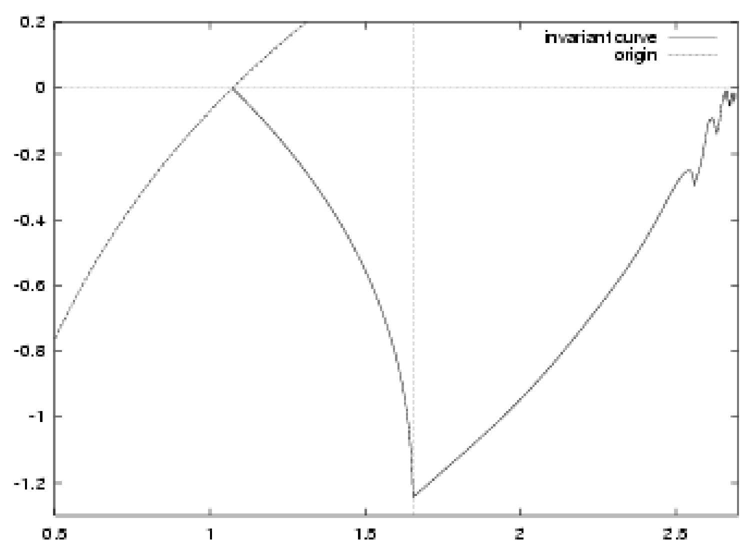


Left: $\alpha = 1.3$. Right: $\alpha = 2.0$.





Left: $\alpha = 2.65$, $\Lambda \approx -0.03884$. Right: $\alpha = 2.665$, $\Lambda \approx -0.00845$.

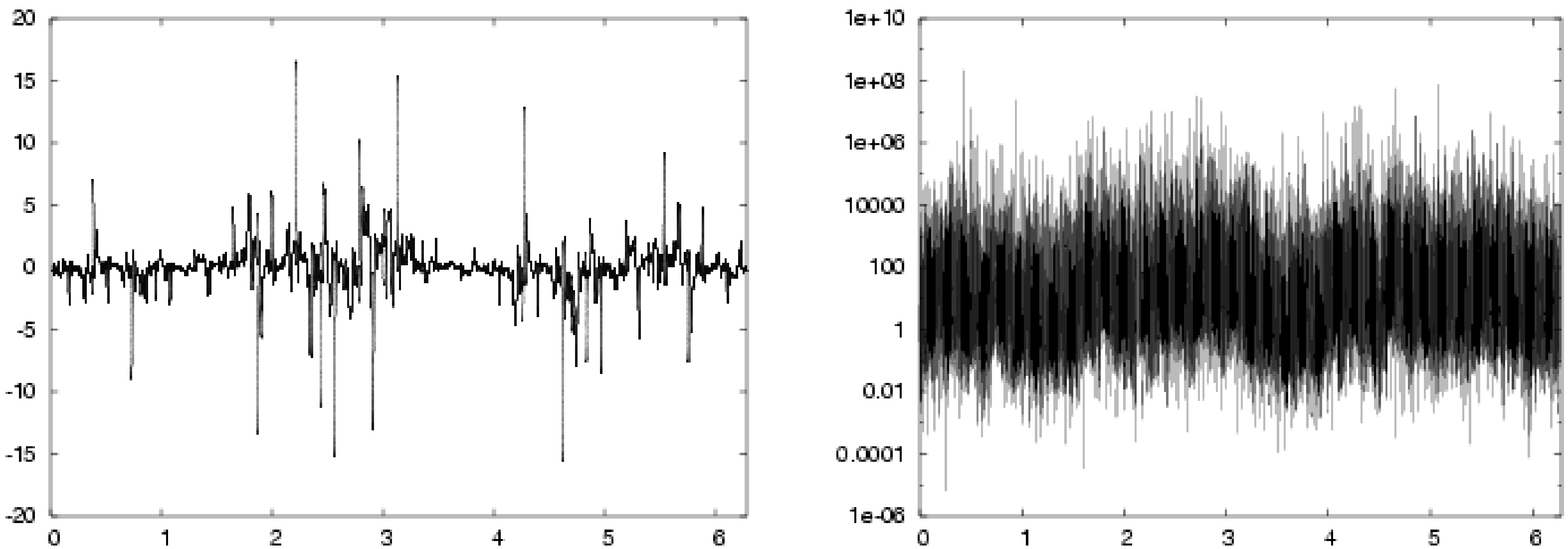


To give more numerical evidence that these “irregular” attracting sets are smooth curves, let us consider the following dynamical system,

$$\left.\begin{array}{rcl} \bar{x} & = & f(x,\theta), \\ \bar{y} & = & D_x f(x,\theta)y + D_\theta f(x,\theta), \\ \bar{\theta} & = & \theta + \omega. \end{array}\right\} \qquad (5)$$

Note that, if $x = x(\theta)$ is a smooth invariant curve, then $(x, y) = (x(\theta), x'(\theta))$ is an invariant curve of the system above. This curve is attracting set of (5) iff $x = x(\theta)$ is an attracting set of the initial system.

Now we will repeat the computations of the attracting sets but on the system (5), to estimate the shape of the derivative of the curve, if there is one. In all the cases we will use the initial condition $y_0 = 1$ for the second equation in (5).



Attracting sets for the variational flow of the quasi-periodically forced logistic map for $\alpha = 2.65$ and 2.665. The horizontal axis refers to θ and the vertical axis refers to y (see (5)). In the last plot we show $|y|$ in a log scale.

To check whether the attractor for $\alpha = 2.665$ is a curve or not, we have performed several magnifications. If the attracting set is a curve, the values of y in (5) once we are on the attracting set can be used to estimate the maximum of the absolute value of the derivative. This quantity gives the amount of magnification needed to see the attractor as a smooth curve.

After a transient of 10^6 iterates, we take the maximum of the derivative for 10^7 extra iterates, to obtain a value of -6.9×10^9 near $\theta_0 = 0.43748252111775532$.

This process is very sensitive to roundoff error, especially from the modulus 2π needed for the variable θ this point later on).

In all our tests the maximum of the derivative is of the order of 10^{10}. These estimates imply that to resolve a neighborhood of θ_0 we need magnifications of the order of 10^{10}, at least.

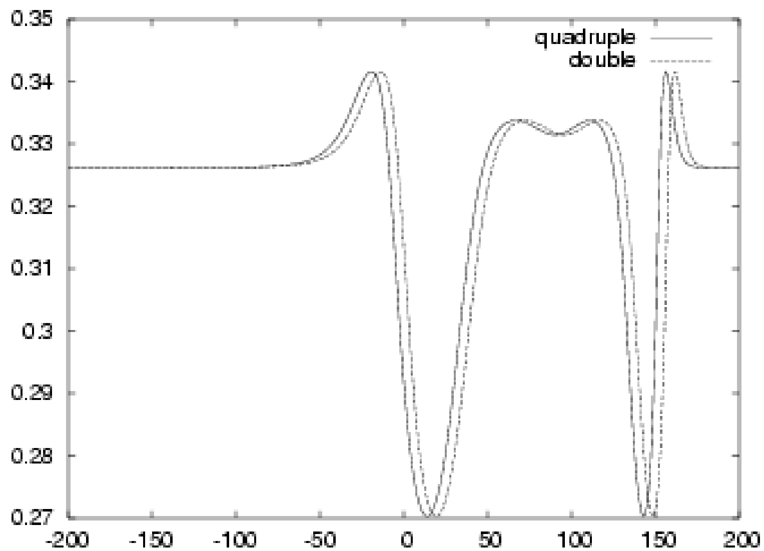
We will take the mesh $\theta_j = \theta_0 + \frac{j}{m}10^{-10}$ for j ranging from $-m$ to m. We have used several values of m between 100 and 1000. Then, we have computed the values $\hat{\theta}_j = \theta_j - n\omega (\mathrm{mod}\, 2\pi)$ for a large n (the concrete values are specified below) and we have iterated forward the points $\theta = \hat{\theta}_j$, $x = 0.4$, n times, to obtain the values $\tilde{\theta}_j$.

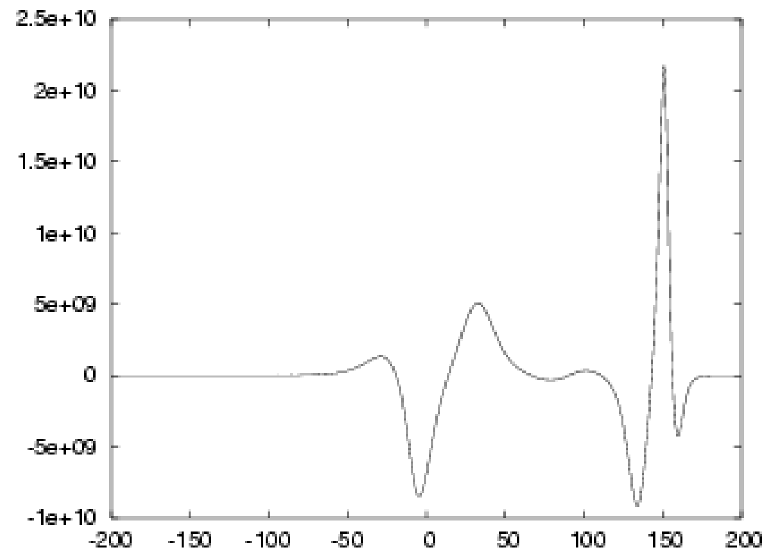
These values should coincide with the initial values θ_j but, due to the roundoff errors (mainly in the operation $\mathrm{mod}\, 2\pi$) they are slightly different. For instance, for $n = 10^5$, the differences $\theta_j - \tilde{\theta}_j$ are close to 2.5×10^{-12}.

To be sure that the results do not depend on the roundoff errors, we have repeated these computations with quadruple precision. Now, for $n = 10^5$ the differences $\theta_j - \tilde{\theta}_j$ are close to 1.7×10^{-23}.

To estimate the effect of the transient in these computations, we have repeated them for $n = 2 \times 10^5$ with no visible differences in the plots. We have also performed this zoom for other values of θ_0 with similar results.

The results are shown below, where we have displayed the index j vs. the corresponding value of x.





Conclusions.

- Any continuous invariant curve with negative maximal Lyapunov exponent is smooth and locally persistent (because it is normally hyperbolic).
- There are simple examples where the process of fractalization consists of the increase of the lenght of the invariant curve, but the smoothness is preserved until the Lyapunov exponent is zero.

  This could be the case of the forced logistic map, as numerical computations seem to confirm.
- From the point of view of operator theory, it seems that this process of fractalization is related to the 'failure' of the IFT when 0 becomes a spectral value that is not an eigenvalue.