HIPERBOLICIDAD NO UNIFORME Y ANCES EN LA RECTA REAL

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During the last years a lot of attention has been paid to the analysis of the so-called Strange Non-chaotic Attractors (SNA).

However the number of conjectures coming from numerical resolution is higher than the number of rigorous proofs.

We consider skew-products flows coming from non-autonomous second order differential and difference equations with quasiperiodic or almost periodic coefficients.

By using methods coming from the topological dynamics, we show the strong connection between these ‘strange’ sets and the well known almost automorphic (but not quasiperiodic) extensions of the base flow.

We establish conditions ensuring the occurrence of this kind of dynamics and sets.

The **Harper flow** is defined on $\mathbb{T}^1 \times \mathbb{R}$ by

$$\boxed{x(n+1) = -\frac{1}{x(n)} - \lambda - 2b\cos 2\pi(\omega + n\alpha)}$$

($\alpha \in \mathbb{R} - \mathbb{Q}$, $b \in \mathbb{R}$ (fixed), $\lambda \in \mathbb{R}$ (varying)**)**.

It comes from the Harper transformation

$$\begin{aligned} \bar{\omega} &= \omega + \alpha \\ \bar{x} &= \frac{1}{x} - \lambda - 2b\cos 2\pi(\bar{\omega}) \end{aligned}$$

and it is of **skew-product** type: the pair $(\omega, x_0) \in \mathbb{T}^1 \times \mathbb{R}$ is sent in time n to the pair $(\omega + n\alpha, x(n))$, where $x(n)$ satisfies the previous equation with $x(0) = x_0$.

The **Harper skew-product flow** is defined on $\mathbb{T}^1 \times \mathbb{R}$ by

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($\alpha \in \mathbb{R} - \mathbb{Q}$, $b \in \mathbb{R}$ (fixed), $\lambda \in \mathbb{R}$ (varying)**)**.

The flow is just local. To cope with the infinity point we take $\varphi = \cot^{-1} x \in [0, \pi)$ and obtain a global flow on $\mathbb{T}^1 \times \mathbb{P}^1$:

$$\boxed{\varphi(n+1) = \cot^{-1}\left(-\frac{1}{\cot\varphi(n)} - \lambda - 2b\cos 2\pi(\omega + n\alpha)\right)}$$

Both flows come from the **almost Mathieu** spectral problem,

$$\boxed{-z(n+1) - z(n-1) - 2b\cos 2\pi(\omega + n\alpha)\, z(n) = \lambda\, z(n)}$$

($x(n) = z(n)/z(n-1)$**)**, and this comes from

$$\boxed{\begin{bmatrix} z(n) \\ z(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\lambda - 2b\cos 2\pi(\omega + n\alpha) \end{bmatrix} \begin{bmatrix} z(n-1) \\ z(n) \end{bmatrix}}$$

$$\boxed{\begin{bmatrix} z(n) \\ z(n+1) \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & -\lambda - 2b\cos 2\pi(\omega + n\alpha) \end{bmatrix} \begin{bmatrix} z(n-1) \\ z(n) \end{bmatrix}}$$

Let $Z_\lambda(n,\omega)$ be its **propagation matrix**: the flow on $\mathbb{T}^1 \times \mathbb{R}^2$ sends $(\omega, \mathbf{z}_0)$ in time n to $(\omega + n\alpha, Z_\lambda(n,\omega)\,\mathbf{z}_0)$.

OSEDELETS: for Lebesgue almost every $\omega \in \mathbb{T}^1$, the limit

$$\lim_{|n|\to\infty} \frac{1}{n} \ln \|Z_\lambda(n,\omega)\| = \beta(\lambda) \geq 0$$

exists and takes the same value. **(**The values $\pm\beta(\lambda)$ are the **Lyapunov exponents** of the system.**)** In addition, if $\beta(\lambda) > 0$, there exists a measurable decomposition $\mathbb{T}^1 \times \mathbb{R}^2 = W_\lambda^+ \oplus W_\lambda^-$ in two invariant one-dimensional subbundles with

$$\lim_{|n|\to\infty} \frac{1}{n} \ln |Z_\lambda(n,\omega)\,\mathbf{z}_0| = \mp\beta(\lambda)$$

for $(\omega, \mathbf{z}_0) \in W_\lambda^\pm$ with $\mathbf{z}_0 \neq 0$.

The family of systems has an **exponential dichotomy over** $\mathbb{T}^1$ if $\beta(\lambda) > 0$ and $\mathbb{T}^1 \times \mathbb{R}^2 = W_\lambda^+ \oplus W_\lambda^-$ as **topological** sum. In this case,

$$\lim_{|n|\to\infty} \frac{1}{n} \ln |Z_\lambda(n,\omega)\,\mathbf{z}_0| = \mp\beta(\lambda)$$

for **every** $\omega \in \mathbb{T}^1$ and $(\omega, \mathbf{z}_0) \in W_\lambda^\pm$ with $\mathbf{z}_0 \neq 0$.

So that the possibilities are:

- $\beta(\lambda) = 0$: **elliptic** case,
- $\beta(\lambda) > 0$ and ED: **uniformly hyperbolic** case,
- $\beta(\lambda) > 0$ and not ED: **non-uniformly hyperbolic** case.

JOHNSON: the spectrum Σ of the (almost Mathieu) operator does not depend on $\omega \in \mathbb{T}^1$, and $\lambda \in \Sigma$ if and only if the two-dimensional system does not admit an exponential dichotomy. (That is, spectrum means ellipticity or non-uniformly hyperbolicity.)

So far the results hold for a much more general setting. What makes the Harper transformation interesting is that

- if $b \neq 0$, Σ is a (non-empty, compact) Cantor set for any $\alpha \in \mathbb{R} - \mathbb{Q}$ (Puig; Ávila & Jitomirskaya).

- if $|b| > 1$, then $\beta(\lambda) > 0$ for every $\lambda \in \mathbb{R}$ (Herman).

So that if $|b| > 1$ the dynamics for Harper transformation is never elliptic, but always hyperbolic: uniformly for $\lambda \notin \Sigma$ and non-uniformly for $\lambda \in \Sigma$. And there exists a countable infinite number of end-points of spectral gaps.

Goal: to prove the existence of SNAs on $\mathbb{R}$ for the transformations corresponding to the "extreme" values of λ.

Consider the Harper flow on $\mathbb{T}^1 \times \mathbb{P}^1$, defined by iterating

$$(\omega, \varphi_0) \mapsto (\omega + \alpha, \varphi_\lambda(1, \omega, \varphi_0))$$

with $\varphi_\lambda(1, \omega, \varphi_0) = \cot^{-1}\left(-\frac{1}{\cot \varphi_0} - \lambda - 2b \cos 2\pi(\omega + \alpha)\right)$.

Definitions. An **invariant curve** is a map $c : \mathbb{T}^1 \to \mathbb{P}^1$ which is measurable, defined everywhere, such that

$$c(\omega \cdot 1) = \varphi_\lambda(1, \omega, c(\omega)) \quad \text{for every } \omega \in \mathbb{T}^1.$$

The curve is **real** if $c : \mathbb{T}^1 \to (0^+, \pi^-)$
($[\delta, \pi - \delta] \subset (0^+, \pi^-)$, $\mathbb{P}^1 \equiv \mathbb{Z}/(n\pi)$**)**.

The corresponding **(real) invariant graph** is the set

$$\operatorname{graph}(c) = \{(\omega, c(\omega)) \mid \omega \in \mathbb{T}^1\} \subset \mathbb{T}^1 \times \mathbb{P}^1 \quad (\mathbb{T}^1 \times (0^+, \pi^-)).$$

The **Lyapunov exponent** of a real c is

$$\beta_s(c) = \int_{\mathbb{T}^1} \ln \frac{\partial \varphi_\lambda}{\partial \varphi_0}(1, \omega, c(\omega))\, d\omega\,.$$

A **strange non-chaotic attractor on** $\mathbb{R}$ is a non-continuous real invariant curve with negative Lyapunov exponent.

Goal: To establish conditions ensuring the existence of SNAs on $\mathbb{R}$ (for those values of λ), such that:

- the curve is discontinuous on a subset of full measure,
- it is continuous on a residual subset,
- the closure of the graph is a minimal subset of $\mathbb{T}^1 \times (0, \pi)$.

Theorem. $M \subset \mathbb{T}^1 \times (0, \pi)$ minimal, $M_\omega = \{x \,|\, (\omega, x) \in M\}$.

1. card $M_\omega = 1$ for the elements $\omega \in R \subset \mathbb{T}^1$ residual (M is an **almost automorphic extension** of the base);
2. if
$$\beta_s(M) = \sup_{\mathrm{graph}(c) \subset M} \widetilde{\beta}_s(c) < 0\,,$$
then card $M_\omega = 1$ for $\omega \in \mathbb{T}^1$ (M is a **copy of the base**, $\mathbb{T}^1$, the graph of a continuous invariant curve).

So that a minimal containing an SNA (its closure) is an almost automorphic extension of the base with $\beta_s(M) \geq 0$. That is,

- its section reduces to a point for a residual set of base points,
- contains a real invariant graph with negative Lyapunov exponent,
- contains a real invariant graph with non-negative **(**it will be positive**)** Lyapunov exponent,
- its sections do not reduce to a point for a full-measure set of base points.

There are well known examples of SNAs not contained in minimal sets, like the one described by KELLER.

As in his case, our SNAs will appear as a result of the **collision** (as a parameter varies) of invariant tori.

Assume $|b| > 1$ and hence $\beta(\lambda) > 0$ for every $\lambda \in \mathbb{R}$, and consider the dynamics on $\mathbb{T}^1 \times \mathbb{P}^1$.

Theorem. If $\lambda \notin \Sigma$ **(**uniformly hyperbolic case**)**,

1. there are exactly two minimal subsets of $\mathbb{T}^1 \times \mathbb{P}^1$, graphs of continuous invariant maps $\mathbb{T}^1 \to \mathbb{P}^1$, $\omega \mapsto \bar{\varphi}_\lambda^{\pm}(\omega)$, which are uniformly attracting and repelling:
$$\mp\beta(\lambda) = \lim_{|n|\to\infty} \frac{1}{n} \ln |\mathbf{z}_\lambda(n, \omega, \varphi_\lambda^{\pm}(\omega))| \quad \text{for all } \omega \in \mathbb{T}^1;$$
2. the orbits of $\mathbb{T}^1 \times \mathbb{P}^1$ starting outside those copies of the base are heteroclinic orbits, going from the graph of $\bar{\varphi}_\lambda^-$ to the graph of $\bar{\varphi}_\lambda^+$.

Note: $\bar{\varphi}_\lambda^{\pm}$ are the projections on $\mathbb{P}^1$ of the **(**closed**)** Oseledets subbundles $W_\lambda^{\pm}$.

Theorem. If $\lambda \in \Sigma$ (non-uniformly hyperbolic case),

1. there is a unique minimal $M_\lambda \subset \mathbb{T}^1 \times \mathbb{P}^1$ which is not a smooth curve: it contains exactly two non-closed invariant subsets, graphs of two non-continuous measurable functions $\bar{\varphi}^\pm_\lambda : \mathbb{T}^1 \to \mathbb{P}^1$;
2. there exists a full measure set $\Omega_\lambda \neq \mathbb{T}^1$ with
$$\mp\beta(\lambda) = \lim_{|n|\to\infty} \frac{1}{n} \ln |\mathbf{z}_\lambda(n, \omega, \varphi^\pm_\lambda(\omega))| \quad \text{for all } \omega \in \Omega_\lambda\,;$$
3. for $\omega \in \Omega_\lambda$ the trajectories corresponding to $(\omega, \bar{\varphi}_0)$ with $\bar{\varphi}_0 \neq \bar{\varphi}^\pm_\lambda(\omega_0)$ are heteroclinic;
4. there is a residual subset of M_λ of points giving rise to orbits on $\mathbb{T}^1 \times \mathbb{R}^2$ oscillating exponentially as $|n| \to \infty$.

As before, $\bar{\varphi}^\pm_\lambda$ are the projections on $\mathbb{P}^1$ of $W^\pm_\lambda$ (non-closed).

Take $J \subset \mathbb{R} - \Sigma$ and let λ^* be a finite extreme point of J.

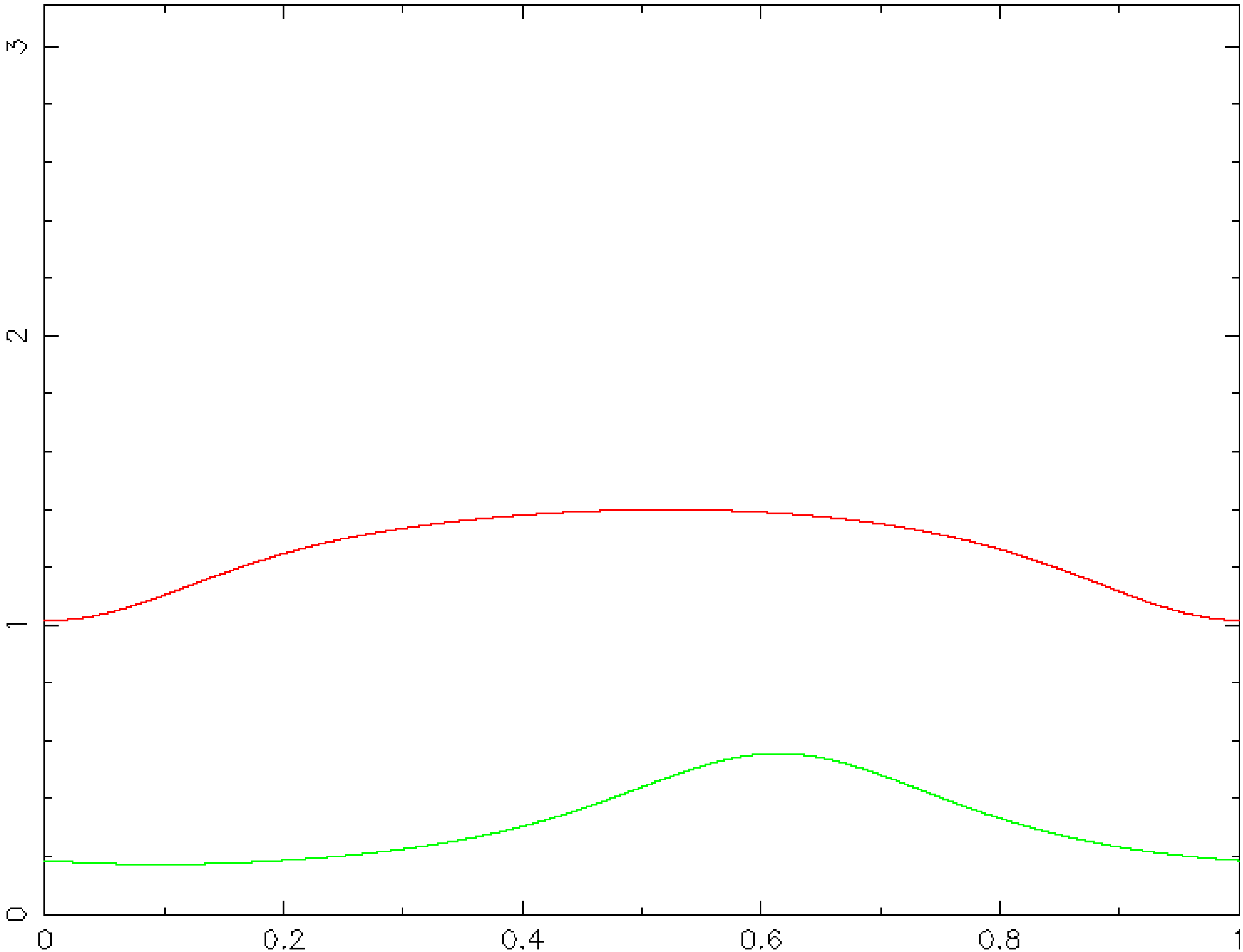
Theorem.

1. As $\lambda \in J$ approaches λ^*, the attracting and repelling curves $\bar{\varphi}^{\pm}_{\lambda}$ approach each other in a monotone and non-uniform way;
2. when λ reaches λ^*, the two curves "collide" on a dense set of points and they stop being continuous;
3. the limiting curves $\bar{\varphi}^{\pm}_{\lambda^*}$ are continuous and coincident on an invariant residual set, whereas they are different and discontinuous in a full measure subset;
4. the closure of the graphs of $\bar{\varphi}^{\pm}_{\lambda^*}$ is the unique minimal subset $M_{\lambda^*} \subset \mathbb{T}^1 \times \mathbb{P}^1$, and this set is an almost automorphic extension of $\mathbb{T}^1$.

In the case that $M_{\lambda^*} \subset \mathbb{T}^1 \times (0, \pi)$, the graph of $\bar{\varphi}^-_{\lambda^*}$ is a real SNA: a non continuous invariant curve with range in $(0^+, \pi^-)$ and Lyapunov exponent $-\beta(\lambda^*)/2$.

It can be proved that this is what happens when λ^* is the first point of the spectrum $\big(J = (-\infty, \lambda^*)\big)$.

The next graphics show the evolution described. All of them correspond to $b = 1.1$, letting the parameter λ vary in small intervals.



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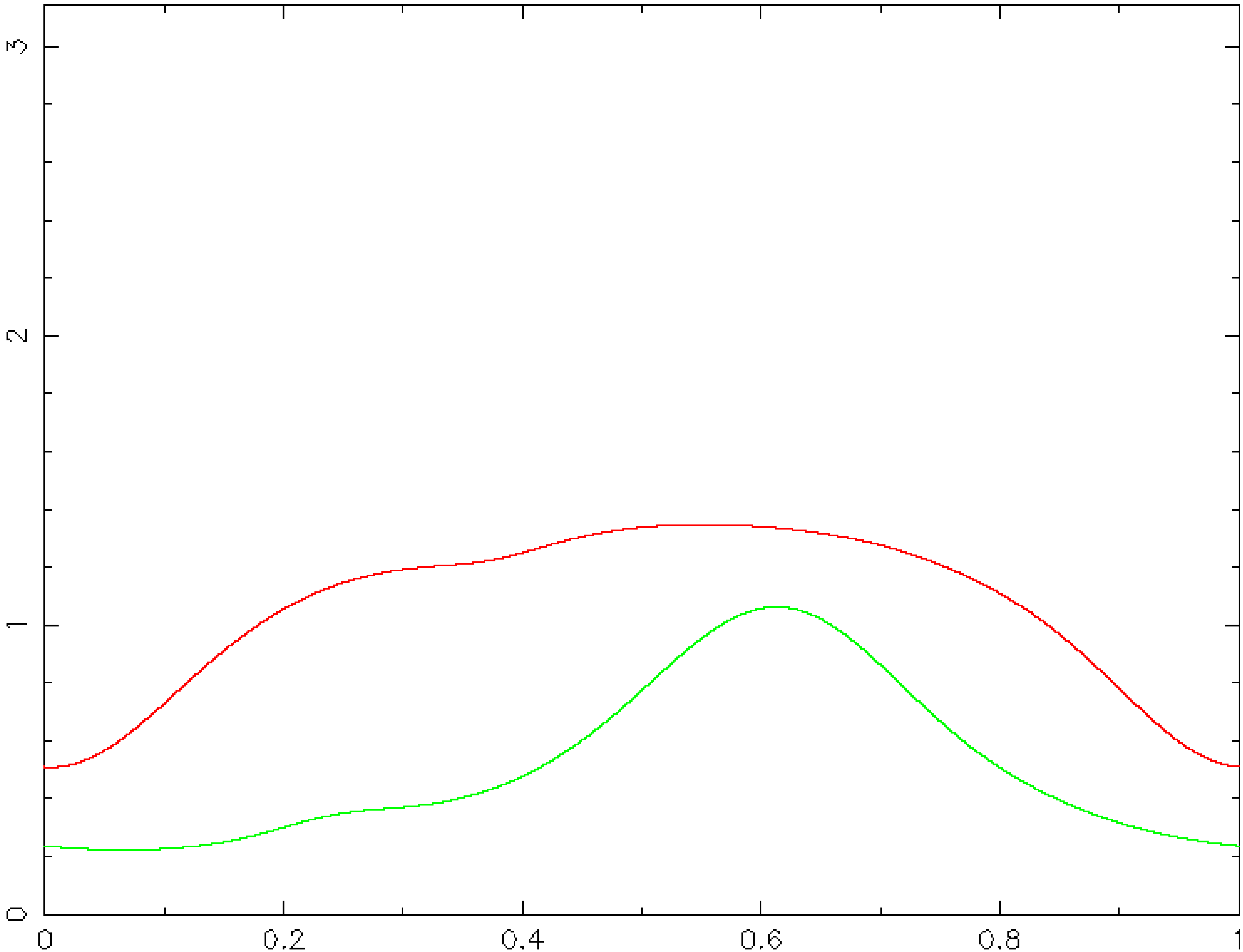

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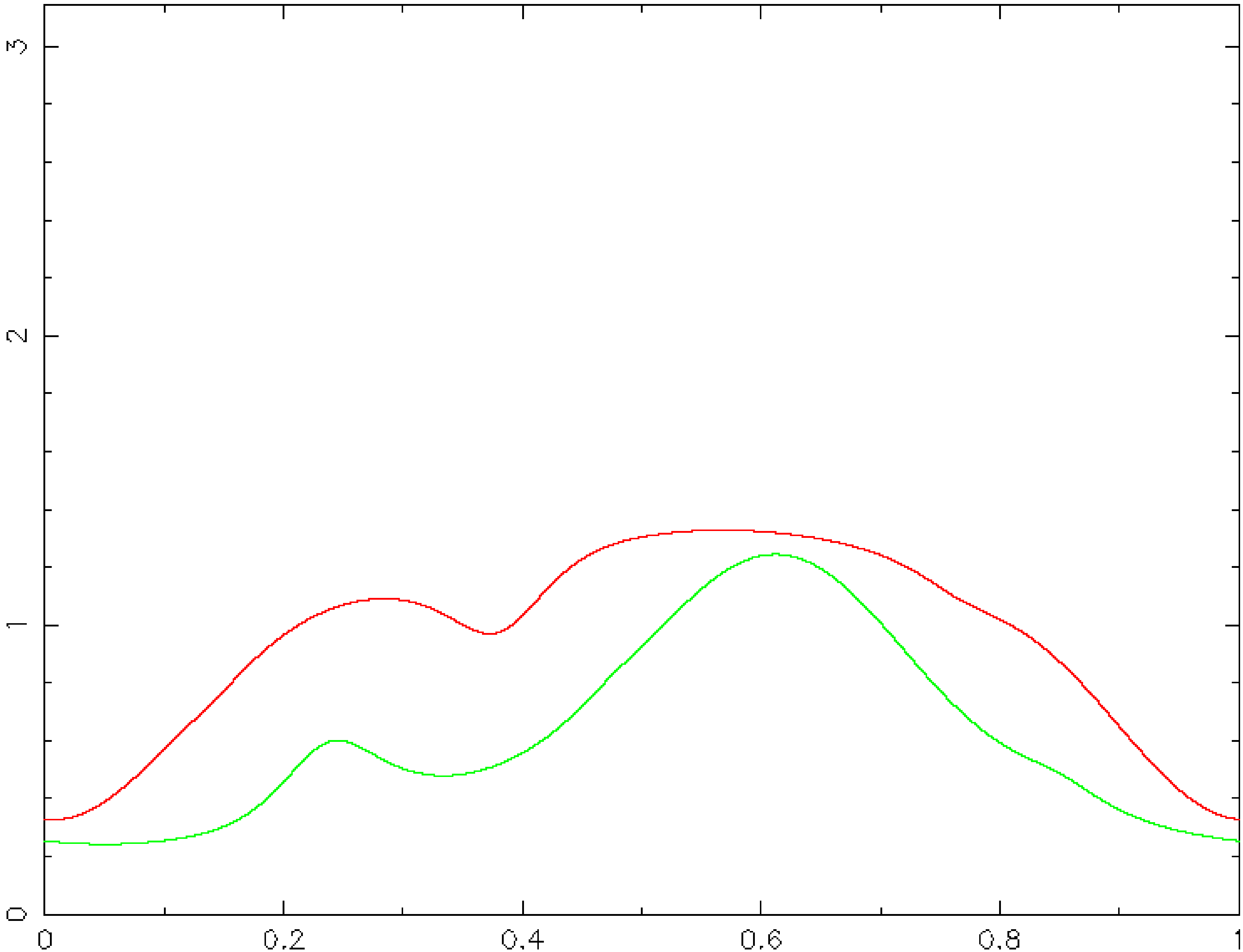

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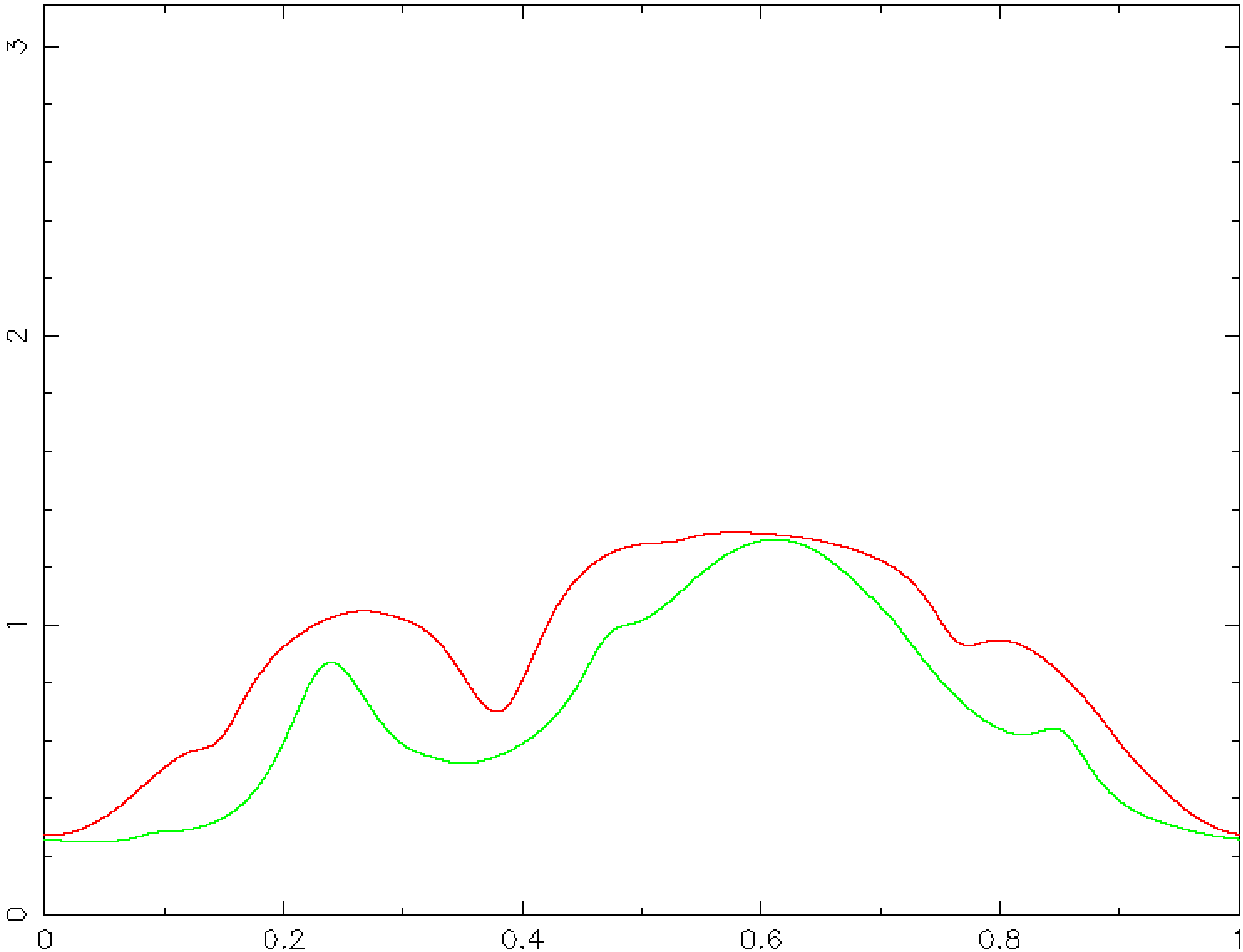

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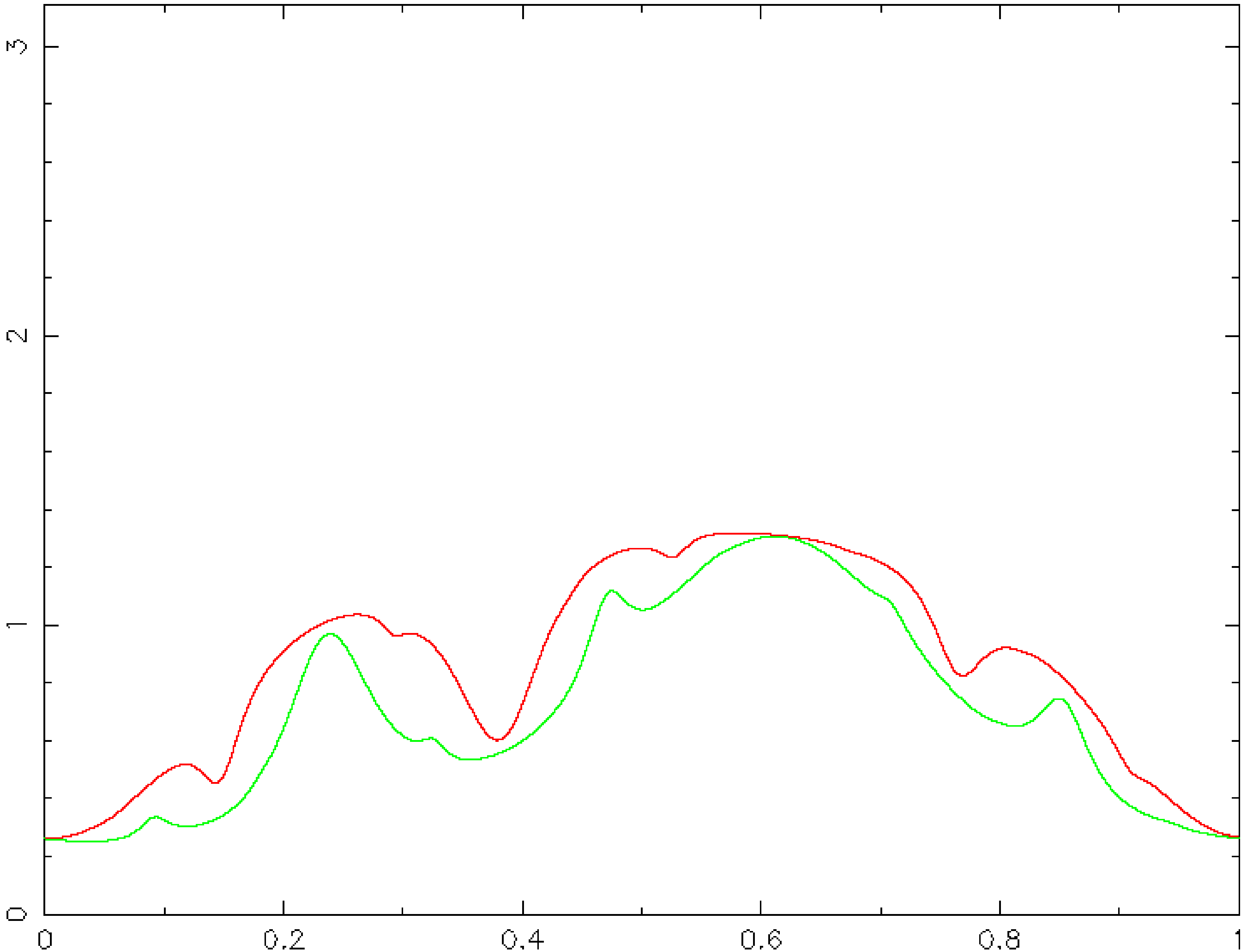

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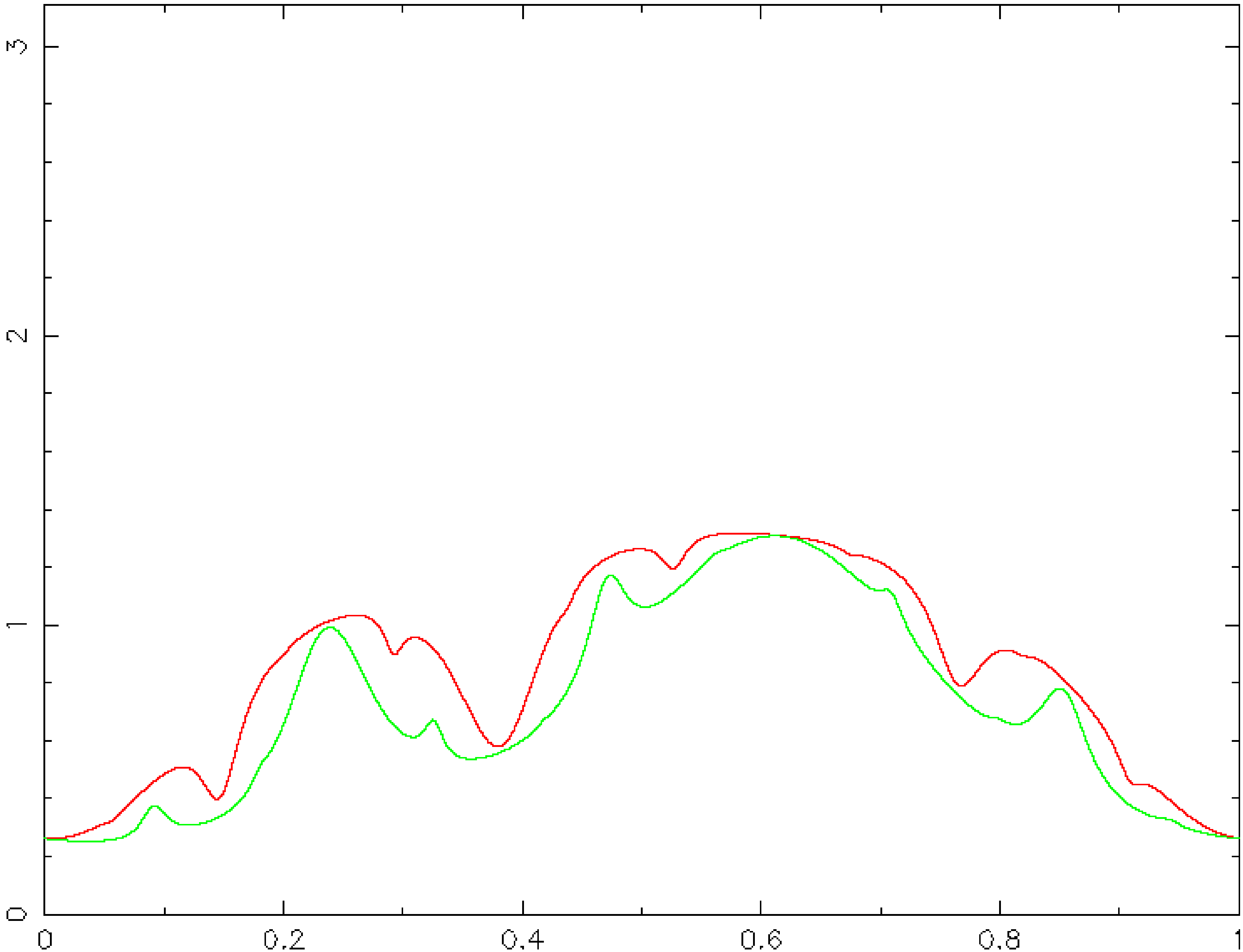

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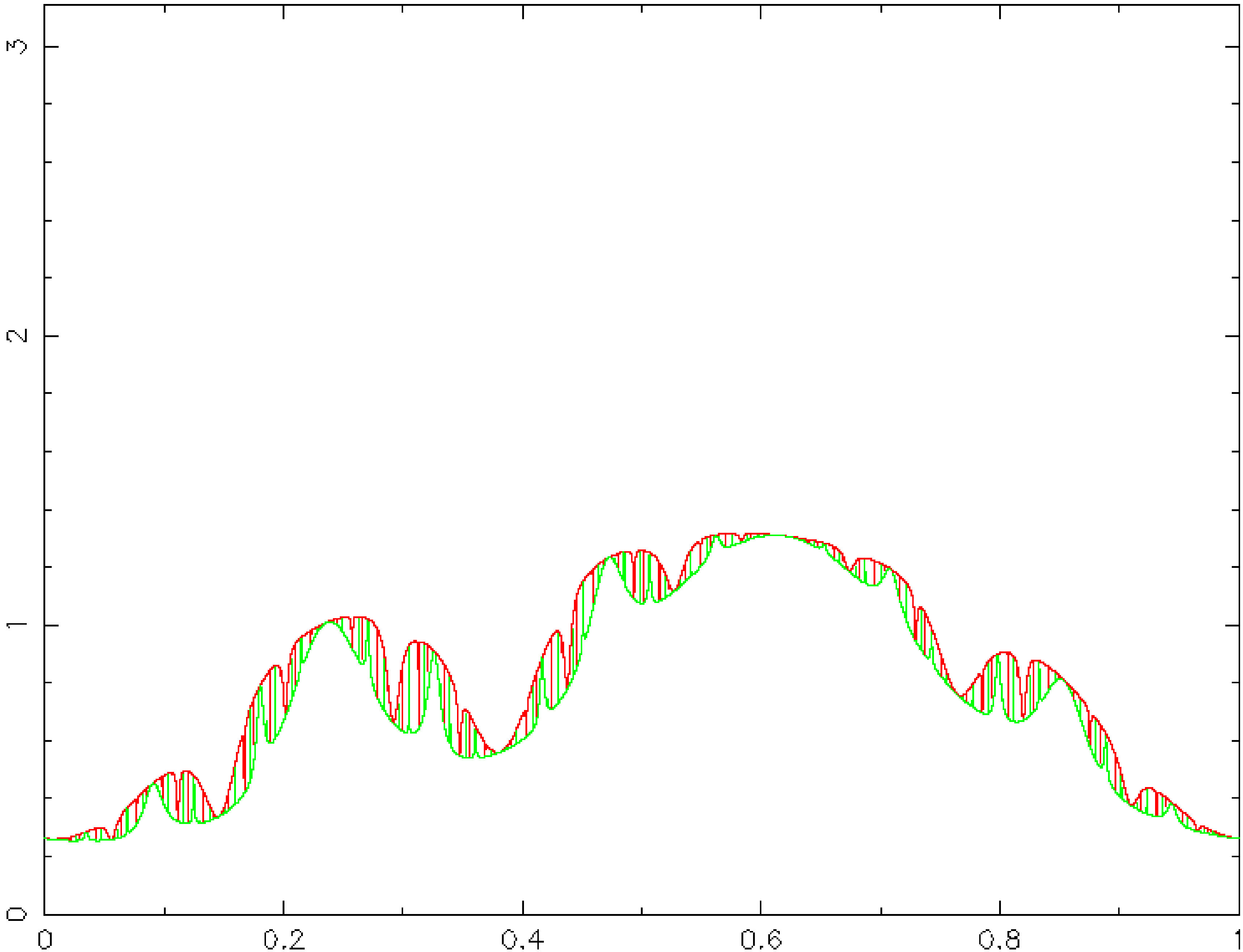

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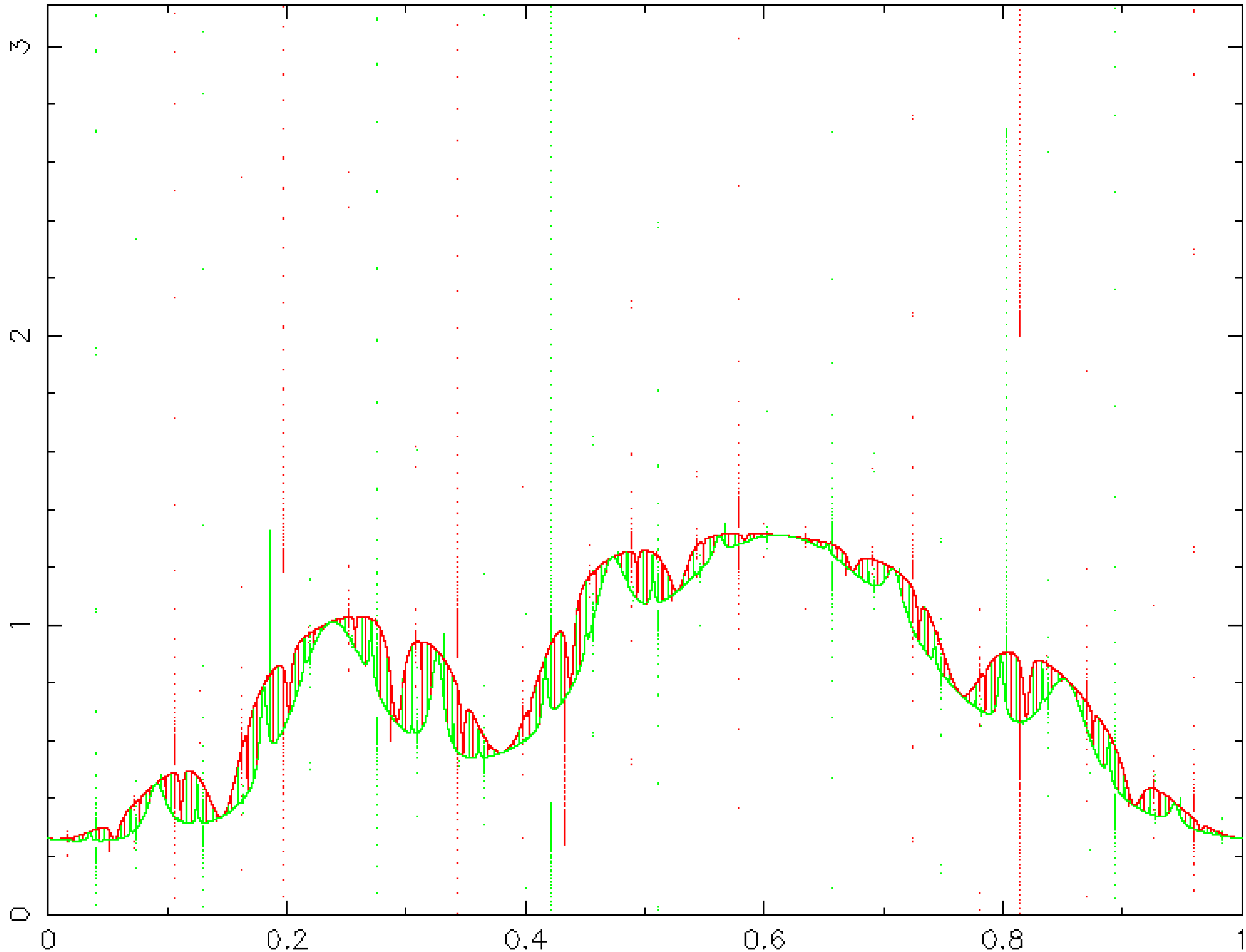

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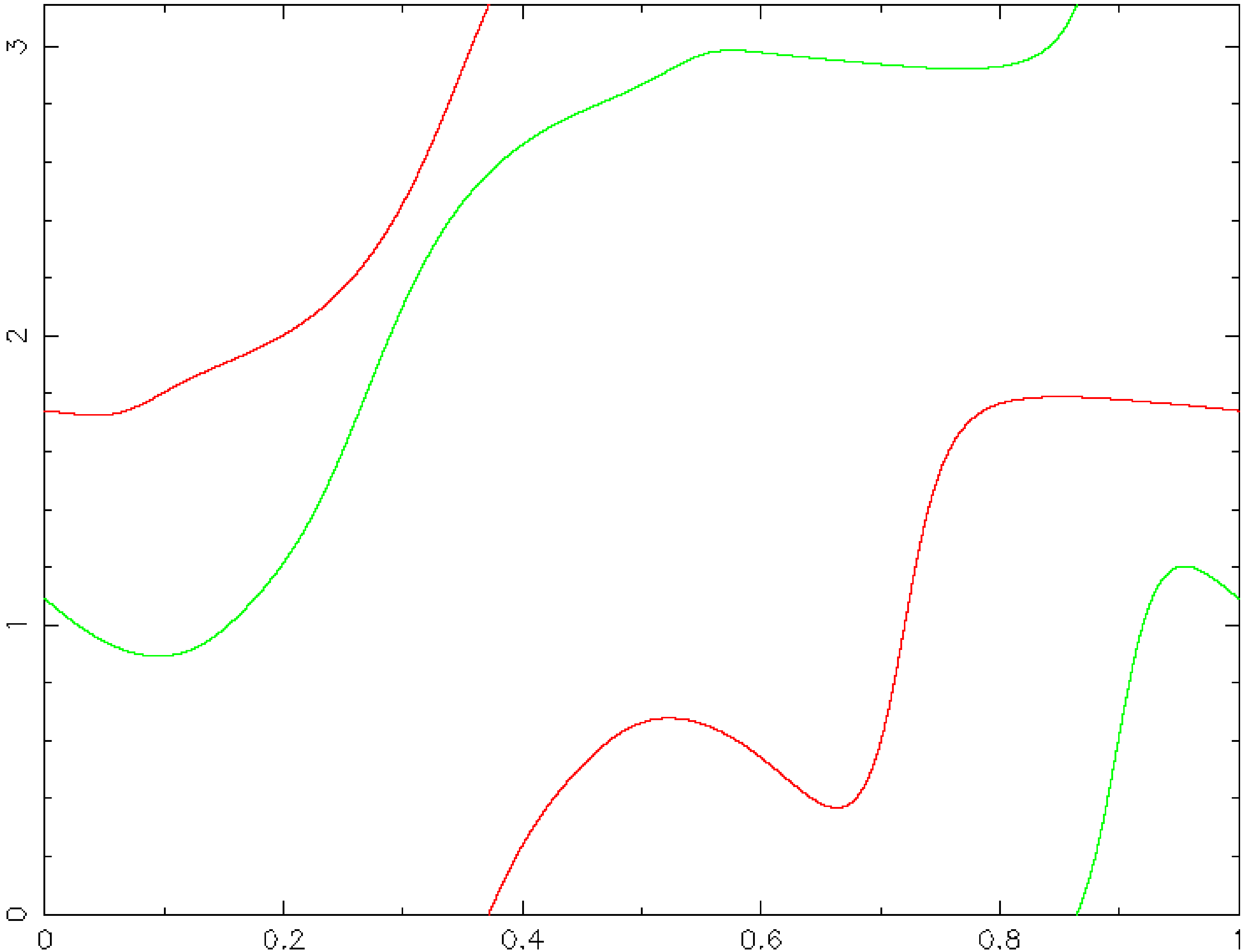
But the situation at any other extreme point of the spectrum may be different: in the case that the projection of M_{λ^*} on $\mathbb{P}^1$ contains the point 0 (the infinity point), the attractor may be winding on $\mathbb{P}^1$, so that it cannot be naturally included in $\mathbb{R}$.



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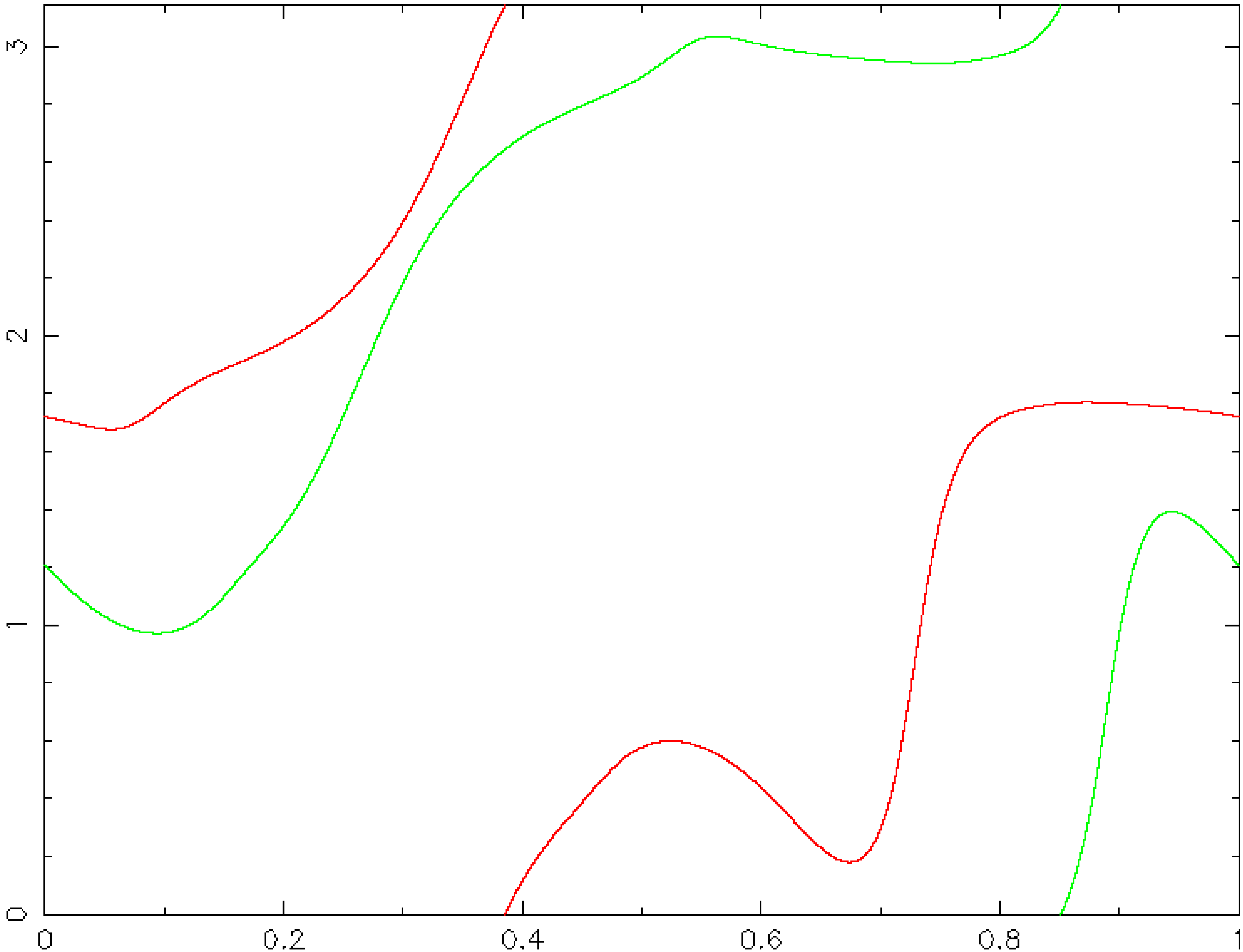

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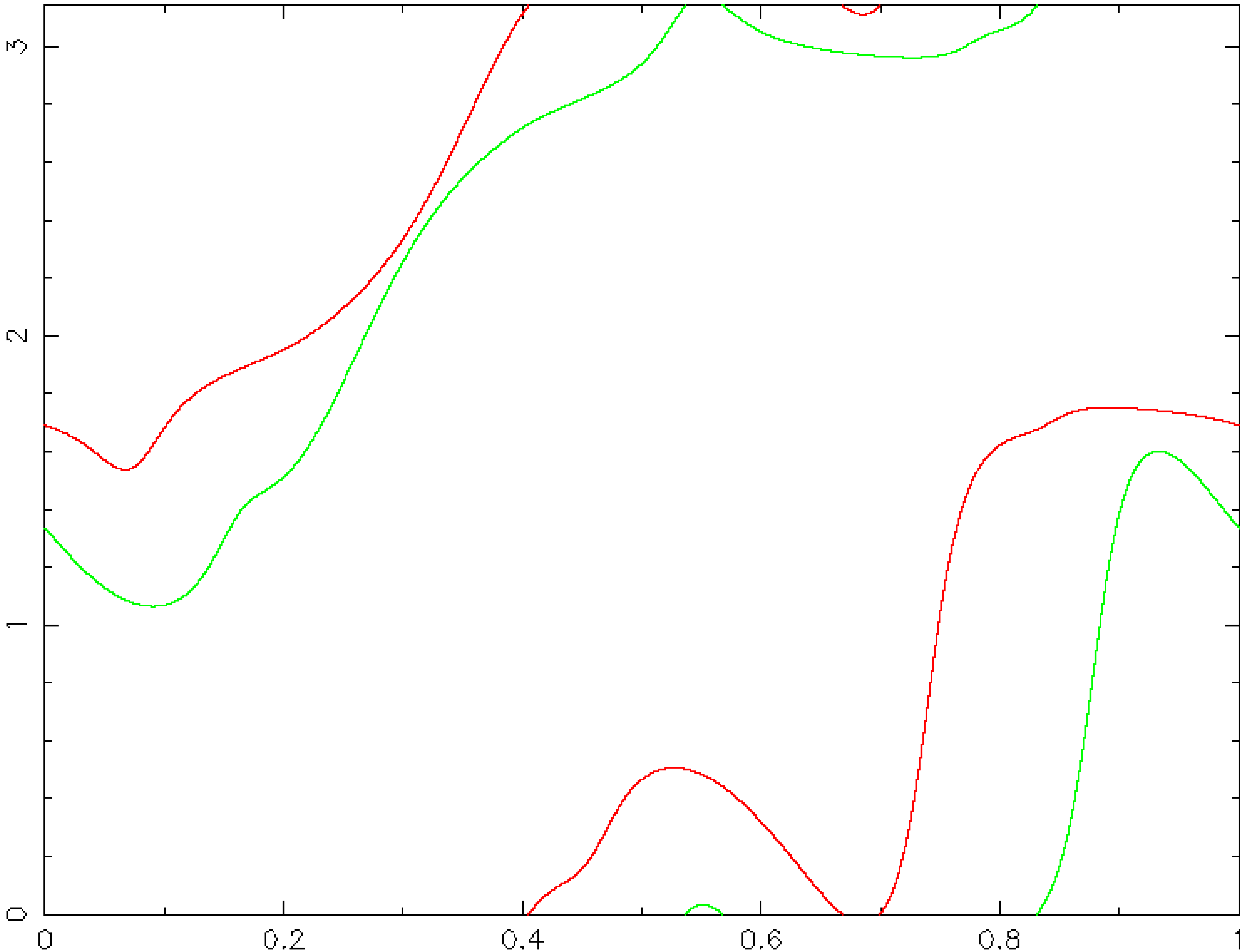

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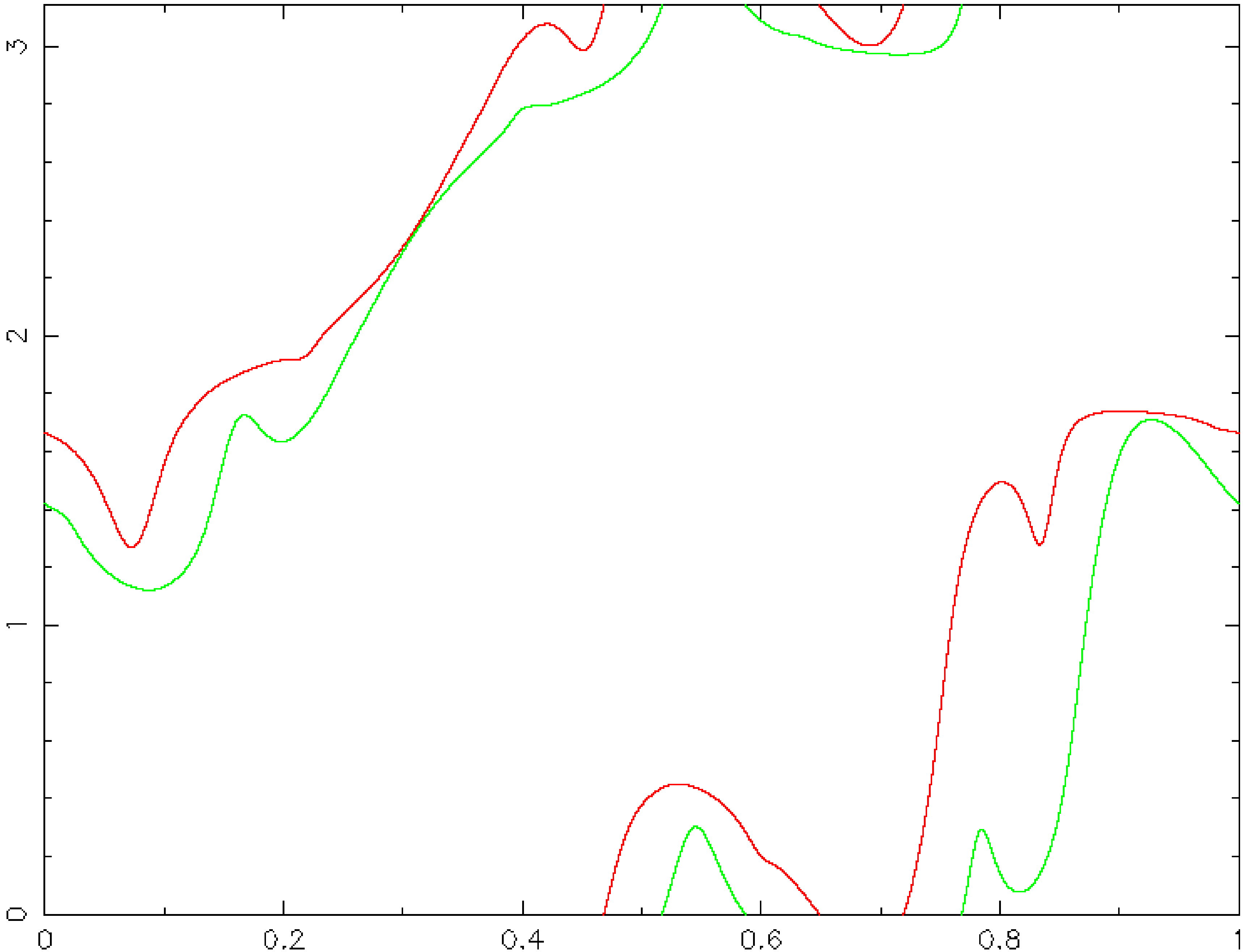

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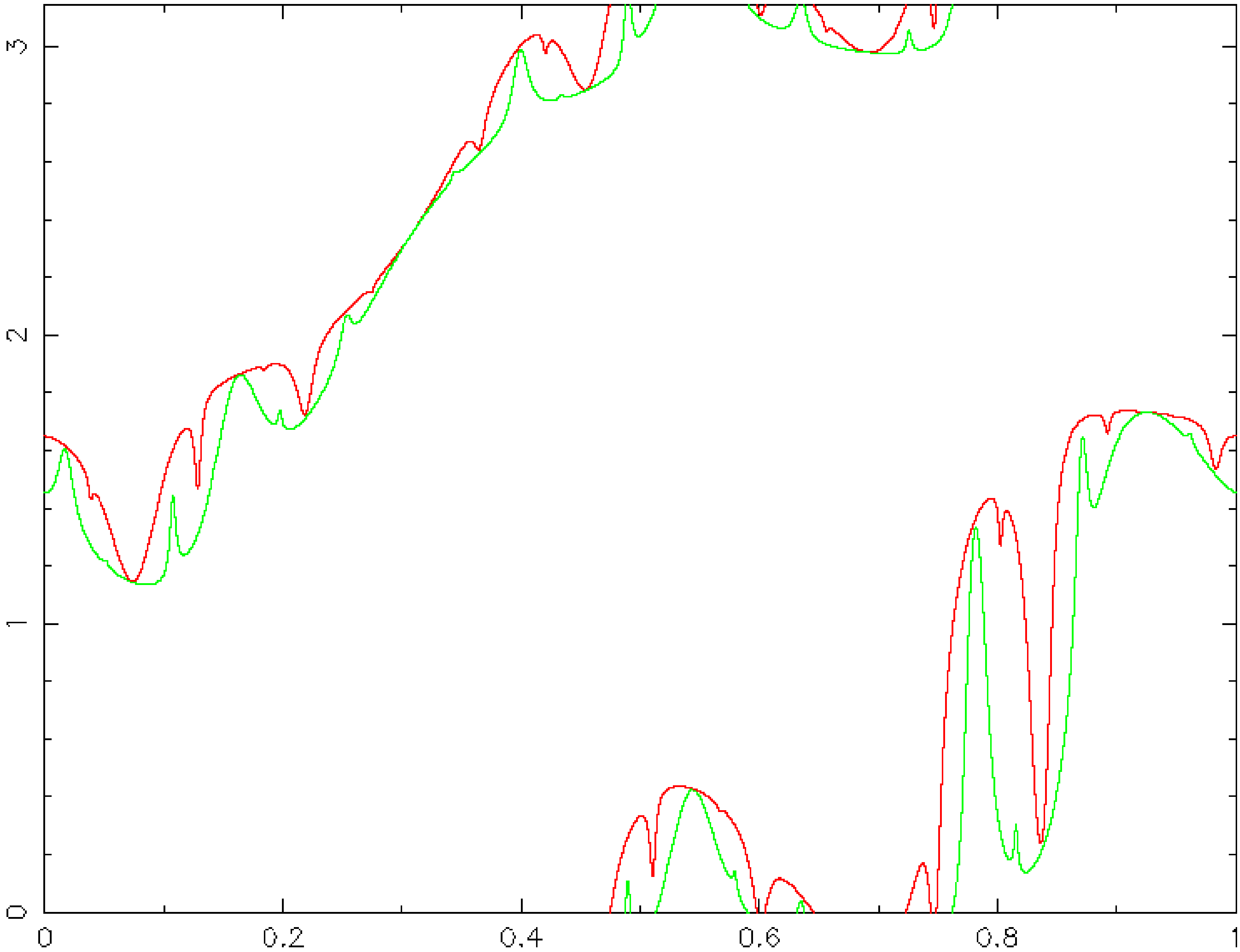

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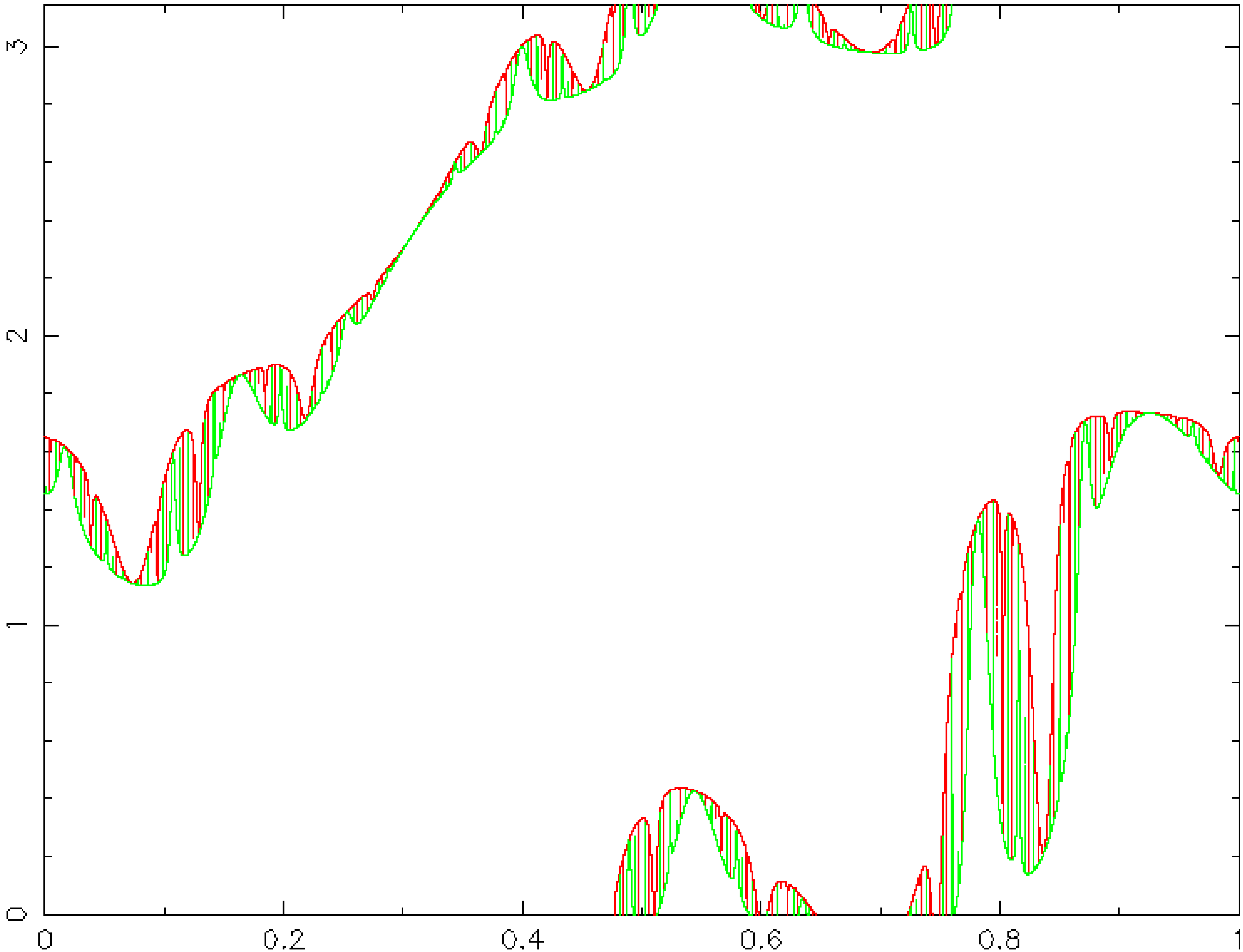

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Theorem. Let J and λ^* be as before, and fix $\lambda_0 \in J$.

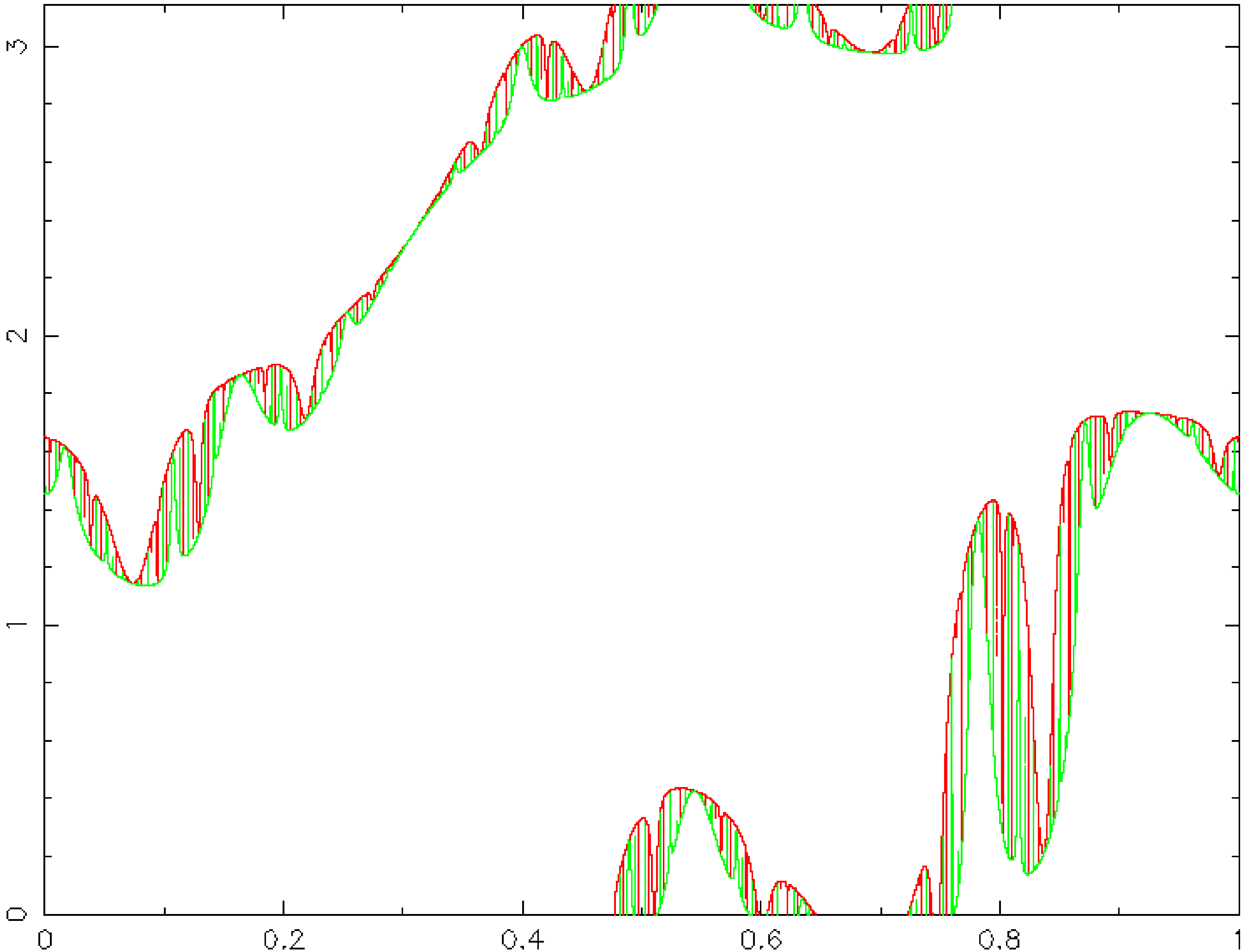
1. The smooth map

$$\mathbb{T}^1 \times \mathbb{P}^1 \to \mathbb{T}^1 \times \mathbb{P}^1, \quad (\omega, \varphi) \mapsto (\omega, \varphi - \bar{\varphi}^-_{\lambda_0}(\omega))$$

defines a flow homeomorphism, and the transformed flow has a unique minimal set $\widetilde{M}_{\lambda^*} \subset \mathbb{T}^1 \times (0, \pi)$ which is not a copy of the base;

2. Let $\varphi^{\pm}_{\lambda^*} = \bar{\varphi}^{\pm}_{\lambda^*} - \bar{\varphi}^-_{\lambda_0}$. Then $\beta_s(\varphi^{\pm}_{\lambda^*}) = \pm\beta(\lambda^*)/2$, and the invariant curve $\varphi^-_{\lambda^*}$ is an SNA on $\mathbb{R}$.

So that the process of **unwinding** the attractor is a smooth transformation (a translation) on $\mathbb{P}^1$ that allows to embed the attractor in $\mathbb{R}$, used to show the existence of SNAs on $\mathbb{R}$.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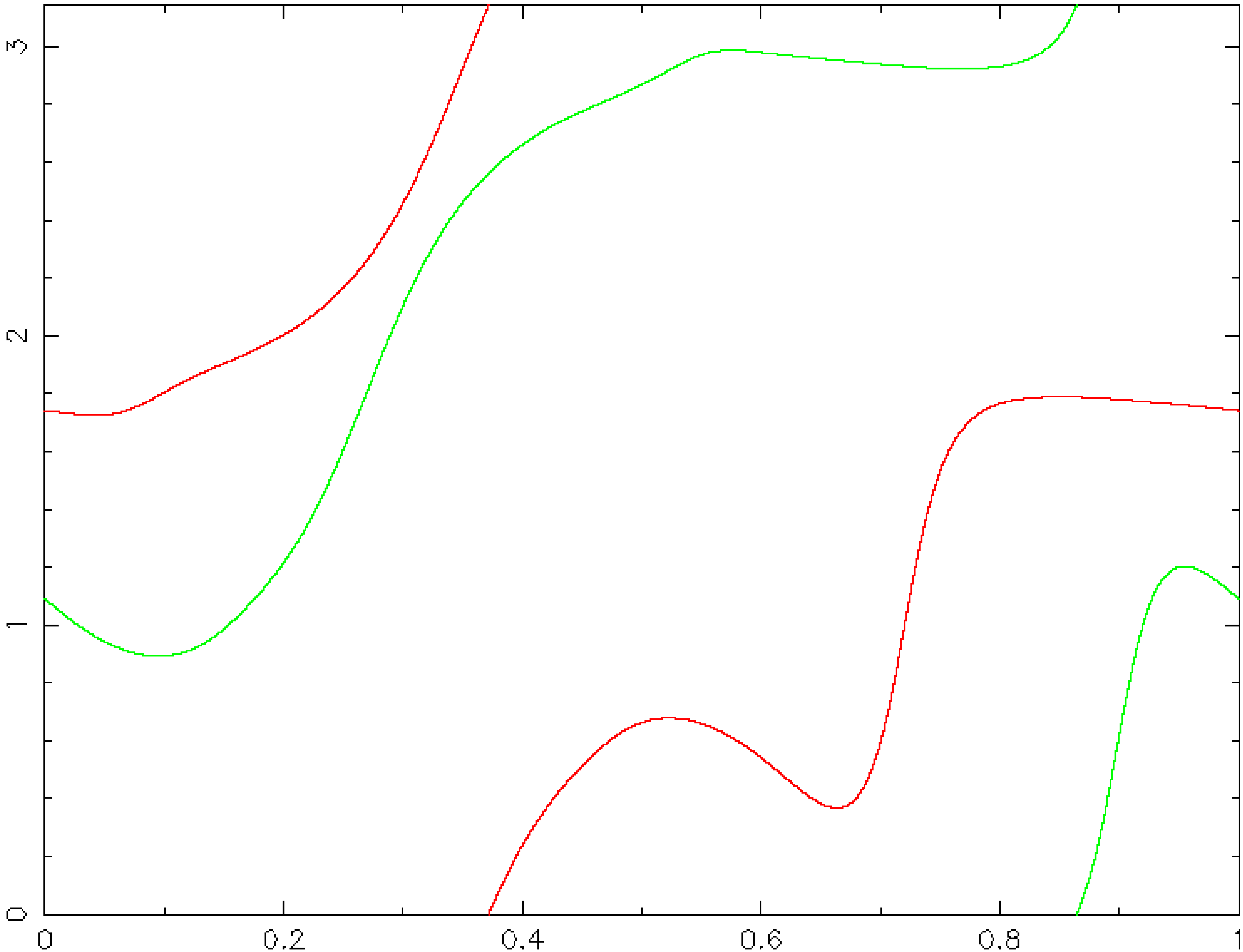

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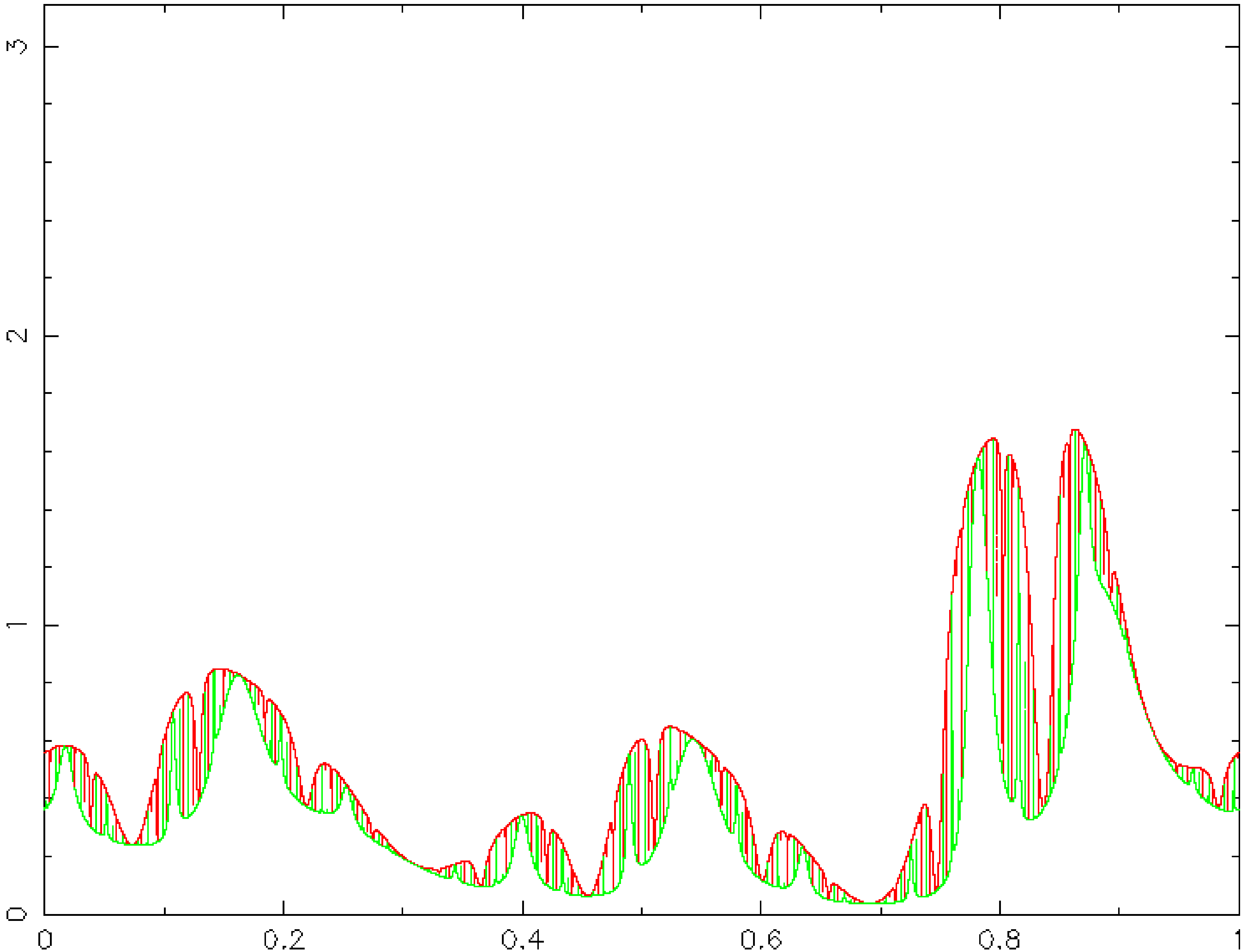

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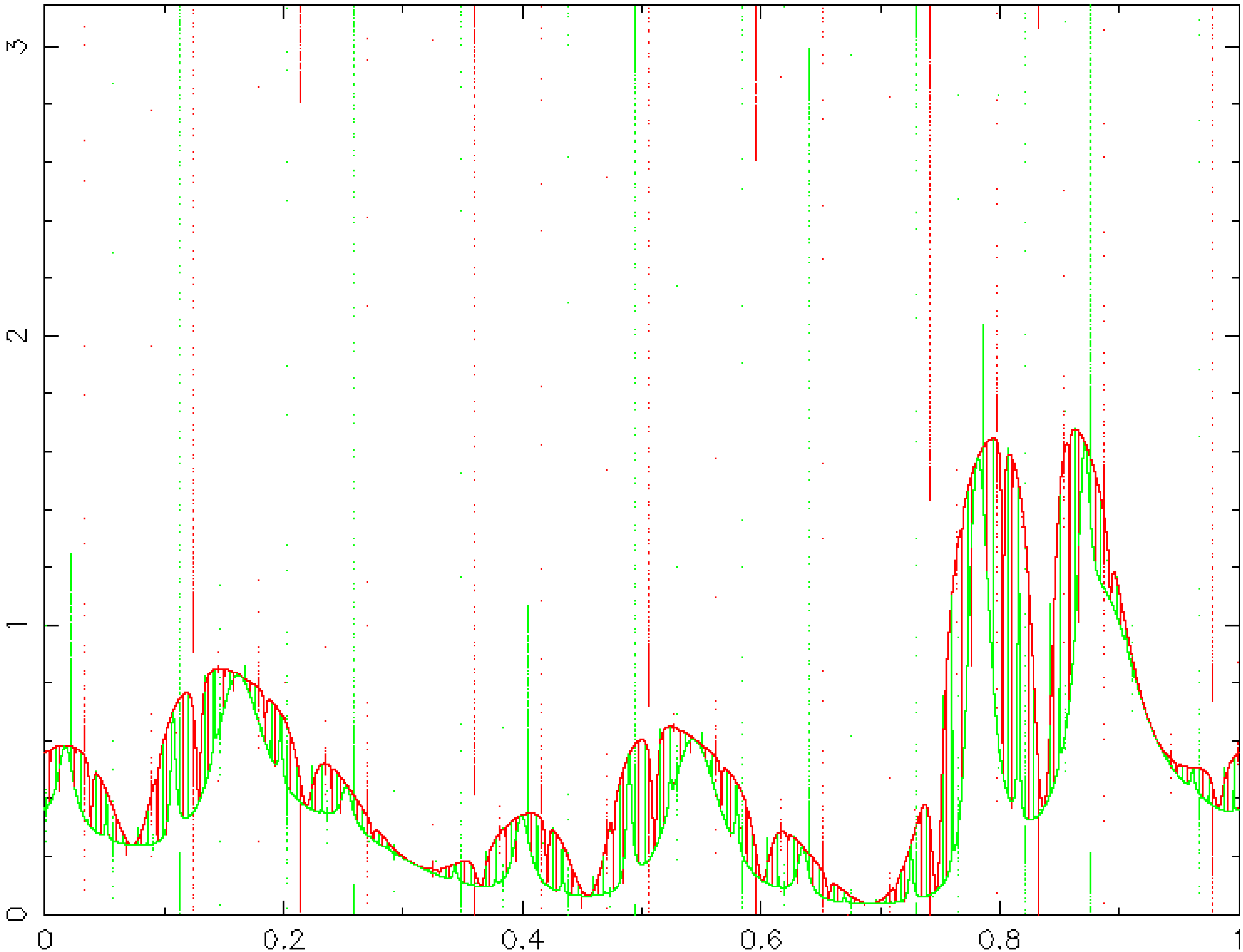

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All these results hold in a much more general setting: projective flows coming from two-dimensional Dirac systems, second order scalar linear Schrödinger equation, and Jacobi transformations, over a continuous quasiperiodic flow on $\mathbb{T}^d$ or even over an almost periodic flow.

Our results show the occurrence of SNAs at those extreme points of spectral gaps in the case that the corresponding dynamics is non-uniformly hyperbolic. (An unwinding procedure is in general required.)

Millionščikov and Vinograd examples fit in our conditions.