Network dynamics and bifurcations

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We discuss some of the mathematical theory for bifurcations of dynamical systems, with application to network dynamics. The presence of special structures in the form of network symmetries constrain but give a much richer set of possible generic bifurcations that the non-symmetric case. We introduce in some detail the tools of generic bifurcation theory for equivariant (symmetric) systems to help understand how onset or less of synchrony appears in networks of coupled oscillators.

Some topics we discuss:

- Bifurcation theory for ODEs, bifurcations of equilibria and genericity
- Centre manifolds, normal forms and symmetries
- Symmetric (equivariant) dynamics: group representation and bifurcations
- Network dynamical systems and applications of symmetric bifurcation theory
- Some examples in networks of coupled oscillators

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Initial discussion: aims, background, expectations

Background

In general, one cannot find solutions $x(t)$ of nonlinear differential equations

$$\dot{x} = f(x)$$

for $x \in \mathbb{R}^n$ and even quite simple functions f with n small. Possible ways forward are:

- Option 1: Use numerical approximation.
- Option 2: Find simple solutions (equilibria, periodic orbits) and determine their stability.

Unfortunately option 1 may give much data but not give much insight, and option 2 may not tell us about the 'typical solutions' that we want to know about.

Bifurcation theory

Bifurcation theory looks at how simple solutions change on changing a *parameter* λ in an autonomous ODE

$$\dot{x} = f(x, \lambda).$$

- $x = (x_1, \cdots, x_n) \in \mathbb{R}^n$ - state variable
- $\lambda = (\lambda_1, \cdots, \lambda_d) \in \mathbb{R}^d$ - bifurcation parameter
- $f : \mathbb{R}^{n+d} \to \mathbb{R}^n$ - smooth function

There is also a coherent bifurcation theory for *iterated maps*

$$x_{n+1} = f(x_n, \lambda)$$

but these is subtly different to what we discuss here.

One parameter bifurcation theory: consider

$$\dot{x} = f(x, \lambda)$$

i.e.

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, \cdots, x_n, \lambda) \\ \vdots & & \vdots \\ \dot{x}_n & = & f_n(x_1, \cdots, x_n, \lambda) \end{array}$$

for

- $x \in \mathbb{R}^n$ - state variable
- $\lambda \in \mathbb{R}$ - bifurcation parameter
- $f : \mathbb{R}^{n+1} \to \mathbb{R}^n$ - right hand side

- Local bifurcation theory deals with *equilibria* (also known as *steady solutions* or *singular points*), i.e. x_0 such that

$$f(x_0, \lambda) = 0.$$

- Equilibria typically come in *branches* i.e. $(X(\lambda), \lambda)$ parametrized by the bifurcation parameter. There may be many branches at any given λ.
- We usually express the bifurcation pattern in a *bifurcation diagram* which plots some measure of the solution x (vertical axis) against the parameter (horizontal axis). A branch is plotted as a smooth line on such a diagram.
- Typical choices for the vertical axis are: one of the coordinates of x; a norm of x (such as L^1 or L^2 norm) but any smooth observable of x may be shown.

Linear stability

Linear stability of equilibrium $(X, 0)$ can be found by examining the $n \times n$ Jacobian matrix J of partial derivatives

$$J = Df(X,0) = \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial x_1} & \cdots & \frac{\partial f_n}{\partial x_n} \end{pmatrix}$$

where all partial derivatives are evaluated at $(X, 0)$.

Consider $J = Df(X,0)$ Jacobian of equilibrium solution:

- If no eigenvalues of J are on the imaginary axis, then we say X is *hyperbolic* and $(X,0)$ is a point on a branch of equilibria.
- If at least one eigenvalues of J are on the imaginary axis, then we say X is at a *bifurcation* and more than one branch may meet at $(X,0)$.

For *generic* or *typical choice* of f the only bifurcations are following:

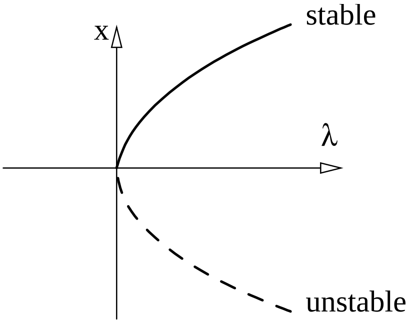
- *Saddle-node bifurcation* where there is a single zero eigenvalue of J.
- *Hopf bifurcation* where there is a single pure imaginary pair of eigenvalues of J.

Saddle-node bifurcation

Example of a saddle-node bifurcation in one dimension

$$\dot{x} = \lambda - x^2 \tag{1}$$

Has two equilibria for $\lambda > 0$, one for $\lambda = 0$ and none for $\lambda < 0$.



Bifurcation diagram

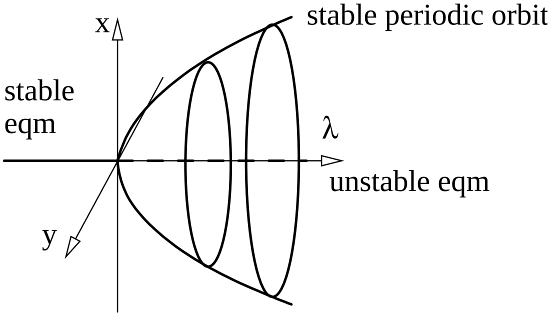
The same bifurcation diagram (up to reflection in x and/or λ) holds for ALL saddle-node bifurcations. Equation (1) is the *normal form* for a saddle-node bifurcation.

Hopf bifurcation

Example of a Hopf bifurcation in two dimensions $z = x + iy$:

$$\dot{z} = (\lambda + i\omega)z - |z|^2 z \qquad (2)$$

Has one equilibrium for all λ. A *periodic orbit* appears on increasing λ through zero.



Bifurcation diagram

The same bifurcation diagram (up to coordinate changes) holds for ALL Hopf bifurcations. Equation (2) is the *normal form* for a Hopf bifurcation.

Genericity

What is meant by a bifurcation problem being *generic*?

There is always an implied context for a system, for instance if

$$\dot{x} = f(x, \lambda)$$

is a model of a simple physical system with $x, \lambda \in \mathbb{R}$ then we hope that the predictions of the model are not sensitive to small details in the specification of f.

Suppose that $f \in C^\infty(\mathbb{R}, \mathbb{R})$ is a particular function, then unless we have a compelling reason to believe that f must have a special property (such as oddness $f(-x) = -f(x)$), we assume that is has no such property.
Formally, let $P(f)$ be some property of a $f \in C^\infty$.

- We say the property P is *generic* if it holds on an open dense subset of C^∞.
- Otherwise we say it is *non-generic*.

Examples of genericity

Let A be the set of all smooth functions $f : \mathbb{R} \to \mathbb{R}$.
Let $A_s \subset A$ be the set of all symmetric (i.e. odd) smooth functions $f : \mathbb{R} \to \mathbb{R}$.

- It is not generic for $f \in A$ to be also in A_s, i.e. to be an odd function.
- It is generic for $f \in A$ to be non-constant.
- It is generic for $f \in A_s$ to have a simple zero at $x = 0$.
- It is not generic for $f \in A$ to have any zero.
- BUT: It is not generic for $f \in A$ to have no zero.

In all cases, genericity depends on context; if we assume there is no special symmetry, the only bifurcations are saddle-node and Hopf. If however we know there is a symmetry in the model, there may be new generic bifurcations.

Bifurcation case study

Case study [G&S (1.1)]

$$\begin{aligned}\dot{x} &= \lambda x - (x+y+z) + x^2\\ \dot{y} &= \lambda y - (x+y+z) + y^2\\ \dot{z} &= \lambda z - (x+y+z) + z^2\end{aligned}$$

where $(x, y, z) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ is a parameter.

- Clearly there is an equilibrium at $x = y = z = 0$ for all λ.
- Jacobian at this equilibrium is:

$$J = \begin{pmatrix} \lambda-1 & -1 & -1 \\ -1 & \lambda-1 & -1 \\ -1 & -1 & \lambda-1 \end{pmatrix}$$

  with eigenvectors and eigenvalues:

$$\begin{aligned} v_0 = (1,1,1)^T &: \; e_0 = \lambda - 3\\ v_1 = (1,-1,0)^T &: \; e_1 = \lambda\\ v_2 = (1,0,-1)^T &: \; e_2 = \lambda \end{aligned}$$

- Bifurcation at $\lambda = 0$ is neither saddle-node nor Hopf!

From

$$\begin{aligned}\dot{x} &= \lambda x - (x+y+z) + x^2\\ \dot{y} &= \lambda y - (x+y+z) + y^2\\ \dot{z} &= \lambda z - (x+y+z) + z^2.\end{aligned}$$

Same equations for all permutations of x, y, z. Equilibria at

$$\lambda x + x^2 = \lambda y + y^2 = \lambda z + z^2 = x + y + z$$

From

$$\begin{aligned}
\dot{x} &= \lambda x - (x+y+z) + x^2 \\
\dot{y} &= \lambda y - (x+y+z) + y^2 \\
\dot{z} &= \lambda z - (x+y+z) + z^2.
\end{aligned}$$

Same equations for all permutations of x, y, z. Equilibria at

$$\lambda x + x^2 = \lambda y + y^2 = \lambda z + z^2 = x + y + z$$

Near $\lambda = 0$ and 0 the is an equilibrium at $x = y = z = 0$ and also equilibria at

$$x = y = \frac{-\lambda + 1 - \sqrt{\lambda^2 - 6\lambda + 1}}{2}, \; z = \frac{-\lambda - 1 + \sqrt{\lambda^2 - 6\lambda + 1}}{2}$$

$$x = z = \frac{-\lambda + 1 - \sqrt{\lambda^2 - 6\lambda + 1}}{2}, \; y = \frac{-\lambda - 1 + \sqrt{\lambda^2 - 6\lambda + 1}}{2}$$

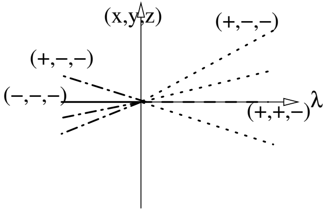
$$y = z = \frac{-\lambda + 1 - \sqrt{\lambda^2 - 6\lambda + 1}}{2}, \; x = \frac{-\lambda - 1 + \sqrt{\lambda^2 - 6\lambda + 1}}{2}$$

(plus four more solutions!)

Near $\lambda = 0$ the first of these solutions can be written

$$\begin{aligned} x &= \lambda + O(\lambda^2) \\ y &= \lambda + O(\lambda^2) \\ z &= -2\lambda + O(\lambda^2) \end{aligned}$$

Computing also stabilities we have the following bifurcation diagram:



Note that (a) several branches of equilibria come together at one point and (b) no branching solution is stable!

Bifurcations of equilibria

We now look at bifurcations of steady solutions (equilibria) in general, with a view to understanding examples such the example with symmetry.
If a bifurcation problem

$$\dot{x} = f(x, \lambda)$$

has constraints, then the typical bifurcations that appear will change substantially.
Examples:

- If f has symmetries then typically get multiple eigenvalues passing through zero.
- If f is divergence free then cannot get bifurcation to attractors.
- If $f = -\nabla\phi$ then cannot get periodic solutions.

Essentially, if the context changes then we get a new set of "generic bifurcations".

Constrained bifurcations

Suppose that the system

$$\dot{x} = f(x, \lambda)$$

for $x, \lambda \in \mathbb{R}$ has a constraint that $x = 0$ is always an equilibrium.

We can infer that $f(x, \lambda) = xg(x, \lambda)$ for some new function g with a removable singularity at $x = 0$.

On varying λ we expect to find equilibria at $x = 0$ and branches $x(\lambda)$ such that $g(x, \lambda) = 0$

Generically, if a branch of solutions $x(\lambda)$ of $g(x, \lambda) = 0$ hits zero, it will pass through "transversally".

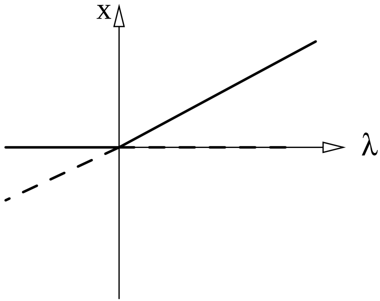
In such a case, near typical bifurcation of $x = 0$ there is a normal form

$$\dot{x} = x(\lambda - x)$$

Transcritical bifurcation

The only generic steady bifurcation for a $1-D$ problem constrained to have an equilibrium at zero is the *transcritical bifurcation* with normal form

$$\dot{x} = x(\lambda - x)$$

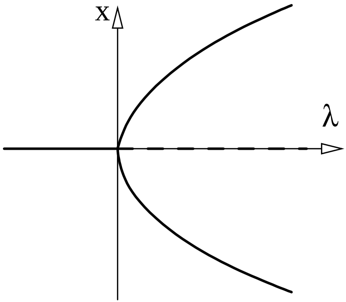


Bifurcation diagram

Pitchfork Bifurcation

The only generic steady bifurcation for a $1-D$ problem with a symmetry $x \mapsto -x$ is the *pitchfork bifurcation* with normal form

$$\dot{x} = x(\lambda - x^2)$$



Bifurcation diagram

Bifurcation equivalence

Previous examples beg the question; What do we mean by *equivalence* and *normal form*? Various approaches exist and we cannot go into full detail here, but give a sketch for the interested student!

(More detailed approaches use *smooth germs of vector fields*; these are equivalence classes of vector fields at a point 0, such that $f \sim g$ if there is a neighbourhood of 0 on which $f \equiv g$.)

For the case of one state variable and one parameter, $(x, \lambda) \in \mathbb{R}^2$, suppose that two bifurcation problems

$$\dot{x} = f(x, \lambda), \quad \dot{x} = g(x, \lambda)$$

have bifurcations we want to compare.

Suppose

$$\dot{x} = f(x, \lambda), \quad \dot{x} = g(x, \lambda)$$

both have bifurcations at $(0, 0)$; namely

$$f(0, 0) = g(0, 0) = f_x(0, 0) = g_x(0, 0) = 0.$$

We say these bifurcations are *strongly equivalent* if there are smooth functions $S(x, \lambda), X(x, \lambda)$ and $\Lambda(\lambda)$ such that:

- The map $(x, \lambda) \mapsto (X(x, \lambda), \Lambda(\lambda))$ is a local diffeomorphism mapping $(0, 0)$ to itself.
- The function $S(x, \lambda)$ is positive in a neighbourhood of $(0, 0)$
- The ODEs are topologically equivalent

  $$f(x, \lambda) = S(x, \lambda)g(X(x, \lambda), \Lambda(\lambda))$$

  in some neighbourhood of $(0, 0)$.

To be more precise, a bifurcation of $\dot{x} = f(x, \lambda)$ is a *pitchfork* at $(0, 0)$ if there are functions X, Λ, S as above such that

$$x(\lambda - x^2) = S(x, \lambda) f(X(x, \lambda), \Lambda(\lambda)).$$

on some neighbourhood of $(0, 0)$.
A more general definition for a pitchfork at (x_0, λ_0) is that there are S, X and Λ with $(x, \lambda) \mapsto (X(x, \lambda), \Lambda(\lambda))$ such that $(x_0, \lambda_0) \mapsto (0, 0)$ and

$$x(\lambda - x^2) = S(x, \lambda) f(X(x, \lambda), \Lambda(\lambda)).$$

on some neighbourhood of (x_0, λ_0).

Finite determinacy

- We say a bifurcation is *k-determined* if it is strongly equivalent to a polynomial function of x, λ of degree at most k.
- Generally we say a bifurcation is *finitely determined* if it k-determined for some finite k.
- It is a useful fact that most bifurcation problems of interest ARE finitely determined, mostly at quite low order!

Finite determinacy allows one to reduce a bifurcation problem to a finite dimensional family.

Taylor expansion and normal forms

Because of finite determinacy of many bifurcation problems, an important tool in the study of bifurcation singularities is the Taylor series expansion; in a multi-variate form one can write near $x = 0$ for $x \in \mathbb{R}^n$ as

$$\dot{x} = f(x) = \sum_{k=0}^{\infty} \sum_{K \in M_k} \frac{f^{(K)}(0)}{K!} x^K$$

where

$$M_k = \{(K_1, \cdots, K_n) \ : \ K_j \in \mathbb{Z}^+ \ \text{ and } \ K_1 + \cdots + K_n = k\}$$

is the set of terms of order k,

$$K! = K_1! K_2! \cdots K_n!, \quad x^K = x_1^{K_1} \cdots x_n^{K_n}$$

and

$$[f^{(K)}]_i = \frac{\partial f_i^k}{\partial^{K_1} x_1 \cdots \partial^{K_n} x_n}$$

Basic question for normal form theory is:

Which of the terms in the Taylor expansion are important to determine the branching behaviour and which are not?

In order to answer this one can consider changes of coordinate that are *near-identity* at the bifurcation in question to transform it into a *normal form*. This means we consider changes in coordinate that remove as many terms as possible from the Taylor series. An examples of a normal form theorem without parameters is the Poincaré normal form theorem:

Theorem

Suppose $\dot{x} = f(x)$ has a hyperbolic equilibrium at $x = 0$ and there are no resonances between eigenvalues of $df(0)$. Then there is a near-identity change of coordinates that removes all nonlinear terms from $f(x)$ near 0.

All of this, and the use of singularity theory to classify generic and degenerate (higher parameter) bifurcation problems in low dimension is treated in some detail a number of texts, and there is an extensive literature on singularity theory for bifurcation problems.

Basic results are in the form of theorems that allow one to *recognize* and *classify* bifurcation problems according to codimension, i.e. number of parameters needed to be added before one obtains genericity.

Some issues remain for singularity theory approach to bifurcation theory:

- Very powerful for one dimension centre manifold, not so good for higher dimension.
- Very powerful for codimension one and fairly simple groups, not so good for higher codimension and more complicated groups.
- Problems with *moduli* for higher codimension.
- Different notions of equivalence can be used to trade between simplicity and accuracy of classification.
- Big challenges for global bifurcations, eg homoclinic bifurcations.

Tuesday

Reduction of bifurcation problems

Singularity theory can become very complicated for higher dimensional subspaces; however one can reduce bifurcations to problems that have lower dimension and then, by finite determinacy, to finite truncation.
Reduction methods can reduce a bifurcation problem at (x_0, λ_0) to one that has dimension

$$\dim W^c(x_0, \lambda_0).$$

Lyapunov-Schmidt reduction

We consider the Lyapunov-Schmidt method for equilibria. Suppose we wish to find equilbria of

$$\dot{y} = F(y, \lambda)$$

for $y \in \mathbb{R}^n$, $\lambda \in \mathbb{R}^k$ and $F(0,0) = 0$, $dF|_{(0,0)}$ is singular. Let

$$\mathcal{K} = \mathrm{ker}(dF|_{(0,0)}), \quad \mathcal{R} = \mathrm{range}(dF|_{(0,0)}).$$

and split

$$\mathbb{R}^n = \mathcal{K} \oplus \hat{\mathcal{K}}, \quad \mathbb{R}^n = \mathcal{R} \oplus \hat{\mathcal{R}}$$

Let $E : \mathbb{R}^n \to \mathbb{R}^n$ be projection onto $\mathcal{R}$ with kernel $\hat{\mathcal{R}}$, i.e. E is the unique linear map such that $E(x) = x$ for $x \in \mathcal{R}$ and $E(x) = 0$ for $x \in \hat{\mathcal{R}}$.
Writing $y = (x, w)$ with $x \in \mathcal{K}$ and $w \in \hat{\mathcal{K}}$ we have

$$F((x, w), \lambda) = 0 \quad (*)$$

if and only if

$$\begin{aligned} EF((x, w), \lambda) &= 0 \\ (1 - E)F((x, w), \lambda) &= 0. \end{aligned}$$

Since $(dF_{(0,0)})|_{\hat{\mathcal{K}}}$ is nonsingular the implicit function theorem implies there is a unique function $w : \mathcal{K} \times \mathbb{R}^k \to \hat{\mathcal{K}}$ such that

$$EF((x, w(x, \lambda)), \lambda) = 0.$$

Hence solving (*) is equivalent to solving

$$f(x, \lambda) := (1 - E)F((x, w(x, \lambda)), \lambda) = 0$$

Note that we have transformed the *n*-state variable problem $F(y, \lambda) = 0$ to a *Lyapunov-Schmidt reduced* bifurcation problem:

$$f(x, \lambda) = 0$$

where $x \in \mathcal{K}$ has dimension of the kernel of dF and $\lambda \in \mathbb{R}^k$ and

$$f(x, \lambda) := (1 - E)F((x, w(x, \lambda)), \lambda) = 0.$$

Another technique is *centre manifold reduction*:

- This is a very powerful method for reduction that preserves not only equilibria at bifurcation but all dynamics near a bifurcation.
- The Centre manifold reduced system is typically not unique and may have lower smoothness than the Lyapunov-Schmidt reduced equations.
- Centre manifold gives access to Hopf and Steady bifurcations at the same time; the Lyapunov-Schmidt method for steady solutions can be adapted to Hopf.
- The Centre manifold reduction can be more involved to compute the reduced equations.
- Both methods give the same solutions and stability for one dimensional reduced problems.

Centre manifold theory relies on the following (or related) result:
Suppose $\dot{y} = F(y, \lambda)$ has $F(0,0) = 0$. Suppose $dF(0,0)$ has a centre eigenspace E^c of dimension $m \leq n$.

Theorem

There exists is a locally invariant manifold, the centre manifold W^c, containing $(0,0)$ that is of dimension $m + k$ on which all bifurcations locally occur.

Dynamics with symmetry

The main content of the course is to examine generic bifurcations of higher dimensional systems where there is a symmetry in the system.

What this means is that we have a bifurcation problem

$$\dot{x} = f(x, \lambda)$$

for $x \in \mathbb{R}^n$, $\lambda \in \mathbb{R}$ and a set of orthogonal transformations $G_1, G_2, \cdots$ (that may be infinite) such that for all k

$$f(G_k x, \lambda) = G_k f(x, \lambda)$$

i.e. the nonlinear map f commutes with the matrices G_k.
Clearly if f commutes with G_1 and G_2 then it commutes with $G_1 G_2$ and with the inverses G_k^{-1}; the set of commuting matrices forms a *group* Γ.

Symmetries: equivariance and invariance

We consider symmetry groups that are *compact Lie groups* (they have manifold and group structure) and in general if they act on $\mathbb{R}^n$ one can assume that these are subgroups of an orthogonal group:

$$O(n) = \{M \in \mathbb{R}^{n\times n} \ : \ M^T M = \mathrm{Id}\}$$

where M^T is the transpose of M; $\det(M) = \pm 1$. Note that:

- $O(2)$ is set of rotations about the origin, and reflections fixing the origin. It has two connected components that for an action on $\mathbb{R}^2$ are generated by:

$$\rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

  and

$$\kappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

- $O(3)$ is the set of rotations about the origin and reflections through the origin in dimensions 2 and 3.

Important subgroup of $O(n)$ is the *special orthogonal group*:

$$SO(n) = \{M \in \mathbb{R}^{n\times n} \ : \ M^T M = \mathrm{Id} \ \text{ and } \ \det(M) = 1\}.$$

Note that:

- $SO(2)$ is set of rotations about the origin in 2d

$$\rho_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

- $SO(3)$ is the set of rotations about the origin in 3d

Other common groups (NB many alternative notations!)

- $\mathbb{Z}_n$ the cyclic group on n elements; a possible representation on $\mathbb{C}$ is
$$\rho(z) = e^{i2\pi/n}z$$
- D_n the dihedral group; symmetries of a regular n-gon; a possible representation on $\mathbb{C}$ is
$$\rho(z) = e^{i2\pi/n}z$$
and
$$\kappa(z) = \overline{z}$$
- S_n the group of all permutations on n elements; a possible representation on $\mathbb{R}^n$ is given by the set of permutation matrices.
- $S^1 = SO(2)$ the group of rotations on a circle.

Case study The dihedral group D_n.

- This group has two generators: ρ and κ.
- We have relations
$$\rho^n = 1,\ \kappa^2 = 1,\ \kappa\rho = \rho^{n-1}\kappa$$
- A list of all $2n$ elements of D_n is
$$\{1, \rho, \rho^2, \cdots, \rho^{n-1}, \kappa, \kappa\rho, \kappa\rho^2, \cdots, \kappa\rho^{n-1}\}.$$
- First n elements are rotations, last n elements are reflections.
- Only subgroups of D_n are
$$\mathbb{Z}_m \text{ for } m|n$$
$$D_m \text{ for } m|n.$$
- Note that there are many copies of D_m for $m < n$, but only one of $\mathbb{Z}_m$.

Normal subgroups

A special class of subgroups are *normal subgroups*: subgroup $H \subset G$ is normal if

$$gH = Hg$$

for all $g \in G$. For D_n example, only $\mathbb{Z}_m$ are normal subgroups

Conjugacy

Suppose $H \subset G$ is a subgroup; then for any $g \in G$ the set if

$$g^{-1}Hg = \{g^{-1}hg \;:\; h \in H\}$$

is a subgroup that is said to be *conjugate* to H. Clearly if H is normal then it is only conjugate to itself.

Group homomorphism
Suppose G and H are groups and $\theta : G \to H$ is a surjective map that respects the group operation, i.e.

$$\theta(g_1 g_2) = \theta(g_1)\theta(g_2)$$

for all $g_1, g_2 \in G$. Then we say θ is a *group homomorphism*.

Isomorphic groups
We say groups G and H are isomorphic groups if there is a bijective group homomorphism from G to H.

Normal subgroups are important because:

- The kernel of θ is a normal subgroup $K \subset G$
- The group H is isomorphic to G/K

One can construct further compact Lie groups by taking direct or semidirect products of compact Lie groups;

- One example is the Viergruppe

$$\mathbb{Z}_2 \times \mathbb{Z}_2$$

  generated by two orthogonal reflections. One can see D_n as a semidirect product of $\mathbb{Z}_n$ and $\mathbb{Z}_2$ (direct when $n = 2$).

- Another example is

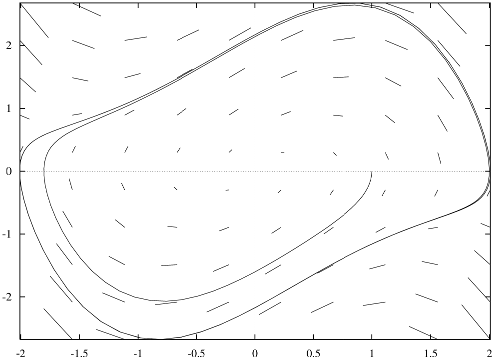
$$\mathbb{T}^n$$

  the group of all translations on an n-torus which is the product of n copies of S^1.

- We say a function $F:\mathbb{R}^n\to\mathbb{R}^n$ is *equivariant* under the action of the matrix group $\Gamma\in O(n)$ if
$$F(gx)=gF(x)$$
for all $g\in\Gamma$.
- We say a function $F:\mathbb{R}^n\to\mathbb{R}$ is *invariant* under the action of the matrix group $\Gamma\in O(n)$ if
$$F(gx)=F(x)$$
for all $g\in\Gamma$.

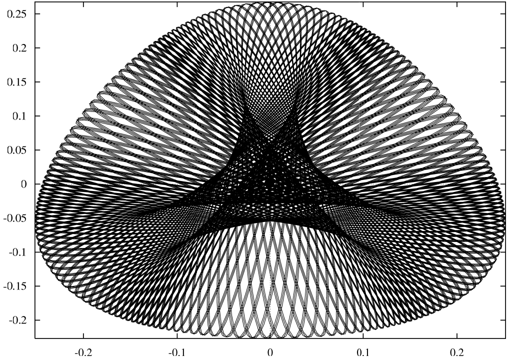
If a system $\dot{x}=F(x)$ has a group of symmetries Γ acting on phase space this means that the function F is equivariant under the action of Γ.

Example: Consider the vector field for the van der Pol system on the plane $\mathbb{R}^2$; $(\dot{x}, \dot{p}) = f(x, p) = (\omega p, \omega(-x + p(1 - x^2))$:



This is symmetric under the $\mathbb{Z}_2$ generated by $(x, p) \mapsto (-x, -p)$; i.e. f is equivariant under this symmetry.

Example: Henon-Heiles $(\dot{x}, \dot{y}, \dot{p}, \dot{q}) = (p, q, -x - 2xy, -y - x^2 + y^2)$ projected onto (p, q):



This is symmetric under the D_3 on $\mathbb{C}^2$ (ρ is cube root of unity) generated by

$$(x + iy, p + iq) \mapsto (\rho(x + iy), \rho(p + iq)), \quad (x + iy, p + iq) \mapsto (x - iy, p - iq))$$

Example: From before

$$
\begin{aligned}
\dot{x} &= \lambda x - (x+y+z) + x^2 \\
\dot{y} &= \lambda y - (x+y+z) + y^2 \\
\dot{z} &= \lambda z - (x+y+z) + z^2.
\end{aligned}
$$

This is symmetric (RHS is equivariant) under the group generated by the orthogonal matrices

$$
\rho = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \kappa = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}.
$$

Exercise: compute the group generated by these matrices and verify that it has 6 elements and is isomorphic to D_3.

Group representations and actions

NB are not interested in groups but rather group representations, i.e. their linear actions on $\mathbb{R}^n$. Often one will use the same symbol for group elements $\gamma \in \Gamma$ and their actions ρ_γ where the context is clear.

However, be aware that a group can act in many different ways; for example $\mathbb{Z}_2$ can act on the plane by:

- Rotation about the origin by π.
- Reflection on an axis through the origin.

For this reason we sometimes label groups as being different $\mathbb{Z}_2^a$, $\mathbb{Z}_2^b$ or $\mathbb{Z}_2(\kappa_1)$, $\mathbb{Z}_2(\kappa_2)$ to distinguish between them.

Isotropy subgroups and fixed point subspaces

A basic property of solutions of equivariant ODEs is that for any solution $x(t)$ one of the following applies: For every $g \in \Gamma$ either

- $gx(t) = x(t)$ for all t (g is a symmetry of the solution), or
- $gx(t) = y(t) \neq x(t)$ for all t and $y(t)$ is also a solution of the system.

Note that the set of symmetries of a solution is a subgroup:

Given a point $x \in \mathbb{R}^n$ we define the *isotropy subgroup* (or just symmetry) of x to be

$$\Sigma_x = \{g \in \Gamma \ : \ gx = x\}.$$

Basic properties of isotropy subgroups:

- Not all subgroups of Γ are necessarily isotropy subgroups.
- Which subgroups are isotropy subgroups depends on the action of Γ.
- Independent of dynamics.
- These groups form a lattice given by $\Sigma_x \to \Sigma_y$ if $\Sigma_x \subset \Sigma_y$.
- If $x(t)$ is a trajectory then
$$\Sigma_{x(0)} = \Sigma_{x(t)}$$
for all t.

Isotropy is about classifying the symmetries of a point in phase space. An obvious converse question is to characterise the set of all points with a certain symmetry. If $H \subset \Gamma$ which acts linearly on $\mathbb{R}^n$ then

$$\mathrm{fix}(H) = \{x \in \mathbb{R}^n \ : \ gx = x \ \text{ for all } g \in H\}$$

is the fixed point space of H.
Without loss of generality, we can assume that H is a subgroup of Γ.

Relationship between isotropy subgroups and fixed point subspaces:

- Typical points $x \in \mathrm{fix}(H)$ have

$$\Sigma_x = H$$

- However, some points may have MORE symmetry.
- If $H \subset K$ isotropy subgroups then

$$\mathrm{fix}(H) \supset \mathrm{fix}(K)$$

Example: D_n acting on $\mathbb{C}$.
Recall action generated by

$$\rho(z) = e^{2\pi i/n} z, \quad \kappa(z) = \bar{z}$$

gives isotropy subgroups that are

- Origin with full symmetry $\Sigma_0 = D_n$.
- Having no symmetry $\Sigma_z = \{e\} = I$ is a generic property.
- Reflection symmetric points e.g. $z = x$ real, $\Sigma_x = \mathbb{Z}_2(\kappa)$.

If we list all the reflection isotropy subgroups for D_n

$$\mathbb{Z}_2^{(k)} = \mathbb{Z}_2(\kappa\rho^k)$$

for $k = 1, \cdots, n$ note that

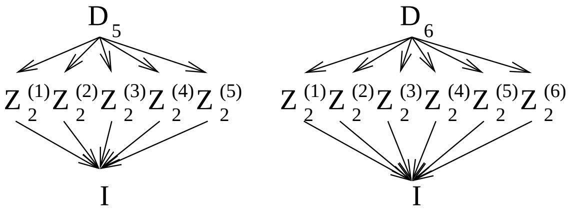
- Each of these is clearly a different but isomorphic subgroup of D_n.
- Because we can conjugate the generator as follows:

$$\rho^{n-\ell}\kappa\rho^k\rho^\ell = \kappa\rho^{k+2\ell}$$

  we have
  - If n is odd then all such groups are conjugate.
  - If n is even then the groups split into two conjugacy classes.

Recall that isotropy subgroups are partially ordered into a lattice, by containment of the groups. It is convenient to identify all conjugate groups and consider isotropy up to conjugacy:

For example, for D_5 and D_6 we have containment of isotropy as below:



We can write the isotropy lattice for general D_n acting on $\mathbb{C}$ as

$$\begin{tikzcd} D_{2n+1} \arrow[d] \\ Z_2 \arrow[d] \\ I \end{tikzcd} \qquad \begin{tikzcd} & D_{2n} \arrow[dl] \arrow[dr] & \\ Z_2^{(1)} \arrow[dr] & & Z_2^{(2)} \arrow[dl] \\ & I & \end{tikzcd}$$

for odd and even n respectively.

The lattice of isotropy subgroups gives a partition of phase space into points of a given isotropy.
More precisely, the set of all points with isotropy $H \subset \Gamma$ can be written

$$\mathrm{fix}(H) \setminus \bigcup_J \mathrm{fix}(J)$$

where the union is over all isotropy subgroups J that contain H. This is important for local bifucations in phase space; they will bifurcate with typical bifurcations for their isotropy.

Recall that

$$\mathrm{fix}(H) = \{x \in \mathbb{R}^n \ : \ gx = x \ \text{ for all } g \in H\}.$$

Lemma

The fixed point spaces $\mathrm{fix}(H)$ *are invariant subspaces for the dynamics.*

There is however no guarantee that the dynamics within a fixed point subspace is stable to perturbations that break the symmetry! However this result does give us a "skeleton" on which to organize the dynamics.

Normalizer and subspace symmetries

One might think that the isotropy is the only constraint to symmetric bifurcations within a subspace, but there are additional constraints:
Define the **normalizer** of a subgroup $H \subset \Gamma$ to be the subgroup

$$N_\Gamma(H) = \{g \in \Gamma \ : \ gH = Hg\}.$$

Note that $N_\Gamma(H)$ is a group

$$H \subset N_\Gamma(H) \subset \Gamma$$

that contains H as a normal subgroup. It is the largest subgroup for which this holds.

Lemma

For any H the fixed point subspace $\mathrm{fix}(H)$ is invariant under the action of the normalizer $N_\Gamma(H)$.

To see this, note that for any $g \in N_\Gamma(H)$

$$\begin{aligned}
g\,\mathrm{fix}(H) &= \{gx \;:\; hx = x \;\text{ for all } h \in H\}\\
&= \{x \;:\; hg^{-1}x = g^{-1}x \;\text{ for all } h \in H\}\\
&= \{x \;:\; ghg^{-1}x = x \;\text{ for all } h \in H\}\\
&= \{x \;:\; hx = x \;\text{ for all } h \in H\}\\
&= \mathrm{fix}(H).
\end{aligned}$$

Example:
Consider D_{2n} with reflection subgroup $\mathbb{Z}_2(\kappa)$. Then

$$N_{D_{2n}}(\mathbb{Z}_2(\kappa)) = \mathbb{Z}_2(\kappa) \times \mathbb{Z}_2(\rho^n).$$

To see this, note that

$$\kappa\rho^n = \rho^{2n-n}\kappa = \rho^n\kappa$$

Elements in $N_\Gamma(H) \setminus H$ are sometimes called *hidden symmetries* because they must be respected by the subspace but are not in the isotropy of points in H.

Irreducible subspaces

Symmetries give a powerful tool for decomposing and understanding the eigenvalues of linear maps.

Back to our favourite example:

$$\begin{aligned}
\dot{x} &= \lambda x - (x+y+z) + x^2 \\
\dot{y} &= \lambda y - (x+y+z) + y^2 \\
\dot{z} &= \lambda z - (x+y+z) + z^2
\end{aligned}$$

has Jacobian at 0 with two equal eigenvalues λ, one $\lambda - 3$. Clearly the symmetries have something to do with the multiple eigenvalues.

Suppose Γ acts orthogonally on $\mathbb{R}^n$. We say $V \subset \mathbb{R}^n$ is a Γ-*invariant subpace* if it is a linear subspace that is mapped to itself by all elements of Γ.

Examples:

- For D_n ($n \geq 3$) acting on $\mathbb{C}$: the only Γ-invariant subspaces are $\{0\}$ and $\mathbb{C}$.
- For $\mathbb{Z}_2$ acting on $\mathbb{C}$ by rotation about the origin by π radians, any line passing through the origin is Γ-invariant.
- For S_4 acting by permutations of axes in $(x_1, x_2, x_3, x_4) \in \mathbb{R}^4$, there are two Γ-invariant subspaces; one of these is the diagonal

$$V = \{(x, x, x, x) \ : \ x \in \mathbb{R}\}$$

while the other is the three-dimensional complement to this:

$$V^c = \{(x_1, x_2, x_3, x_4) \ : \ x_1 + x_2 + x_3 + x_4 = 0\}$$

Last result is a special case of following lemma: Let $\langle u, v\rangle$ represent the inner product of vectors u, v.

Lemma

Suppose $V \subset \mathbb{R}^n$ is an Γ-invariant subspace for Γ acting orthogonally. Then V^c is also invariant.

Proof: Suppose that $V \subset \mathbb{R}^n$ is Γ-invariant and consider

$$V^c = \{w \in \mathbb{R}^n \ : \ \langle w, v\rangle = 0 \text{ for all } v \in V\}.$$

Then for any $w \in V^c$, $v \in V$ and $g \in \Gamma$

$$\langle gw, v\rangle = \langle w, g^{-1}v\rangle$$

The fact that V is Γ-invariant means that the latter is zero.

We can also combine Γ-invariant subspaces through vector addition:

Lemma

Suppose $V, W \subset \mathbb{R}^n$ are Γ-invariant subspaces for Γ acting orthogonally. Then $V + W$ is also Γ-invariant.

A Γ-*irreducible* subspace is a Γ-invariant subspace that contains no smaller nontrivial Γ-invariant subspace.
The above lemma means that any vector space on which Γ acts can be usefully decomposed into a sum of Γ-irreducible subspaces $V_1, \cdots V_k$.

Note that we can use this result to decompose $\mathbb{R}^n$ into a number of irreducible subspaces:

$$\mathbb{R}^n = V_1 \oplus V_2 \oplus \cdots \oplus V_k$$

However this is not usually a unique decomposition.

Example:
For $\mathbb{Z}_2$ acting on $\mathbb{C}$ by rotation by π we can decompose into any pair of one-dimensional orthogonal subspaces.

- For D_n ($n \geq 3$) acting on $\mathbb{C}$, $\mathbb{C}$ is Γ-irreducible.
- For D_2 acting on $\mathbb{C}$, $\mathbb{C}$ is not Γ-irreducible as it contains Γ-invariant real and imaginary axes.
- For $\mathbb{Z}_2$ acting on $\mathbb{C}$ by rotation about the origin by π radians, any line passing through the origin is Γ-irreducible.
- For S_4 acting on $\mathbb{R}^4$, both the diagonal and its orthogonal complement are Γ-irreducible.

Wednesday

Commuting linear maps

We now consider the consequences of Γ-irreducible subspaces for a linear map that commutes with a group action: We say the linear map $A : \mathbb{R}^n \to \mathbb{R}^n$ commutes with the action Γ if

$$gA = Ag$$

for all $g \in \Gamma$.

- Note that $\ker(A)$ is Γ-invariant as $Av = 0$ implies that

  $$Agv = gAv = 0$$

  so gv is also in $\ker(A)$.
- If A commutes with Γ then A^{-1} (if it exists) also commutes with Γ.

We now give a classification of irreducible subspaces. Suppose that Γ acts on $\mathbb{R}^n$ such that $\mathbb{R}^n$ is irreducible; then we say Γ has an irreducible representation (irrep) on $\mathbb{R}^n$.

Lemma

(Schur) *For Γ acting irreducibly on $\mathbb{R}^n$, the set of linear maps $GL_\Gamma \subset GL(\mathbb{R}^n)$ that commute with Γ is isomorphic to one of:*

$$\mathbb{R}, \quad \mathbb{C}, \quad \mathbb{H}$$

Each compact Lie group has a finite number of irreps; in almost all cases these give commuting matrices equal to $\mathbb{R}$ or $\mathbb{C}$.
If GL_Γ is isomorphic to $\mathbb{R}$ then:

- We say Γ acts *absolutely irreducibly*.
- The set of commuting maps is

$$GL_\Gamma = \{c \,\mathrm{Id} \;:\; c \in \mathbb{R}\}$$

  i.e. the only commuting matrices are scalar multiples of identity.

Consider a steady equivariant bifurcation problem, with f equivariant for Γ:

$$\dot{x} = f(x, \lambda).$$

Suppose there is a trivial solution $f(0, \lambda) = 0$ that has steady bifurcation at $\lambda = 0$. Let $A_0 = (df)_{0,0}$.

Theorem

(G&S, Thm 1.27) *For the above conditions, generically we have*

- *The only eigenvalue of A_0 on the imaginary axis is 0.*
- *The generalized eigenspace corresponding to 0 is $\ker(A_0)$.*
- *The group Γ acts absolutely irreducibly on $\ker(A_0)$.*

This result allows us without loss of generality to only consider steady bifurcations in cases where Γ acts absolutely irreducibly!

Equivariant Branching Lemma

We now arrive at a fundamental result of Vanderbauwhede and Cicogna: Suppose Γ acts orthogonally and absolutely irreducibly on $\mathbb{R}^n$ and we have a steady bifurcation

$$\dot{x} = f(x, \lambda)$$

that is Γ-equivariant and has a bifurcation at $\lambda = 0$. Then we can write

$$(df)_{0,\lambda} = c(\lambda)I$$

with $c(0) = 0$ and generically $c'(0) \neq 0$.

Theorem

(Equivariant Branching Lemma) *For such a bifurcation problem pick any isotropy subgroup $H \subseteq \Gamma$ with*

$$\dim(\text{fix}(H)) = 1.$$

Then there is a branch of steady solutions with symmetry H that bifurcates from the origin at $\lambda = 0$.

Proof of the Equivariant Branching Lemma:
Because $\dim(\mathrm{fix}(H)) = 1$ we write $\mathrm{fix}(H)$ as scalar multiples of a single nonzero vector $v \in \mathbb{R}^n$. In this subspace we have

$$f(tv, \lambda) = h(t, \lambda)v$$

by invariance of $\mathrm{fix}(H)$, where $h : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$. Moreover

$$f(0, \lambda) = h(0, \lambda) = 0$$

so by Taylor's theorem we have $h(t, \lambda) = tk(t, \lambda)$ with $k(0, 0) = c(0) = 0$ and $k_\lambda(0, 0) = c'(0) \neq 0$.
The implicit function theorem means there is a unique $\lambda(t)$ such that $\lambda(0) = 0$ and

$$k(t, \lambda(t)) = 0$$

meaning that $f(t, \lambda(t)) = 0$. Note that the solutions clearly have isotropy H for $t \neq 0$.

Example:
Consider $\Gamma = \mathbb{Z}_2$ on $\mathbb{R}$ with action $x \mapsto -x$.

As $\dim(\mathrm{fix}(H)) = 1$ for the trivial group $H = I$ we can apply the Equivariant Branching Lemma to verify that at any bifurcation of 0 in a $\mathbb{Z}_2$-symmetric system there will be at least one branch of symmetry broken equilibrium solutions.

Moreover, considering the group orbit of these solutions there must be a pair of these.

In detail, consider $f(x,\lambda)$ such that $f(-x,\lambda) = -f(x,\lambda)$ and write as

$$f(x,\lambda) = x\,a(x,\lambda)$$

where $a(x,\lambda) = a(-x,\lambda)$. Then one can write $a(x,\lambda) = b(x^2,\lambda)$ if a is smooth. Generically we have $b_{x^2}(0,0) = B \neq 0$ and $b_\lambda(0,0) = L \neq 0$ and so we can write using Taylor expansion that

$$f(x,\lambda) = Bx^3 + Lx\lambda + O(x^5, x^3\lambda, \lambda^2)$$

By a suitable change of coordinates we can make f strongly equivalent to

$$g(x,\lambda) = \pm x^3 \pm \lambda x$$

which is the normal form for a pitchfork bifurcation.

Example:

Now consider $\Gamma = D_m$ acting on $\mathbb{C} = \mathbb{R}^2$ as before and suppose we have an equivariant bifurcation problem

$$\dot{z} = f(z, \lambda)$$

that has $f(0, \lambda) = 0$ and $df(0, 0)$ has nontrivial kernel, and the eigenvalues pass from negative to positive half plane as λ increases through zero with nonzero speed.

Then because Γ acts absolutely irreducibly on $\mathbb{C}$ we have $df(0, \lambda) = c(\lambda)I$. The nonzero speed means $c'(0) > 0$.

Taking $\mathbb{Z}_2^{(k)} = \mathbb{Z}_2(\kappa\rho^k)$ for any k we have $\dim(\mathrm{fix}(\mathbb{Z}_2^{(k)})) = 1$.

Applying the Equivariant Branching Lemma means that there must be a branch of solutions with symmetry $\mathbb{Z}_2^{(k)}$.

Linear and nonlinear equivariant systems

Summarizing so far, we are interested in understanding bifurcations of steady solutions $y \in \mathbb{R}^k$

$$\dot{y} = F(y, \lambda)$$

with symmetry Γ.

We can generically reduce to a bifurcation where the kernel is a subspace of dimension equal to one of the absolutely irreducible representations of Γ.

Using Lyapunov-Schmidt reduction, we can reduce the original system to

$$\dot{x} = f(x, \lambda)$$

where $x \in \mathbb{R}^n$, $n \leq k$ and f is equivariant for Γ.

Isotypic decomposition

We now look at implication of symmetries on the Jacobian of

$$\dot{x} = f(x, \lambda)$$

where f is equivariant under an orthogonal action of Γ. Clearly

$$f(gx, \lambda) = gf(x, \lambda)$$

means that for any x_0 and $g \in \Sigma_{x_0}$ we have

$$(df)_{x_0,\lambda} g = g(df)_{x_0,\lambda}.$$

In particular

$$(df)_{0,\lambda} g = g(df)_{0,\lambda}.$$

for all $g \in \Gamma$.

Suppose Γ acts on V and on W and recall that we say V is Γ-*isomorphic* to W, $V \cong W$ if there is a linear isomorphism $A : V \to W$ such that

$$A(gx) = gA(x).$$

for all $g \in \Gamma$ and $x \in V$.

Recall one can decompose any $\mathbb{R}^n$ with a Γ action into a sum

$$\mathbb{R}^n = V_1 \oplus \cdots \oplus V_k$$

where each V_j is a Γ-irreducible, but this decomposition is not unique.

Definition

Pick a Γ-irreducible representation $V \subseteq \mathbb{R}^n$ and let $\hat{V}$ denote the sum of all Γ-irreducible representations that are isomorphic to V.

The subspace $\hat{V}$ is called the *isotypic component* of $\mathbb{R}^n$ corresponding to the irreducible rep V.

By definition, isotypic components are unique, although they will be composed of Γ-irreducibles that are not unique.

Some results about isotypic components [G& S p38]

Lemma

Let V be an irreducible subspace. Then there exist irreducible subspaces V_j isomorphic to V such that $\hat{V} = V_1 \oplus \cdots \oplus V_s$

[G& S p38] Let $V_1 = V \subseteq \hat{V}$.

If $W \cong V$ is contained in V_1 we are done, otherwise choose $V_2 \subseteq V$ with $V_2 \not\subseteq V_1$.

Irreducibility means that $V_1 \cap V_2 = 0$ meaning $V_1 + V_2 = V_1 \oplus V_2$. Hence at some stage we can decompose

$$V' = V_1 \oplus V_2 \oplus \cdots \oplus V_p \subseteq V$$

with all $V_k \cong V$. If every $W \subseteq V$ is contained in V' then we are done otherwise suppose we have a $V_{p+1} \cong V$ with $V_{p+1} \not\subseteq \cap V' = 0$.

Irreducibility means that $V_{p+1} \cap V' = 0$ meaning we have a direct sum. This means the dimension increases at each stage and hence the process must stop after a finite number of steps.

Lemma

Let U, V be irreps of Γ on $\mathbb{R}^n$ and suppose $U \subseteq \hat{V}$. Then $U \cong V$.

Lemma

Choose Γ-irreducible subspaces $V_j \subseteq \mathbb{R}^n$, $j = 1, \cdots, s$ so that every irreducible subspace is isomorphic to exactly one V_j. Then

$$\mathbb{R}^n = \hat{V}_1 \oplus \cdots \oplus \hat{V}_s.$$

[G& S p38] Note that $\mathbb{R}^n$ must be the sum of the V_j; the issue is that the sum is direct which means we need to show that $(\hat{V}_1 \oplus \cdots \oplus \hat{V}_j) \cap \hat{V}_{j+1} = 0$ for $j = 1, \cdots, s-1$. If there was an intersection containing something more then let U be an irreducible in this intersection. We would have $U \cong V_{j+1}$. However irreducibility of U means it must be contained $U \subseteq \hat{V}_i$, $i \leq j$. This would give an isomorphism between irreducibles in different isotypic components and hence it would give a contradiction.

A key result is the following: one can use the unique isotypic decomposition

$$\mathbb{R}^n = \hat{V}_1 \oplus \cdots \oplus \hat{V}_s$$

to block diagonalize a Jacobian (or any commuting linear map):

Lemma

If $A : \mathbb{R}^n \to \mathbb{R}^n$ is linear map that commutes with Γ then $A(\hat{V}) \subseteq \hat{V}$.

To see this pick any Γ-irreducible $U \subseteq \hat{V}$ and consider $A' : U \to \mathbb{R}^n$, the restriction. Then $\ker(A')$ is Γ-invariant. Irreducibility means that either $\ker(A') = 0$ in which case $A(U) \cong U$ or $\ker(A') = U$ in which case $A' = 0$. In either case we have $A(\hat{V}) \subseteq \hat{V}$.

Consider a Γ-equivariant bifurcation problem and an arbitrary steady solution $x_0 \in \mathbb{R}^n$, $\lambda_0 \in \mathbb{R}$.

Theorem

The isotypic decomposition of $\mathbb{R}^n$ with respect to Σ_{x_0} can be used to block-diagonalize $(df)_{x_0,\lambda_0}$.

The larger the isotropy subgroup Σ_{x_0} is, the more of a constraint this is.

When an equilibrium x_0 does a generic steady bifurcation, it will bifurcate with a centre eigenspace that is an absolutely irreducible subspace in only one of the isotypic components of Σ_{x_0}.

Back to our favourite example [GS (1.1)]

$$\begin{array}{rcl}\dot{x} & = & \lambda x - (x+y+z) + x^2\\ \dot{y} & = & \lambda y - (x+y+z) + y^2\\ \dot{z} & = & \lambda z - (x+y+z) + z^2\end{array}$$

where $(x, y, z) \in \mathbb{R}^3$ and $\lambda \in \mathbb{R}$ is a parameter and (df) at origin is

$$J = \begin{pmatrix} \lambda - 1 & -1 & -1 \\ -1 & \lambda - 1 & -1 \\ -1 & -1 & \lambda - 1 \end{pmatrix}$$

We note that only irreducible subspaces are

$$V_1 = \{(x, x, x)\}, \quad V_2 = \{(a, b, c) \ : \ a + b + c = 0\}$$

and these are not isomorphic. Hence this is also the isotypic decomposition and J in a basis for these subspaces will be diagonal.

Example:

Consider ring of six diffusively coupled cells

$$\dot{x}_k = f(x_k) + C(x_{k+1} + x_{k-1} - 2x_k)$$

for $k = 1, \cdots, 6$ and $(x_1, \cdots, x_6)$ (subscripts taken mod 6).
This is equivariant under the action of D_6 acting by permutation on $\mathbb{R}^6$, i.e. $[\rho(x)]_k = x_{k+1}$ and $[\kappa(x)]_k = x_{6-k}$. This decomposes into four absolutely irreducible subspaces

$$V_1 = \{(x, x, x, x, x, x)\}$$

$$V_2 = \{(x, -x, x, -x, x, -x)\}$$

$$V_3 = \{(x, x\cos\pi/3, x\cos 2\pi/3, -x, -x\cos\pi/3, -x\cos 2\pi/3) + (0, y\sin\pi/3, y\sin 2\pi/3, 0, -y\sin\pi/3, -y\sin 2\pi/3)\}$$

$$V_4 = \{(x, x\cos 2\pi/3, x\cos 4\pi/3, x, x\cos 2\pi/3, x\cos 4\pi/3) + +(0, y\sin 2\pi/3, y\sin 4\pi/3, 0, y\sin 2\pi/3, y\sin 4\pi/3)\}.$$

Note that each irreducible is non-isomorphic, meaning the isotypic decomposition is just

$$\mathbb{R}^6 = V_1 \oplus V_2 \oplus V_3 \oplus V_4$$

where D_6 acts on

- V_1 trivially.
- V_2 as a homomorphism into $\mathbb{Z}_2$.
- V_3 as a homomorphism into D_3.
- V_4 faithfully (kernel of the action is the identity).

This decomposition is invariant for the Jacobian of any solution with full symmetry meaning a generic steady bifurcation will occur in precisely one of the V_i.

Using the previous decomposition we can then use the Equivariant Branching Lemma to classify the steady instabilities in this system according to the isotypic component containing the kernel of the Jacobian:

V_1: There will be branches bifurcating in the full-symmetry subspace

$$\mathrm{fix}(D_6) = \mathbb{R}(1,1,1,1,1,1)$$

V_2: There will be branches bifurcating in

$$\mathrm{fix}(D_3) = \mathbb{R}(1,-1,1,-1,1,-1)$$

V_3 and V_4: There will be branches bifurcating tangent to all fixed point spaces with $\dim \mathrm{fix} = 1$ in these representations.

For example if V_4 goes unstable then there will be a branch in

$$\mathrm{fix}(\mathbb{Z}_2(\kappa)) = \{(a,b,c,c,b,a)\}$$

that is tangent to the vector in V_4

$$(-1,0,1,1,0,-1).$$

Nonlinear terms will typically "bend" the centre manifold into generic points in $\mathrm{fix}(\mathbb{Z}_2(\kappa))$, i.e. typical points on the branch will be

$$(a,b,c,c,b,a).$$

Example:
Suppose that $\mathbb{Z}_2$ acts orthogonally on $\mathbb{R}^n$ for some n. Then we can write the generator as

$$\kappa(x) = Mx$$

where $M^2 = I$. Since $\mathbb{Z}_2$ has only two irreducible representations both of dimension 1: one where there is trivial action and one where the action is $x \mapsto -x$.

Hence we can decompose into two isotypic components

$$\mathbb{R}^n = V_1 \oplus V_2.$$

with $\dim(V_1) = k$, $\dim(V_2) = n - k$.

Relative to this basis for this, we can write any linear A that commutes with this group in block diagonal form.

Linearization and group orbits

Given a point $x \in \mathbb{R}^n$ and Γ acting on $\mathbb{R}^n$ the *group orbit* of x under Γ is the set

$$\Gamma x = \{gx \ : \ g \in \Gamma\}.$$

From previous results we have that if x an equilibrium then all points in its group orbit are equilibria.

More generally, for any dynamically invariant set $A \subseteq \mathbb{R}^n$, its group orbit

$$\Gamma A = \{gA \ : \ g \in \Gamma\}$$

will be composed of invariant sets.

Note that if A is compact and Γ a compact Lie group then ΓA also compact.

We introduce a concept that is especially useful for problems with *continuous symmetry*.

Suppose x is a point such that Γx is dynamically invariant. Then we say all points on Γx are *relative equilibria*.

We do not require that x is an equilibrium, though it may be.

For a continuous group of dimension $\dim(\Gamma) > 0$, the group orbit Γx of a point will be a manifold of dimension at most $\dim(\Gamma)$.

If Γ is finite we have a dichotomy that there is a $\delta > 0$ such that for every $g \in \Gamma$ either

$$gx = x \quad \text{or} \quad |gx - x| > \delta.$$

This means that x is *isolated* from the rest of its group orbit.

Hence to understand the constraints of symmetry on linearization near an equilibrium x for a *finite group* we need only consider the action of Σ_x.

If Γ is a continuous group there will be other constraints on the linearization, notably [G&S p39]:

Theorem

A Jacobian $(df)_{x_0,\lambda_0}$ *will have* $\dim\Gamma - \dim\Sigma_{x_0}$ *eigenvalues equal to zero, and the corresponding eigenvectors correspond to infinitesimal motion on the group orbit of* x_0*.*

To see this, consider a parametrized path $g(s)$

$$g : \mathbb{R} \to \Gamma$$

with $g(s) = I$ and a point x_0 that we assume to be a relative equilibrium.

This means that $f(x_0, \lambda_0)$ must be tangent to the group orbit and so $f(g(s)x_0, \lambda_0)$ is also tangent to the group orbit for any fixed s.

We calculate

$$0 = \frac{d}{ds} f(g(s)x_0, \lambda_0)|_{s=0} = (df)_{x_0,\lambda_0}(g'(0)x_0)$$

and so any infinitesimal perturbation $v = g'(0)$ on the group orbit will lead to a zero eigenvalue.

Note that $\dim(\Sigma_{x_0})$ of these will give $vx_0 = 0$ so only $\dim(\Gamma) - \dim(\Sigma_{x_0})$ will give non-zero eigenvectors.

The following result [G&S p173] characterizes typical motion of relative equilibria:

Theorem

Suppose that Γx is a relative equilibrium with isotropy Σ_x. Then generically the dynamics of x is quasiperiodicity with k independent frequencies, where k is the dimension of the largest torus group in $N(\Sigma_x)/\Sigma_x$.

This motion on a relative equilibrium is sometimes called *drift on the group orbit*.

Example:

Consider $O(2)$ acting on $\mathbb{R}^2$. Group orbits are circles around the origin and these are generically fixed points.

Consider $SO(2)$ acting on $\mathbb{R}^2$. Group orbits are also circles around the origin but generically there is a periodic motion on the group orbit.

Nonlinear commuting maps

We now wish to understand the structure of polynomial invariants and equivariants.

A polynomial $F : \mathbb{R}^n \to \mathbb{R}^n$ is *equivariant* for $\Gamma \in O(n)$ if

$$F(gx) = gF(x)$$

for all $g \in \Gamma$. A polynomial $F : \mathbb{R}^n \to \mathbb{R}$ is *invariant* for $\Gamma \in O(n)$ if

$$F(gx) = F(x)$$

for all $g \in \Gamma$.

Let $\mathcal{P}(\Gamma)$ denote the set of Γ-invariant polynomials for a particular action. This forms a commutative ring under addition and multiplication.

We say the ring of invariant functions $\mathcal{P}(\Gamma)$ has a *Hilbert basis* $\{u_1(x), \cdots, u_k(x)\}$ if every polynomial $h \in \mathcal{P}$ can be written as a polynomial function of $u_1, \cdots u_k$, i.e. if there is a p such that

$$h(x) = p(u_1(x), \cdots, u_k(x)).$$

Theorem

(Hilbert-Weyl) *Let Γ be a compact Lie group acting on $\mathbb{R}^n$. Then there exists a finite Hilbert basis for $\mathcal{P}(\Gamma)$.*

Hilbert-Weyl can be generalized in a very powerful way to hold not only for invariant polynomials but for invariant smooth functions. Let $\{u_1, \cdots, u_k\}$ be a Hilbert basis for $\mathcal{P}(\Gamma)$.

Theorem

(Schwartz) *Let $f : \mathbb{R}^n \to \mathbb{R}$ be any C^∞ invariant function. Then there is a C^∞ function h such that*

$$f(x) = h(u_1(x), \cdots, u_k(x)).$$

Similarly, for equivariants one can find

$$X_1(x), \cdots X_\ell(x)$$

that allow one to write any polynomial equivariant function in the form

$$p_1(x)X_1(x) + \cdots p_\ell(x)X_\ell(x)$$

where the p_j are Γ-invariant polynomial functions and the X_j are a basis of polynomial equivariants.

The main result that is used for decomposition of equivariant vector fields is [G&S p42]. We pick a group Γ acting on some $\mathbb{R}^n$.

Theorem

(Poénaru) *Let $u(x) = (u_1(x), \cdots, u_k(x))$ be a Hilbert basis for the Γ-invariant functions and $X_1(x), \cdots, X_\ell(x)$ be a basis for the module of Γ-equivariants functions. Then any C^∞ Γ-equivariant $f(x)$ can be written as*

$$f(x) = p_1(u(x))X_1(x) + \cdots + p_\ell(u(x))X_\ell(x).$$

Example: D_n

We now consider a case study of computing the invariants and equivariants for nonlinear maps that commute with the action of D_n on $\mathbb{C} = \mathbb{R}^2$.

Consider a polynomial invariant $I : \mathbb{C} \to \mathbb{R}$; then we can write

$$I(z) = \sum c_{p,q} z^p \overline{z}^q$$

with the sum over some set of non-zero complex coefficients $c_{p,q}$.

Invariance means we require

$$I(z) = I(\overline{z})$$

$$I(ze^{2\pi i/n}) = I(z)$$

while real-valuedness means that

$$I(z) = \overline{I(z)}.$$

For invariance we need

$$I(z) = \sum c_{p,q} z^p \overline{z}^q = \sum c_{p,q} z^q \overline{z}^p$$

$$I(z) = \sum c_{p,q} z^p \overline{z}^q = \sum \overline{c_{p,q}} z^q \overline{z}^p$$

and

$$I(z) = \sum c_{p,q} z^p \overline{z}^q = \sum c_{p,q} z^p \overline{z}^q e^{2\pi i(p-q)/n}$$

for all z.

This means that

$$c_{p,q} = c_{q,p}$$

and

$$c_{p,q} = \overline{c}_{p,q}$$

Using these properties we can conclude that

$$I(z) = \sum_{q \leq p} c_{p,q} (z\overline{z})^q (z^{p-q} + \overline{z}^{p-q}).$$

Using the other invariance we have $c_{p,q} = 0$ unless

$$p = q \pmod{n}$$

so we can write the invariant function as

$$I(z) = p_1(|z|^2)(z^n + \overline{z}^n) + p_2(|z|^2)(z^{2n} + \overline{z}^{2n}) + \cdots$$

with p_1, p_2 real-valued arbitrary polynomials. Noting that

$$\begin{aligned} z^{n(k+1)} + \overline{z}^{n(k+1)} &= (z^{nk} + \overline{z}^{nk})(z^n + \overline{z}^n) - z^{nk}\overline{z}^n - \overline{z}^{nk} z^n \\ &= (z^{nk} + \overline{z}^{nk})(z^n + \overline{z}^n) - |z|^{2n}(z^{n(k-1)} + \overline{z}^{n(k-1)}) \end{aligned}$$

inductively we can write all terms as functions of

$$|z|^2,\ z^n + \overline{z}^n$$

Hence (Lemma 2.25 of [GS]) a Hilbert basis for the invariants is

$$\{|z|^2, z^n + \overline{z}^n\}.$$

For the equivariants of the action of D_n on $\mathbb{C}$, suppose that $E : \mathbb{C} \to \mathbb{C}$

$$E(z) = \sum d_{q,p} z^p \overline{z}^q$$

is equivariant. Then

$$\sum d_{p,q} z^p \overline{z}^q = \sum \overline{d_{p,q}} z^p \overline{z}^q$$

which says that $d_{p,q}$ is real, and

$$e^{2\pi i/n} \sum d_{p,q} z^p \overline{z}^q = \sum d_{p,q} z^p \overline{z}^q e^{2\pi i(p-q)/n}$$

for all z.

The second condition says that

$$\sum d_{p,q} z^p \overline{z}^q = \sum d_{p,q} z^p \overline{z}^q e^{2\pi i(p-q-1)/n}$$

and so

$$p - q = 1 \pmod{n}.$$

This means that a minimal set of equivariant generators is:

$$\{z,\ \overline{z}^{n-1}\}.$$

Summarizing, we have [G&S Theorem 2.24]:

Theorem

One can write any smooth equivariant function for D_n acting on $\mathbb{C}$ as

$$f(z) = p_1(|z|^2, z^n + \overline{z}^n)z + p_2(|z|^2, z^n + \overline{z}^n)\overline{z}^{n-1}$$

for some smooth real functions p_1, p_2.

The previous calculations can be used to find the general normal form for D_n vector fields near $z = 0$.

For example, consider D_4 and let $u = |z|^2$, $v = z^4 + \overline{z}^4$. Then

$$\dot{z} = p_1(u, v)z + p_2(u, v)\overline{z}^3.$$

Expanding up to and including fifth order terms we have

$$\dot{z} = (a_1 + a_2|z|^2 + a_3(z^4 + \overline{z}^4) + a_4|z|^4)z + (a_5 + a_6|z|^2)\overline{z}^3$$

This can be written

$$\dot{z} = a_1 z + a_2 |z|^3 z + a_5 \bar{z}^3 + a_3 (z^5 + z\bar{z}^4) + a_4 |z|^4 z + a_6 |z|^2 \bar{z}^3.$$

to fifth order.

Compare to the general order Taylor expansion:

$$g(z) = g_0 + g_1 z + g_2 \bar{z} + g_3 z^2 + g_4 z\bar{z} + g_5 \bar{z}^2 + g_6 z^3 + g_7 z^2 \bar{z} + g_8 z\bar{z}^2 + g_9 \bar{z}^3 + \cdots$$

and note that:

- Many of the terms from a general expansion are missing
- Some of the terms that are present are related.
- Generically we can assume that all coefficients are non-zero.

We now use this to investigate bifurcation of the normal form with a parameter λ. In this case the Taylor expansion is

$$\begin{aligned} \dot{z} &= a_1(\lambda)z + a_2(\lambda)|z|^2 z + a_5\overline{z}^3 + a_3(\lambda)(z^5 + z\overline{z}^4) \\ &\quad + a_4(\lambda)|z|^4 z + a_6(\lambda)|z|^2\overline{z}^3 \end{aligned}$$

and at bifurcation we can assume $(d_\lambda a_1)(0) = 0$.

To cubic order in (λ, z) we have

$$\dot{z} = f(z, \lambda) = b_1\lambda z + b_2|z|^2 z + b_3\overline{z}^3$$

with b_1, b_2, b_3 real valued constants that are generically non-zero.

One can solve this setting $z = re^{i\theta}$ to give

$$r = 0$$

or

$$b_1\lambda + r^2(b_2 + b_3 e^{-4i\theta}) = 0.$$

Solving the latter we have $\theta_k = k\pi/4$, $k = 0, \cdots, 7$ and so

$$r^2 = -\frac{b_1\lambda}{b_2 + (-1)^k b_3}.$$

This corresponds to pitchfork bifurcations occuring in each of the subspaces $\mathrm{fix}(\mathbb{Z}_2)$ for D_4 as long as we have non-degeneracy conditions

$$b_1 \neq 0, \quad b_2 \neq 0, \quad b_3 \neq 0, \quad b_2 \neq b_3, \quad b_2 \neq -b_3.$$

Hence the only bifurcating branches for this problem will generically be those predicted by the Equivariant Branching Lemma.

On the other hand, for D_5 the normal form to third order is

$$\dot{z} = f(z, \lambda) = b_1 \lambda z + b_2 |z|^2 z$$

with b_1, b_2 real valued constants that are generically non-zero.

In this case, there is a ring of equilibria for one sign of λ. This is not robust to perturbations implying that the normal form, if finitely determined, is only determined at order four or higher.

In fact D_n bifurcation for $n \geq 4$ will generically be determined at order $n - 1$.

Similar calculations are available in the literature for many group actions.

The basic recipe for analysing equivariant steady bifurcations (including those of higher dimension) is:

- Reduce to a centre eigenspace.
- Compute the most general normal form the commutes with the action on the centre eigenspace, including any parameters.
- Analyse this for generic choice of normal form coefficients.
- Check that all branches and stabilities are finitely determined.

For any specific system, one must then determine the actual normal form coefficients.

Periodic solutions with symmetry

Thus far we have limited ourselves to steady bifurcation problems. These are essentially reducible to algebraic problems.

Periodic orbits give a number of new challenges of interest to symmetric systems notably they are properly time-dependent solutions. Moreover they can appear at generic bifurcation (Hopf bifurcation) from a trivial solution.

Symmetries of periodic orbits

Suppose for $x \in \mathbb{R}^n$

$$\dot{x} = f(x)$$

and f is equivariant under the action of the orthogonal group Γ.

Suppose $P(t)$ is a periodic orbit, i.e. P is non-constant but $P(t+T) = P(t)$ for all $t \in \mathbb{R}$. We define the *instantaneous symmetry* of the periodic orbit to be the isotropy subgroup

$$K = \Sigma_{P(0)} = \{g \in \Gamma \ : \ gP(0) = P(0)\}$$

Recall that isotropy is preserved along orbits and so

$$K = \Sigma_{P(s)}$$

for all $s \in \mathbb{R}$.

We define the *spatio-temporal symmetry* of the periodic orbit to be

$$H = \{g \in \Gamma \ : \ gP(0) = P(sT) \ \text{ for some } s\}.$$

Note that $K \subseteq H$ and periodicity of P means that we can write any symmetry in H as (g, s) for $s \in [0, 1)$; there is a group homomorphism

$$\theta : H \to S^1.$$

One can show that

$$K \subseteq H \subseteq \Gamma$$

and $K = \ker(\theta)$ is a normal subgroup of H, with

$$H/K \cong S \subseteq S^1.$$

Note that H can be interpreted as the group of symmetries that fix the periodic orbit *as a set*.

Recall, if $P(t)$ is a T-periodic orbit of an equivariant vector field then we define the *instantaneous symmetry* of the periodic orbit to be the isotropy subgroup

$$K = \Sigma_{P(0)} = \{g \in \Gamma \ : \ gP(0) = P(0)\}$$

and we define the *spatio-temporal symmetry* of the periodic orbit to be

$$H = \{g \in \Gamma \ : \ gP(0) = P(sT) \ \text{ for some } s\}.$$

Typically, only a small number of subgroups of Γ are isotropy subgroups; this depends on the action of Γ. On the other hand, more possible subgroups are available as spatio-temporal symmetries of periodic orbits. A nice characterization of possible spatio-temporal symmetries is [G&S, Theorem 3.4]:

Theorem

Suppose Γ is a finite group acting on $\mathbb{R}^n$. Then there is a Γ-equivariant vector field on $\mathbb{R}^n$ with a periodic orbit of spatial symmetry K and spatio-temporal symmetry H if and only if:

(a) *H/K is cyclic*

(b) *K is an isotropy subgroup*

(c) $\dim \mathrm{fix}(K) \geq 2$.

(d) *H fixes a connected component of*

$$\mathrm{fix}(K) \setminus \bigcup_{g \notin K} \mathrm{fix}(g).$$

Hopf bifurcation with symmetry

One can adapt methods for steady bifurcations, including the Lyapunov-Schmidt method, to calculate branching at equivariant Hopf bifurcations.

In these cases (see [G&S Chapter 4]) one can show the the linearization of an equivariant bifurcation problem will generically have an action of Γ that is called Γ-simple:

We say a vector space W is Γ-*simple* if either

- $W = V \oplus V$ where V is an absolutely irreducible representation of Γ, or
- Γ acts irreducibly but not absolutely irreducibly on W.

The main result at the linear level is [G&S Theorem 4.5]: the action on the centre eigenspace is Γ-simple.

Note that as a consequence of [G&S Lemma 4.7], if there is a Γ-simple action on $\mathbb{R}^n$ and $\dot{x} = f(x)$ is an equivariant ODE with $(df)_0$ having eigenvalues $\pm i\omega$, then these eigenvalues will have multiplicity $m = n/2$ and there is a linear invertible mapping $S : \mathbb{R}^n \to \mathbb{R}^n$commuting with Γ such that

$$(df)_0 = \omega SJS^{-1}$$

and

$$J = \begin{bmatrix} 0 & -I_m \\ I_m & 0 \end{bmatrix}$$

where I_m is the $m \times m$ identity matrix.

As a consequence one can change coordinates such that $(df)_0$ is linearly in normal form, i.e. so that $(df)_0 = J$. Lemma 4.7 shows that one may choose this change of coordinates depending on a parameter λ in a sensible way.

The linearization J induces an action of S^1 on $\mathbb{R}^n$ by

$$\rho_\theta x = \exp(-i\theta J)x$$

and one can *preserve this action* in all terms up to order n of any Taylor series near an equivariant Hopf bifurcation.

This is just a special case [Theorem 4.16] of the more result for general Poincaré-Birkhoff normal forms [G&S Theorem 4.14] that states that one can to any finite order remove any terms that do not commute with the group $\exp(s(df)_0)$.

This gives a normal form that commutes with $\Gamma \times S^1$ symmetry. The remainder terms are Γ-equivariant, but generically not S^1 equivariant.

In some cases one finds that symmetries force Hopf bifurcation to be the *only* generic symmetry-breaking bifurcation.

For example, consider $\mathbb{Z}_3$ acting on $\mathbb{R}^3$ by permutation; then the linearization of a fully symmetric state will be

$$\begin{pmatrix} \lambda & \alpha & \beta \\ \beta & \lambda & \alpha \\ \alpha & \beta & \lambda \end{pmatrix}$$

which has eigenvalues

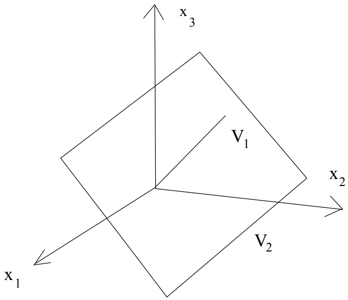
$$\lambda+\alpha+\beta, \quad \lambda-\frac{\alpha+\beta}{2}\pm\frac{i}{2}\beta\sqrt{3}.$$

and for $\beta\neq 0$ this includes a complex pair. Hence the only generic symmetry breaking bifurcation will be a Hopf bifurcation!

Note that for $\mathbb{Z}_3$ acting on $\mathbb{R}^3$ by permutation we have isotypic decomposition

$$\mathbb{R}^3 = V_1 \oplus V_2$$

where V_1 is the trivial action in 1D and V_2 is the action by rotation in 2D.



However steady bifurcations can occur for this example at codimension two.

At the nonlinear level, one can obtain an analogous result to the Equivariant Branching Lemma for steady bifurcations, namely the Equivariant Hopf Theorem: Consider

$$\dot{x} = f(x, \lambda)$$

Theorem

Suppose Γ acts Γ-simply, orthogonally and nontrivially on $\mathbb{R}^{2m}$ and that

(a) $f : \mathbb{R}^{2m+1} \to \mathbb{R}^{2m}$ *is equivariant,* $f(0, \lambda) = 0$ *and* $(df)_{0,\lambda}$ *has eigenvalues* $\sigma(\lambda) \pm i\rho(\lambda)$ *of multiplicity m.*

(b) $\sigma(0) = 0$, $\rho(0) = 1$, $\sigma'(0) \neq 0$.

(c) $\Sigma \subset \Gamma \times S^1$ *has* $\dim \mathrm{fix}(\Sigma) = 2$.

Then there is a unique branch of periodic solutions with period close to 2π branching from the origin, with spatio-temporal symmetry Σ.

The proof of the Equivariant Hopf Theorem proceeds by working within the two dimensional subspace $z \in \mathbb{C}$; the centre eigenspace within the invariant fixed point subspace. Using either Lyapunov-Schmidt or Poincaré-Birkhoff normal form we can reduce to a map with an extra S^1 symmetry:

$$f(z) = p(|z|^2, \lambda)z + q(|z|^2, \lambda)iz$$

and then showing that this will generically have a branch of periodic solutions when p passes through zero.

More generally, Lyapunov-Schmidt reduction can be used to look for an orbit close to 2π-periodic that is fixed by the dynamics; i.e. a 2π-periodic loop $u(s)$ such that

$$0 = (1 + \tau)\dot{u} - f(u, \lambda)$$

where we view the right hand side as a map from $C^1_{2\pi}$ to $C^0_{2\pi}$ i.e. between Banach spaces of 2π-periodic functions in $\mathbb{R}^n$. This can be used to reduce to a problem on a centre eigenspace.

Poincare-Birkhoff normal form

Now returning to the normal form, suppose that we have for $y \in \mathbb{R}^n$

$$\dot{y} = Ly + f_2(y) + f_3(y) + \cdots + f_k(y) + \cdots$$

where each of the terms f_k is homogeneous of order k. Let $\mathcal{P}_k$ denote the set of all kth-order homogeneous polynomials from $\mathbb{R}^n$ to $\mathbb{R}^n$ that are Γ-equivariant. Define

$$\mathrm{ad}_L(P_k)(y) = (LP_k)(y) - (dP_k)_y Ly$$

and note that this is a linear operator on $\mathcal{P}_k$. There is an inner product on $\mathcal{P}_k$ such that one can orthogonally decompose

$$\mathcal{P}_k = \mathcal{G}_k \oplus \mathrm{im}\,\mathrm{ad}_L\,.$$

The Poincaré-Birkhoff Normal Form Theorem [G&S Thm 4.14] states then:

Theorem

For each k there is a polynomial near-identity change of coordinates $y \mapsto x$ such that

$$\dot{x} = Lx + g_2(x) + g_3(x) + \cdots + g_k(x) + r_{k+1}(x)$$

with $g_j \in \mathcal{G}_j$ and r_{k+1} is of degree at least $k+1$.

Example: Hopf bifurcation with $O(2)$ symmetry

As an example we consider a Γ-simple action of $O(2)$ on $\mathbb{C}^2$ given by

$$(z_1, z_2) \mapsto (z_2, z_1), \quad (z_1, z_2) \mapsto (e^{-i\phi} z_1, e^{i\phi} z_2)$$

and then S^1 acts by

$$(z_1, z_2) \mapsto (e^{i\theta} z_1, e^{i\theta} z_2).$$

One can compute the general Poincaré-Birkhoff normal form that commutes with this action as:

$$\begin{aligned}
\dot{z}_1 &= (p + iq)z_1 + (r + is)(|z_2|^2 - |z_1|^2)z_1 \\
\dot{z}_2 &= (p + iq)z_2 - (r + is)(|z_2|^2 - |z_1|^2)z_2
\end{aligned}$$

where p, q, r, s are smooth functions of

$$|z_2|^2 + |z_1|^2, \quad (|z_2|^2 - |z_1|^2)^2.$$

This normal form can be reduced to phase-amplitude equations: let $x = |z_1|$ and $y = |z_2|$ and observe that above implies that

$$\begin{aligned}\dot{x} &= (p + r(y^2 - x^2))x \\ \dot{y} &= (p - r(y^2 - x^2))y\end{aligned}$$

where p, r are functions of $N = x^2 + y^2$ and $\Delta = (y^2 - x^2)^2$.

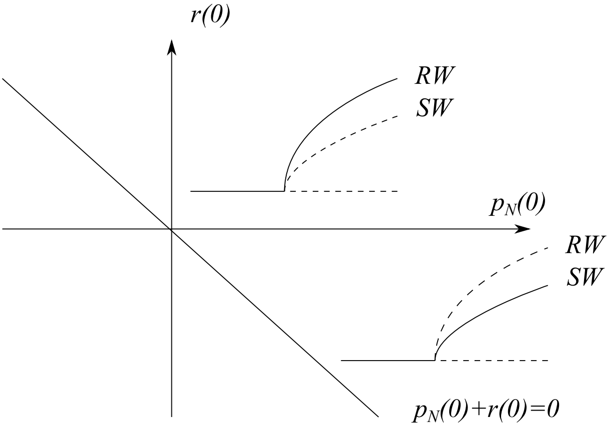
If there is a bifurcation and we wish to find periodic solutions, this means we need to find equilibria of the above. One can show:

- There will be branches of RW solutions $(x, 0)$ and $(0, x)$ if $p_\lambda(0) \neq 0$ and $p_N(0) + r(0) \neq 0$.
- There will be a branch of SW solutions (x, x) if $p_\lambda(0) \neq 0$ and $p_N(0) \neq 0$.
- No other branches will bifurcate generically.

In fact solutions $(x, 0)$ and $(0, x)$ can be interpreted as rotating waves (RW), and (x, x) as standing waves (SW). one can obtain [G&S Thm 4.19]:

Theorem

At generic $O(2)$ Hopf bifurcation the only branches are rotating and standing waves. Moreover, one can only obtain stable periodic solutions as branches if both branches are supercritical, and in this case only one of the branches can be stable.


r(0)
RW
SW
pN(0)
RW
SW
pN(0)+r(0)=0

So far we have considered the constraints that symmetries pose on bifurcation patterns through analysis of normal forms at generic one-parameter bifurcations.

We now briefly consider:

- Higher codimension bifurcations
- Invariant subspaces and robust heteroclinics
- Other bifurcations with symmetry

Codimension two bifurcations

Suppose we have

$$\dot{x} = f(x, \lambda, \mu)$$

where $x \in \mathbb{R}^n$ and $\lambda, \mu \in \mathbb{R}$ and f is equivariant under the action of some compact Lie group Γ and suppose that $f(0, \lambda, \mu) = 0$.

If we consider λ as a "preferred" or "distinguished" parameter, for generic choices of μ we will have a generic bifurcation with one parameter, but for exceptional cases we may have a further degeneracy. For the sake of argument suppose that there is a degenerate steady bifurcation where $\lambda = \mu = 0$, i.e.

$$\dim \ker(df)_{(0,0,0)} \geq 1$$

with some degeneracy.

Then one of the following occurs:

- There is an additional second linear instability; this means that the action of Γ on $W = \ker(df)_{(0,0,0)}$ is no longer absolutely irreducible. In such cases there are several generic possibilities:
  - $W = V_1 \oplus V_2$ for two absolutely irreducible representations in the same isotypic component of $\mathbb{R}^n$.
  - $W = V_1 \oplus V_2$ for two absolutely irreducible representations in different isotypic components of $\mathbb{R}^n$ (this is often called a *mode interaction*).
  - $W = V$ is a non-absolutely irreducible subspaces in $\mathbb{R}^n$ (in which case (λ, μ) generically has no steady bifurcations for nearby $\mu \neq 0$!
- There is a nonlinear degeneracy of the normal form but $W = V$ is an absolutely irreducible representations of Γ.

This division into linear and nonlinear degeneracies informs the analysis of such higher codimension bifurcations; for a *linear degeneracy* one typically arrives at a normal form of higher dimension, but with generic choices of nonlinear terms.

On the other hand nonlinear degeneracies can be dealt with by including higher order terms to ensure finitely determinacy.

Further generic codimension two points involve, for example, Hopf and Steady bifurcation lines coinciding.

Example: linear degeneracy Consider

$$\begin{aligned}\dot{x} &= (\lambda + ax^2 + by^2)x \\ \dot{y} &= (\mu + cx^2 + dy^2)y\end{aligned}$$

which is the third order normal form for a $\mathbb{Z}_2 \times \mathbb{Z}_2$ equivariant bifurcation problem on $(x, y) \in \mathbb{R}^2$.

Note that there are bifurcations at $\lambda = 0$ and $\mu = 0$. The dimension of the kernel of the Jacobian is one at all bifurcations except at the codimension two point $\lambda = \mu = 0$.

We can represent the bifurcations near $\lambda = \mu = 0$

Equilibria are at

$$\begin{aligned} 0 &= (\lambda + ax^2 + by^2)x \\ 0 &= (\mu + cx^2 + dy^2)y \end{aligned}$$

meaning that $x = 0$ or $ax^2 + by^2 = -\lambda$,
and $y = 0$ or $cx^2 + dy^2 = -\mu$.

In consequence there are branches:

- Trivial solution $(x, y) = (0, 0)$
- Primary solutions
$$(x, 0) \quad \text{where } x^2 = -\frac{\lambda}{a}$$
for $-\lambda/a \geq 0$ and
$$(0, y) \quad \text{where } y^2 = -\frac{\mu}{d}.$$
for $-\mu/d \geq 0$
- *Mixed-mode solutions*
$$(x, y)$$
where $ax^2 + by^2 = -\lambda$, $cx^2 + dy^2 = -\mu$, such that x^2 and y^2 both positive.

Example:

$$
\begin{aligned}
0 &= (\lambda + x^2 + \frac{1}{2}y^2)x \\
0 &= (\mu - \frac{1}{2}x^2 + y^2)y
\end{aligned}
$$

has pure mode solutions $(x, 0)$ in the region

$$\lambda < 0,$$

pure mode solutions $(0, y)$ in the region

$$\mu < 0$$

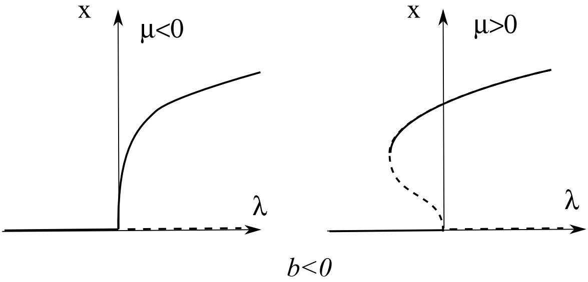
and mixed mode solutions (x, y) in region

$$\lambda - \frac{1}{2}\mu < 0 \quad \text{and} \quad \frac{1}{2}\lambda + \mu < 0.$$

Example: nonlinear degeneracy Consider the bifurcation problem on $x \in \mathbb{R}$ with $\mathbb{Z}_2$ symmetry:

$$\dot{x} = \lambda x + \mu x^3 + bx^5 + cx^7 + \cdots$$

Note that if $\lambda = 0$ there is a steady bifurcation that is a supercritical pitchfork for $\mu < 0$ and a subcritical pitchfork for $\mu > 0$; there is a *change of criticality* at $\mu = 0$.



For $\mu = 0$ there is a codimension two point where branching is determined by the fifth order terms.

Other equivariant bifurcations

Chaotic atractors also have bifurcations that can be classified according to symmetry breaking.

Suppose that $A \subset \mathbb{R}^m$ is a chaotic attractor for some Γ-equivariant ODE $\dot{x} = f(x)$.

As with periodic orbits one can distinguish the *instantaneous symmetry* of A

$$K = \{g \in \Gamma \ : \ gx = x \ \text{ for all } x \in A\}$$

and the *average symmetry* of A

$$H = \{g \in \Gamma \ : \ gA = A\}.$$

In fact some average symmetries cannot be achieved by periodic orbits but can be achieved by chaotic attractors!

Some bifurcations get much simpler in a symmetric context.

For instance bifurcation of periodic and quasiperiodic attractors if they arise as of relative equilibria.

Various generalisations, e.g. to non-compact groups are considered in [G&S].

Invariant subspaces and bifurcation to heteroclinic cycles

The presence of invariant subspaces $\mathrm{fix}(H)$ for $H \subset \Gamma$ subgroups forces a strong structure not only onto local dynamics and bifurcations, but also on global dynamics. This is partly because they are not constrained to be normally hyperbolic.

Suppose $N \subset \mathbb{R}^n$ is a closed flow-invariant manifold. We say N is *normally hyperbolic* if the expansion/contraction for the flow withing the manifold is dominated by the expansion/contraction normal to the manifold (see texts for precise statements).

Standard results imply that normally hyperbolic invariant manifolds persist under perturbations to the flow, i.e. they are robust. Conversely, invariant manifolds that are not normally hyperbolic do not persist under perturbations of the flow.

By contrast all fix(H) are trivially robust to all perturbations that preserve the symmetry. This can have consequences that invariant manifolds may behave in an unusual manner for the dynamics.

A stranger consequence is that one can bifurcate straight from a trivial solution, via a generic steady bifurcations with certain symmetries to robust heteroclinic attractors or even to chaotic attractors ("instant chaos").

Only fairly weak assumptions are necessary on the group action to get such attractors appearing as robust attractors even if they do not bifurcate directly from the origin.

Example: bifurcation to a heteroclinic attractor

Consider the third order normal form for generic bifurcation with $(\mathbb{Z}_2)^3 \times_2 \mathbb{Z}_3$ for $(x, y, z) \in \mathbb{R}^3$:

$$\begin{aligned}
\dot{x} &= (\lambda + a(x^2 + y^2 + z^2) + by^2 + cz^2)x \\
\dot{y} &= (\lambda + a(x^2 + y^2 + z^2) + bz^2 + cx^2)y \\
\dot{z} &= (\lambda + a(x^2 + y^2 + z^2) + bx^2 + cy^2)z
\end{aligned}$$

For an open set of a, b, c there is a bifurcation from stable $(0, 0, 0)$ for $\lambda < 0$ to an attracting heteroclinic cycle for $\lambda > 0$. No other solutions are stable for $\lambda > 0$.

We consider the case where $\lambda > 0$ $a < 0$ and b, c close to zero. Note that by a rescaling of time and state we can set $\lambda = 1$ and $a = -1$:

$$\begin{aligned} \dot{x} &= (1-(x^2+y^2+z^2)+by^2+cz^2)x \\ \dot{y} &= (1-(x^2+y^2+z^2)+bz^2+cx^2)y \\ \dot{z} &= (1-(x^2+y^2+z^2)+bx^2+cy^2)z \end{aligned}$$

for some scaled b, c. For any $|b|, |c| \ll 1$ this has equilibria at $(x, y, z) = (0, 0, 0)$ and

$$(\pm 1, 0, 0),\ (0, \pm 1, 0),\ (0, 0, \pm 1).$$

We compute the Jacobian at (x, y, z) to be

$$\begin{pmatrix} \begin{matrix} 1-3x^2+(b-1)y^2 \\ +(c-1)z^2 \end{matrix} & 2(b-1)yx & 2(c-1)zx \\ 2(c-1)xy & \begin{matrix} 1-3y^2+(b-1)z^2 \\ +(c-1)x^2 \end{matrix} & 2(b-1)yz \\ 2(b-1)xz & 2(c-1)yz & \begin{matrix} 1-3z^2+(b-1)x^2 \\ +(c-1)y^2 \end{matrix} \end{pmatrix}$$

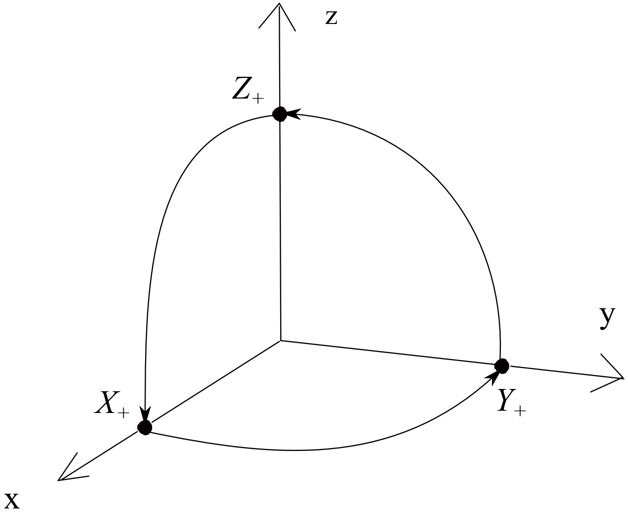
Hence for $(0,0,0)$ we have

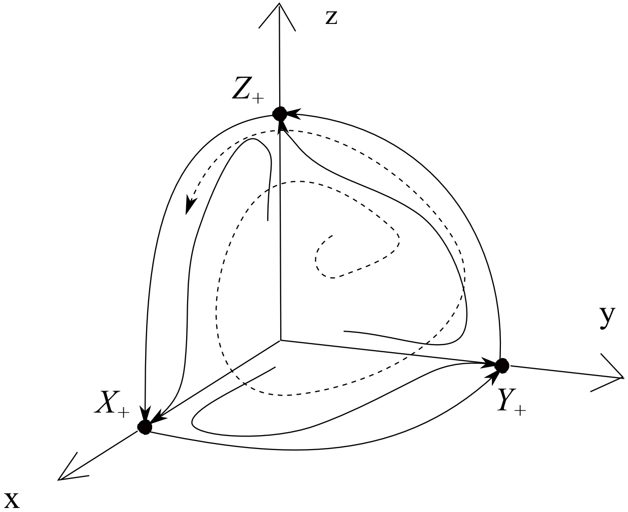
$$J = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

(unstable: source) while for $X_{\pm} = (\pm 1, 0, 0)$ we have

$$\begin{pmatrix} -2 & 0 & 0 \\ 0 & c & 0 \\ 0 & 0 & b \end{pmatrix}$$

etc. We choose $b < 0$ and $c > 0$ then $X_{\pm}$ etc are all saddle points with one dimensional unstable manifolds. However the unstable manifold of X_{+} is contained within the invariant subspace $(x, 0, y)$ and can show within this subspace that the unstable manifold limits to a sink at $Z_{\pm} = (0, 0, \pm 1)$.


z
Z_+
y
X_+
Y_+
x


z
Z_+
y
X_+
Y_+
x

As a consequence there is a network of heteroclinic connections for an open set of a, b, c and this may be an attractor (need $b < |c|$).

- The attractor is robust to all perturbations from higher order terms; the bifurcation is 3-determined.
- Such heteroclinic cycles cannot occur robustly for generic dissipative systems.
- The network can be robustly asymptotically stable.
- Trajectories approaching the network do not behave ergodically.
- Similar phenomena occur in a wide range of examples.

Coupled oscillators, synchrony and symmetry

We discuss some examples of application of dynamical systems with symmetry to coupled oscillators, in particular those arising in neural systems modelling. Such models attempt to explain how a coupled network of simply oscillatory systems can give complex emergent behaviour in the network.

Oscillators and isochrons

Q: What is an oscillator?
A: A dynamical system that produces periodic behaviour.

For example, in $\mathbb{R}^d$:

$$\dot{x}_1 = f_1(x_1, \cdots, x_d), \quad \dot{x}_d = f_d(x_1, \cdots, x_d)$$

with a periodic orbit

$$P(t) = (p_1(t), \cdots, p_d(t))$$

with period $T > 0$, i.e.

$$P(t + T) = P(t)$$

such that T is smallest possible choice of periodicity of all components.
We consider *stable limit cycle oscillators* of ODEs for any initial condition x that starts close enough to $P(t)$ in all components we have

$$|x(t) - P(t + \phi)| \to 0$$

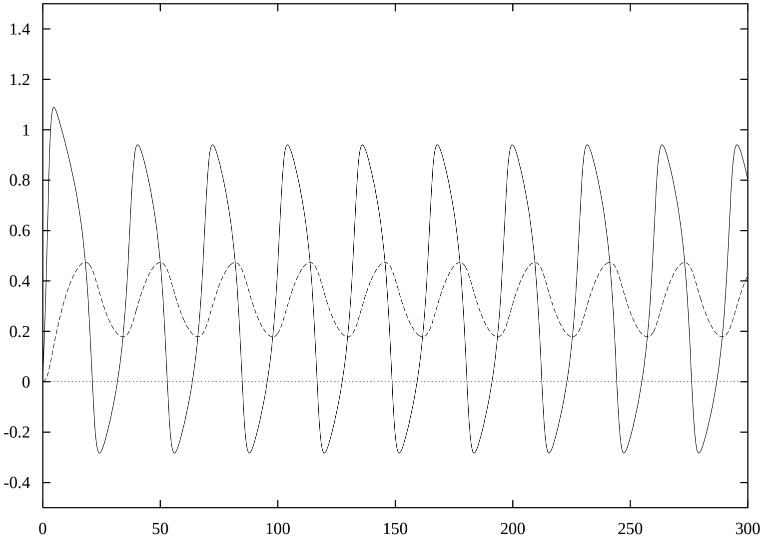
as $t \to \infty$ for some ϕ.

Consider the Fitzhugh-Nagumo system

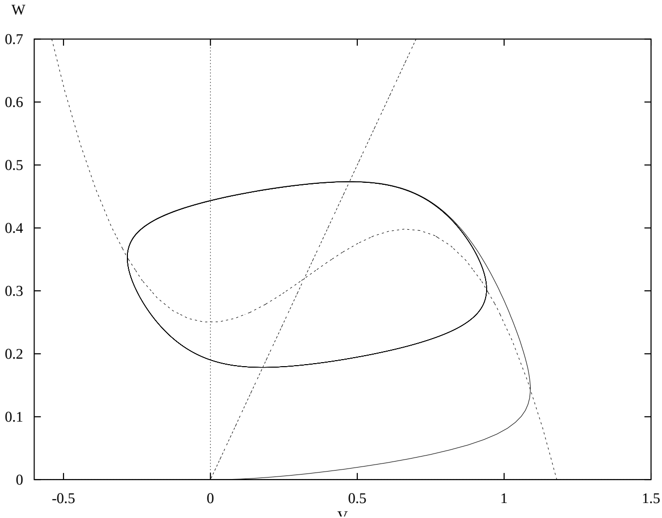
$$\begin{aligned} \dot{V} &= F(V) - W + I \\ \dot{W} &= \epsilon(V - \gamma W) \end{aligned}$$

with $F(V) = V(1 - V)(V - A)$ and parameters

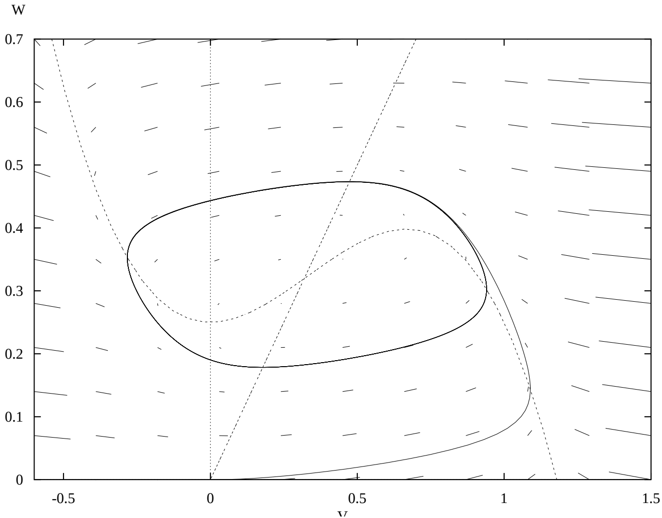
$$A = .25,\ \epsilon = .05,\ \gamma = 1,\ I = .25$$



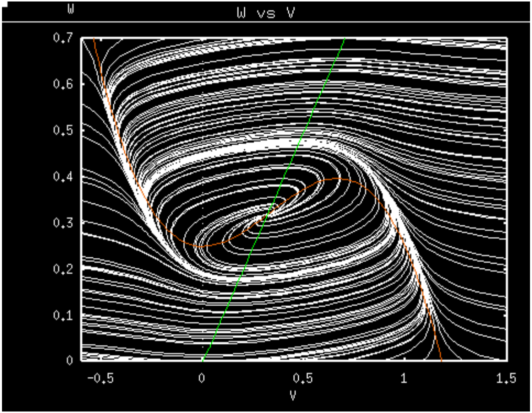
Time-series of typical solution.



Phase plane of typical solution.



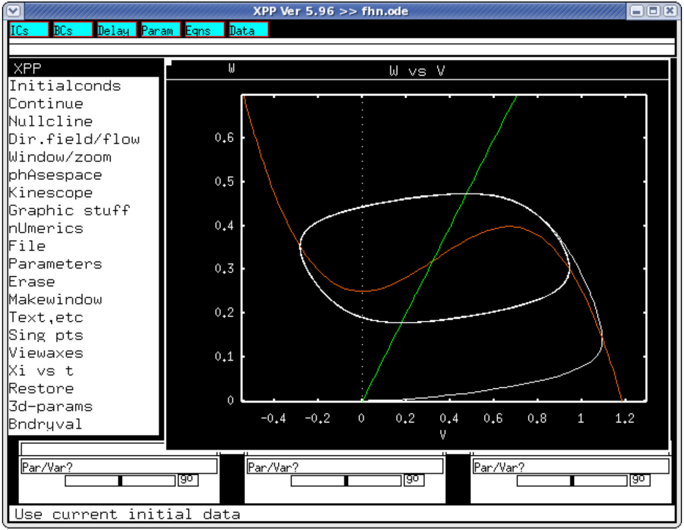
Phase plane of typical solution with directions of flow.



Phase plane of typical solution with flow added.

Code for xppaut [Ermentrout]:
http://www.math.pitt.edu/~bard/xpp/xpp.html

```
dv/dt = f(v)-w+s(t)+I_0
dw/dt = eps*(v-gamma*w)
f(v)=v*(1-v)*(v-a)
s(t)=al*sin(omega*t)
param a=.25,eps=.05,gamma=1,I_0=.25
param al=0,omega=2
@ total=100,dt=.2,xhi=100
done
```


XPP Ver 5.96 >> fhn.ode
ICs
BCs
Delay
Param
Eqns
Data
XPP
Initialconds
Continue
Nullcline
Dir.field/flow
Window/zoom
phAsespace
Kinescope
Graphic stuff
nUmerics
File
Parameters
Erase
Makewindow
Text,etc
Sing pts
Viewaxes
Xi vs t
Restore
3d-params
Bndryval
W
W vs V
0.6
0.5
0.4
0.3
0.2
0.1
0
-0.4
-0.2
0
0.2
0.4
0.6
0.8
1
1.2
V
Par/Var?
go
Par/Var?
go
Par/Var?
go
Use current initial data

Different types of periodic oscillations:

- Weakly nonlinear oscillations, e.g. Near-onset small amplitude oscillations, Hopf bifurcation.
- Relaxation oscillations, e.g. Fitzhugh Nagumo, Hodgkin-Huxley models
- Hybrid/switched system oscillations, e.g. leaky integrate-and-fire models

In all cases can be modelled as a *phase oscillator*

$$\dot{\theta} = \omega$$

for θ modulo 2π when transients have decayed, where frequency

$$\omega = \frac{2\pi}{T}$$

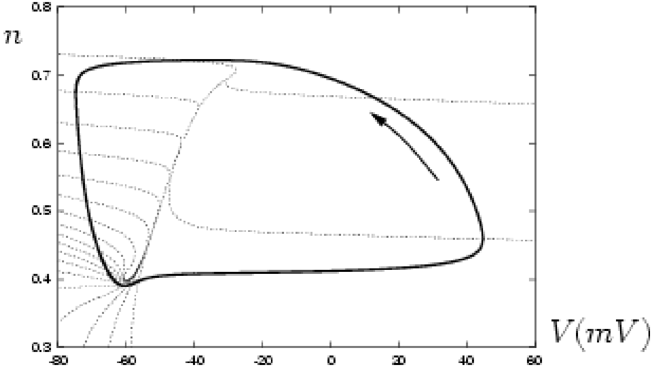
related to the period T. Off the limit cycle, however, other dynamics are at work.

Suppose $X \in \mathbb{R}^d$ with
$$\dot{X} = F(X)$$
has a stable limit cycle $P(t)$, period T. We define the set of points with eventual phase ϕ to be
$$I_\phi = \{Y \in \mathbb{R}^d \ : \ |Y(t) - P(t+\phi)| \to \infty\}$$
The sets I_ϕ are called the *isochrons* of the limit cycle. For a stable limit cycle:

- They are manifolds of dimension $d-1$.
- They foliate a neighbourhood of the cycle.
- They can be used to understand the behaviour of forced or coupled oscillators.

K. Josic, E. Brown, J. Moehlis:
http://www.scholarpedia.org/article/Isochron

E Izhikevich:
http://www.izhikevich.com



Isocrons for a Hodgkin-Huxley neuron
(http://www.scholarpedia.org/article/Isochron)

Phase response curves

The *phase response curve* is a way of measuring the response to sudden change in one variable; if we consider a vector perturbation $Z \in \mathbb{R}^d$ then

$$PRC(\theta) = \{\phi : I_\phi \text{ contains } P(\theta) + Z\}.$$

Equivalently, starting at $P(\theta)$ we impulsively change to

$$X(0) = P(\theta) + Z$$

and allow the system to evolve forwards in time. We choose PRC so that

$$|X(t) - P(t + \theta + PRC(\theta))| \to 0$$

as $t \to \infty$.

The phase response curve models the change in phase exactly, even for large perturbations if a long enough settling time between perturbations is allowed.

Can apply to continuous perturbations using various equivalent approaches to obtain the *infinitesimal phase response curve*.

Suppose that

$$\dot{X} = F(X) + \epsilon G(t)$$

where $G(t)$ represents forcing and F has an attracting limit cycle $P(t)$.

Assume that unperturbed oscillator $P(t)$ has period 2π.

Kuramoto's approach: We define phase $\Theta(X)$ of all points $X \in \mathbb{R}^d$ by inverting the isochron map I_ϕ:

$$\phi = \Theta(X) \;\Leftrightarrow\; X \in I_\phi.$$

Note that for the system with $\epsilon = 0$ we have

$$\frac{d}{dt}\left[\Theta(X(t))\right] = \nabla\Theta \cdot \frac{dX}{dt} = \nabla\Theta . F(X)$$

But $\dot{\phi} = 1$ so

$$\nabla\Theta \cdot F(X) = 1.$$

Hence for the case $\epsilon \neq 0$ we have

$$\dot{\phi} = 1 + \epsilon\nabla\Theta \cdot G(t).$$

Adjoint approach (Malkin): Note that if the unperturbed oscillators is linearly stable, then the perturbed equation is to first order in ϵ given by

$$\dot{\theta} = 1 + \epsilon Q(\theta) \cdot G(t)$$

where $Q(t)$ is the solution to the *adjoint variational equation*

$$\dot{Q} = -\{DF(P(t))\}^T Q, \quad \text{such that } Q(0) \cdot F(P(0)) = 1.$$

Note that

$$\begin{aligned} \frac{d}{dt}(Q \cdot F) &= \dot{Q} \cdot F + Q \cdot \dot{F} \\ &= -(DF)^T Q \cdot F + Q \cdot (DF)F = 0. \end{aligned}$$

Hence the solutions of the AVE satisfy

$$Q(t) \cdot F(P(t)) = 1$$

for all t.

Coupled phase dynamics

Consider two symmetrically coupled oscillators

$$\begin{aligned} \dot{X}_1 &= F(X_1) + \epsilon G_1(X_2, X_1) \\ \dot{X}_2 &= F(X_2) + \epsilon G_2(X_1, X_2) \end{aligned}$$

for ϵ small (weak coupling) and $\mathbb{Z}_2$ symmetry.
In terms of phases we have approximately

$$\begin{aligned} \dot{\theta}_1 &= 1 + \epsilon Q(\theta_1) \cdot G_1(P(\theta_2), P(\theta_1)) \\ \dot{\theta}_2 &= 1 + \epsilon Q(\theta_2) \cdot G_2(P(\theta_1), P(\theta_2)). \end{aligned}$$

Because the phase difference evolves on a slower timescale that the phases, we get an interaction function that expresses the effect of X_2 on X_1 that can be written

$$H_1(\theta) = \frac{1}{T}\int_0^T Q(t) G_1(X_0(t+\theta), X_0(t))\, dt$$

where Q is the solution of the adjoint variational equation.
Method of *averaging* allows us to write previous equation (to $O(\epsilon^2)$) as

$$\begin{aligned} \dot{\theta}_1 &= 1 + \epsilon H_1(\theta_2 - \theta_1) \\ \dot{\theta}_2 &= 1 + \epsilon H_2(\theta_1 - \theta_2). \end{aligned}$$

For two identically coupled oscillators we set $\phi = \theta_2 - \theta_1$ and obtain

$$\dot{\phi} = -\epsilon(H_1(\phi) - H_2(-\phi)) = \epsilon g(\phi)$$

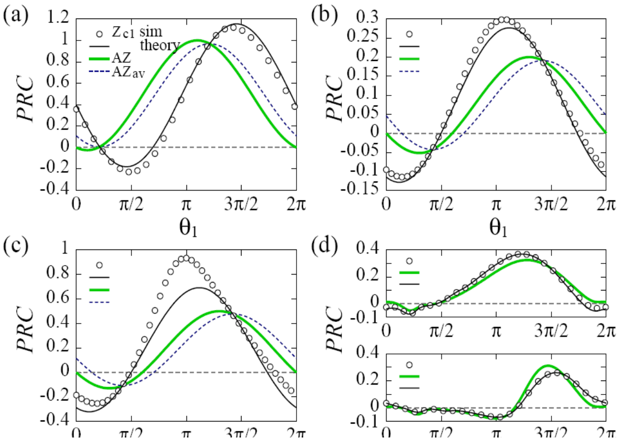
Similarly, starting at a set of N weakly coupled identical phase oscillators $\theta_1, \cdots, \theta_N$, we can reduce to a set of $N-1$ phase differences

$$\phi_i = \theta_i - \theta_N$$

and can obtain

$$\dot{\phi}_i = \epsilon g_i(\phi_1, \cdots, \phi_{N-1})$$

with evolution on a slow timescale.



Example of $g(\theta)$ for various coupling functions. (d) are for gap junction-coupled Morris-Lecar Neurons (from T.-W. Ko and G.B. Ermentrout, PRE 78:016203, 2008)

Synchrony and coupling

Now consider N coupled oscillators reduced to phases:

$$\dot{\theta}_i = \omega + G(\theta_1 - \theta_i, \cdots, \theta_N - \theta_i).$$

Simple model with "additive coupling" is

$$\dot{\theta}_i = \omega + \sum_{j \neq i} K_{ij} g(\theta_i - \theta_j)$$

with K_{ij} coupling strengths; $K_{ij} = K$ for global (mean field) coupling.

For global coupling, note that the system

$$\dot{\theta}_i = \omega + \sum_{j \neq i} g(\theta_i - \theta_j)$$

has full permutation symmetry $S_N \times \mathbb{T}$.

This symmetry causes a number of interesting effects such as

- Synchrony breaking bifurcations have $N-1$-dimensional centre manifolds.
- Typically N branches appear at bifurcation.
- Bifurcation directly to large amplitude periodic orbits.
- For $N \geq 4$ bifurcation directly to robust heteroclinic cycles.

See *[A, Coombes and Nicks, J Math. Neurosci 2016]* for a review.

Some consequences of $S_N \times \mathbb{T}$ symmetry:

Theorem (A & Swift 1992)

Every isotropy subgroup of a general $S_N \times \mathbb{T}$-equivariant vector field on $\mathbb{T}^N$ is of the form a rotating block *where*

$$(S_{k_1} \times \cdots \times S_{k_\ell})^m \times_s \mathbb{Z}_m$$

where $N = m(k_1 + \cdots + k_\ell)$, and $\times_s$ denotes the semi-direct product.

Example Find all isotropy subgroups of $S_6 \times \mathbb{T}$ acting on $\mathbb{T}^6$ and organize them into a lattice by containment.

Example Sketch the isotropy subspaces for phase differences of the action of $S_3 \times \mathbb{T}$ on $\mathbb{T}^3$.

Simple choices for phase response curve $g(\phi)$:

- Kuramoto
$$g(\phi) = -\sin(\phi)$$
- Kuramoto-Sakaguchi
$$g(\phi) = -\sin(\phi - \alpha)$$
- Hansel-Mato-Meunier
$$g(\phi) = -\sin(\phi - \alpha) + r\sin(2\phi)$$
- General two harmonic
$$g(\phi) = -\sin(\phi - \alpha) + r\sin(2\phi - \beta)$$

https://www.frontiersin.org/articles/10.3389/fams.2016.00007/ful

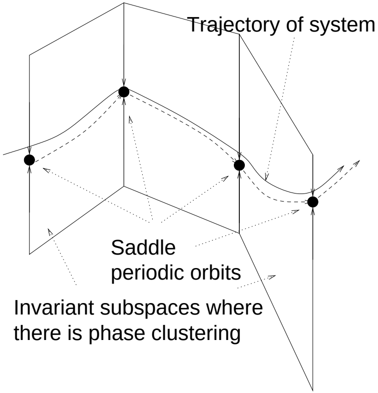
More general: Daido et al:

$$g(\phi) = \sum_n (a_n \cos n\phi + b_n \sin n\phi)$$

Cluster switching/ slow oscillations

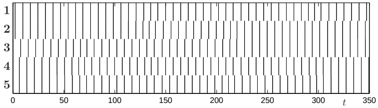
Can find open regions in parameter space (for all $N \geq 4$) where the only attractors consist of robust heteroclinic networks made up of:

- Periodic orbits with nontrivial clustering.
- Unstable manifolds of these periodic orbits.
- *Winnerless competition* between cluster states (Afraimovich, Huerta, Laurent, Nowotny, Rabinovich et al)
- *Slow oscillations/switching dynamics* (Hansel et al, Kori and Kuramoto)

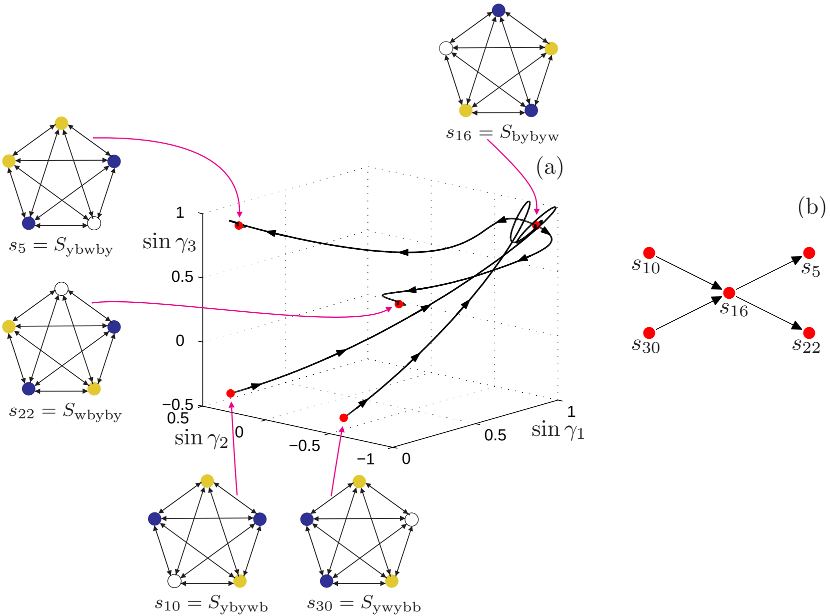

Trajectory of system
Saddle
periodic orbits
Invariant subspaces where
there is phase clustering

Example of transient clustering dynamics:
$N = 5$, $\alpha = 1.8$, $r = 0.2$, $\beta = -2$

$$g(\phi) = -\sin(\phi + \alpha) + r\sin(2\phi + \beta)$$



[G. Orosz, J. Wordsworth, S. Townley & A. Reliable switching between cluster states for globally coupled phase oscillators, SIAM J Applied Dynamical Systems **6**:728-758 (2007)]



$s_{16} = S_{\mathrm{bybyw}}$
(a)
$s_5 = S_{\mathrm{ybwby}}$
$s_{22} = S_{\mathrm{wbyby}}$
$s_{10} = S_{\mathrm{ybywb}}$
$s_{30} = S_{\mathrm{ywybb}}$
$\sin \gamma_3$
$\sin \gamma_2$
$\sin \gamma_1$
1
0.5
0
−0.5
−1
(b)
s_{10}
s_5
s_{16}
s_{30}
s_{22}

Stable clustering

$$\dot{\theta}_i = \omega + \frac{1}{N}\sum_{j=1}^{N} g(\theta_i - \theta_j)\,, \tag{3}$$

Consider M clusters where $1 \leq M \leq N$. Corresponding M-cluster partition $\mathcal{A} = \{A_1, \ldots, A_M\}$ of $\{1, \ldots, N\}$ such that

$$\{1, \ldots, N\} = \bigcup_{p=1}^{M} A_p\,, \tag{4}$$

where A_p are pairwise disjoint sets ($A_p \cap A_q = \emptyset$ if $p \neq q$). NB if $a_p = |A_p|$ then

$$\sum_{p=1}^{M} a_p = N\,. \tag{5}$$

For partition $\mathcal{A}$ associate a subspace

$$\mathbb{T}^N_{\mathcal{A}} = \{\theta \in \mathbb{T}^N \colon \theta_i = \theta_j \;\Leftrightarrow\; \text{there is a } p \text{ such that } i, j \subset A_p\}\,, \tag{6}$$

and we say a given $\theta \in \mathbb{T}^N_{\mathcal{A}}$ *realizes the partition* $\mathcal{A}$.

Denote phase of the p-th cluster by $\psi_p := \theta_i = \theta_j = \theta_k = \ldots$ for $\{i, j, k, \ldots\} \subset A_p$ we obtain

$$\dot{\psi}_p = \omega + \frac{1}{N} \sum_{q=1}^{M} a_q\, g(\psi_p - \psi_q) \tag{7}$$

for $p = 1, \ldots, M$.
We say $\theta \in \mathbb{T}^N_{\mathcal{A}}$ *realizes the partition* $\mathcal{A}$ *as a periodic orbit* if

$$\psi_p = \Omega\, t + \phi_p \tag{8}$$

for $p = 1, \ldots, M$ and all $\phi_p \pmod{2\pi}$ are different.

Substituting (8) into (7) gives

$$\Omega = \omega + \frac{1}{N}\sum_{q=1}^{M} a_q\, g(\phi_p - \phi_q) \tag{9}$$

for $p = 1, \ldots, M$. By subtracting the last equation ($p = M$) from each of the preceding equations ($p = 1, \ldots, M-1$) we obtain

$$0 = \sum_{q=1}^{M} a_q \left(g(\phi_p - \phi_q) - g(\phi_M - \phi_q)\right) \tag{10}$$

for $p = 1, \ldots, M-1$. Can determine $M-1$ phases out of ϕ_p, $p = 1, \ldots, M$ while one phase can be chosen arbitrarily, and (9) determines the frequency Ω.

Can compute linear stability in a similar way to above and show [Orosz, A. 2009]:

Theorem

There is a coupling function g for the system (3) such that for any N and any given M-cluster partition $\mathcal{A}$ of $\{1,\ldots,N\}$ there is a linearly stable periodic orbit realizing that partition (and all permutations of it). Moreover, all nearby g in the C^2 norm have a stable periodic orbit with the same partition.

Final Comments

In summary, there are practical and numerical ways of reducing and understanding the dynamics of coupled limit cycle oscillators of general type to coupled phase oscillators. This can be useful because:

- Reduces dimension of phase space
- Gives framework for understanding effects of coupling (e.g. pattern formation) on oscillators
- For identical oscillators, can reduce limit cycle problems to equilibrium problems
- Phase dynamics can be highly nontrivial even for quite simple coupling
- Can extend to cases of frequency synchrony breaking and “chimera states”.

Tools include:

- Numerical simulation/solution continuation.
- Isochrons/phase response curves/phase transition curves
- Averaging method
- Use of adjoint variational equation
- Analysis of coupled ODEs on a torus
- Studying the synchronization properties
- Symmetric dynamics and bifurcation theory
- Coupling structures in coupled systems

Review: P. Ashwin, S. Coombes, R. Nicks. Mathematical Frameworks for Oscillatory Network Dynamics in Neuroscience, *J. Math. Neurosci.* 2016:

https://mathematical-neuroscience.springeropen.com/articles/10.1186/s13408-015-0033-6

The method of reduction to phase oscillators works well for sufficiently weak coupling, but needs to be treated with respect for:

- Strong coupling
- Weakly attracting/neutrally stable limit cycles
- Chaotic "oscillators"
- Non-smooth systems
- Be careful when averaging in multi-frequency systems

Thank you for listening to the end and good luck for the future.

Please let me know if you find any interesting dynamics, oscillations or bifurcations!

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