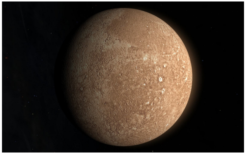
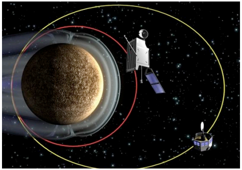
Spin-orbit resonances

- Very common situation
- Resonance between the orbital motion of a body and its rotation spin : 87-88 days - 58 days
- Examples : Moon, Galilean satellites, Titan, ... and Mercury (3/2)
- Full system : orbital and rotational motion
- Known orbit (function of time) in the rotational dynamics
- Eccentricity role is essential for 3/2, not for 1/1 resonances
- Mercury blocked in a $3/2$ Spin-Orbit resonance : 58 days / 88 days
- Mercury was the *Forgotten Planet* (Mariner 10)

The context

- Space missions : Messenger & BepiColombo
- Complete model of rotation for MORE
- Academic and practical study
- Rigid body - Fluid core - Multi-layers core
- Long or Short periodic terms ?
- Resonances : classical and unexpected
- Suitable reference frames
- Namur : D'Hoedt, Dufey, Lhotka, Noyelles, Sansottera (from 04 to 16)
- King : Peale (+ Yseboodt + Margot) (65, 72, 74, 76, 97, 01, 05, 06, 07, 08, 09)





Capture of Mercury into the 3/2 Spin-Orbit resonance

- The capture is assumed
- Only known case of capture in a $3/2$: why not a $1/1$?
- Connected to the long time evolution of the Solar System orbital and rotational motions
- A. Correia and J. Laskar : 2004, Nature **429**, 884

  Mercury's capture into the 3/2 spin-orbit resonance as a result of its chaotic dynamics

  Probability of capture in the $3/2$: 52%
- A. Correia and J. Laskar : 2009, Icarus **201**, 1

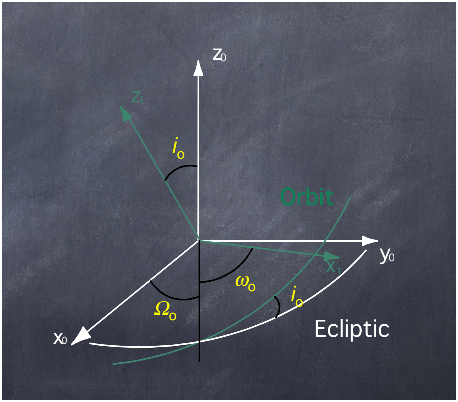
  Mercury's capture into the 3/2 spin-orbit resonance including the effect of coreĐmantle friction

  cascade of captures - final capture in spin-orbit for 98%
  $5/2$: 22%, $2/1$: 32% and $3/2$: 26%
  increased to 55% if $e < 0.025$, to 73% if $e < 0.005$ in the past.

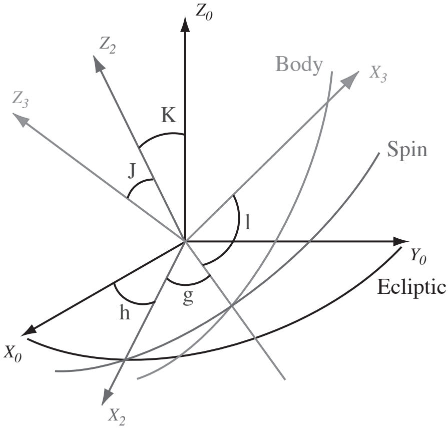
The first hypotheses

- Mercury is considered as a rigid body
- Two coefficients of the gravitational potential are known C_0^2 and C_2^2 with uncertainty of 50%
- Mercury's orbit is keplerian
- Hamiltonian formalism to describe the rotational dynamics
- Three dimensional problem : 3 Euler's angles with their proper frequencies
- Four reference frames - origin = center of mass of Mercury
  - Inertial one (ecliptic at some epoch J2000)
  - Orbital frame (orbit of the Sun around Mercury)
  - Spin frame (rotational angular momentum)
  - Figure or body frame (principal axes of inertia)

The orbital frame



The reference frames



- (h, K, g) between the ecliptic frame and the spin frame
- $(l, J, -)$ between the spin frame and the body frame
- $\vec{G}$ angular momentum in the direction of Z_2
- Conventions
  - Inertial : 0
  - Orbital : 1
  - Spin : 2
  - Body : 3
- K ecliptic obliquity

The three dimensional Hamiltonian

$$\mathcal{H} = T_{\text{rotational}} + V_{\text{gravitational}}$$

Andoyer - Deprit set of canonical variables and momenta

Variables q_i	Momenta p_i
l	$L = G\cos J \quad (J \simeq 0)$
g	$G =$ norm of the angular momentum $\vec{G}$
h	$H = G\cos K \quad (K$ the ecliptic obliquity $)$

a_o the semi-major axis
i_o the inclination
e_o the eccentricity
l_o the mean anomaly, linear function of time
v_o the true anomaly
ω_o the argument of the pericenter
h_o the longitude of the ascending node

$$\dot{q}_i = \frac{\partial H}{\partial p_i}$$
$$\dot{p}_i = -\frac{\partial H}{\partial q_i}$$

Non singular variables and kinetic energy

l and h : slow variables
g spin : fast variable (58 days)

$$\begin{aligned} \lambda_1 &= l + g + h & \Lambda_1 &= G \\ \lambda_2 &= -l & \Lambda_2 &= G - L = G\,(1 - \cos J) \\ \lambda_3 &= -h & \Lambda_3 &= G - H = G\,(1 - \cos K) \end{aligned}$$

$$T = \frac{(\Lambda_1 - \Lambda_2)^2}{2 I_3} + \frac{1}{2}(\Lambda_1^2 - (\Lambda_1 - \Lambda_2)^2)(\frac{\sin^2 \lambda_2}{I_1} + \frac{\cos^2 \lambda_2}{I_2})$$

I_1, I_2 and I_3 : moments of inertia of the planet $I_1 < I_2 < I_3$.

$$T = T(\Lambda_1, \lambda_2, \Lambda_2).$$

Remark :

$J \simeq 0 \quad \rightarrow \quad$ Spin $\equiv$ Body $\quad \rightarrow \Lambda_2 \simeq 0 \quad \rightarrow \quad T \simeq \frac{\Lambda_1^2}{2 I_3}$

The potential V_G

$$V_G = -\frac{GM}{r}\left(\frac{R_e}{r}\right)^2 \left[C_2^0\, P_2(\sin\theta) + C_2^2\, P_2^2(\sin\theta)\cos 2\varphi\right]$$

- $2C_2^0 = I_1 + I_2 - 2I_3$ and $4C_2^2 = I_2 - I_1$
- P_2 and P_2^2 : Legendre's polynomials
- R_e : Mercury's equatorial radius,
- r, θ and φ : position of the Sun in the body frame (3).
- Corresponding normalized cartesian coordinates :

$$\bar{x}_3 = \cos\varphi\cos\theta \quad \bar{y}_3 = \sin\varphi\cos\theta \quad \bar{z}_3 = \sin\theta$$

$$V_G = -\frac{GM}{r^3}R_e^2\left[\frac{C_2^0}{2}\,(2\bar{z}_3^2 - \bar{x}_3^2 - \bar{y}_3^2) + 3C_2^2\,(\bar{x}_3^2 - \bar{y}_3^2)\right]$$

The rotations

$$\begin{pmatrix} \bar{x}_3 \\ \bar{y}_3 \\ \bar{z}_3 \end{pmatrix} = R_3(-\lambda_2)R_1(J)R_3(\lambda_1+\lambda_2+\lambda_3)R_1(K)R_3(-\lambda_3)\times$$

$$R_3(-h_o)R_1(-i_o)R_3(-g_o)\begin{pmatrix} \cos v_o \\ \sin v_o \\ 0 \end{pmatrix}$$

Orbit frame $\to$ Inertial frame $\to$ Spin frame $\to$ Body frame.

Keplerian orbit : $V_G = V_G(\lambda_1, \Lambda_1, \lambda_2, \Lambda_2, \lambda_3, \Lambda_3, l_o(t))$

$$\boxed{\mathcal{H} = T(\Lambda_1, \lambda_2, \Lambda_2, \Lambda_3) + V_G(\lambda_1, \Lambda_1, \lambda_2, \Lambda_2, \lambda_3, \Lambda_3, l_o) + n_o L_o}$$

The resonant variables

The spin-orbit resonance : $\boxed{\sigma = \frac{2\lambda_1 - 3l_o}{2}}$ slow variable

$$\begin{array}{ll} \sigma_1 = \sigma - h_o - g_o & \Lambda_1 \\ \sigma_3 = \lambda_3 + h_o & \Lambda_3 \\ l_o \quad \text{fast variable} & \Lambda_o = L_o + \frac{3}{2}\Lambda_1 \\ \sigma_2 = \lambda_2 & \Lambda_2 \end{array}$$

$$\mathcal{H} = \mathcal{H}(\sigma_1, \Lambda_1, \sigma_2, \Lambda_2, \sigma_3, \Lambda_3, l_o, \Lambda_0)$$

- Truncature in e_o and i_o
- Average over the fast variable l_o : $\bar{\Lambda}_o$ is a constant
- First order averaging : $\bar{\mathcal{H}} = \bar{\mathcal{H}}(\bar{\sigma}_1, \bar{\Lambda}_1, \bar{\sigma}_2, \bar{\Lambda}_2, \bar{\sigma}_3, \bar{\Lambda}_3)$
- Equilibria : $\dfrac{\partial \mathcal{H}}{\partial \bar{\sigma}_i} = 0 = \dfrac{\partial \mathcal{H}}{\partial \bar{\Lambda}_i}$

The Cassini's equilibrium

Description of the equilibrium corresponding to Mercury

- $\bar{\sigma}_1 = 0$: spin-orbit resonance
- $\bar{\sigma}_3 = 0$: node commensurability (h and h_o)
- $\bar{\Lambda}_1^\star$ and $\bar{\Lambda}_3^\star$: $K^\star = i_o$
- $\bar{\sigma}_2 = 0$ and $\Lambda_2 = 0$ or $J = 0$: Spin axis $\equiv$ axis of inertia

Small librations around the exact equilibrium

- Quadratic Taylor's development of $\mathcal{H}$ in cartesian coordinates
- Apparition of crossed terms (1 and 3) : untangling transformation
- Rescaling of the variables and action-angle variables

$$\mathcal{H} = \nu_1 J_1 + \nu_2 J_2 + \nu_3 J_3 + \dots$$

3 proper (free) frequencies :

ν_1 (free) longitude of the libration

ν_2 (free) wobble

ν_3 (free) precession

Comments

- Model introduced by Peale (1973) :
  - no wobble
  - inertial frame = orbital frame + precession of the orbital node
- Introduction of a CONSTANT precession of the node (over some period of time) : $K^\star = i_o + \epsilon$,
  1 Arcmin $\leq \epsilon \leq$ 2 Arcmin.
- Introduction of a CONSTANT precession of the pericenter : less interesting
- D'Hoedt & Lemaitre (2004) : data taken Anderson et al (1987)
- 3 proper periods : $$T_1 = 15.8573 \text{ years}, \quad T_2 = 583.989 \text{ years} \quad T_3 = 1065.08 \text{ years}$$

Libration of Mercury about the exact 3/2 resonance

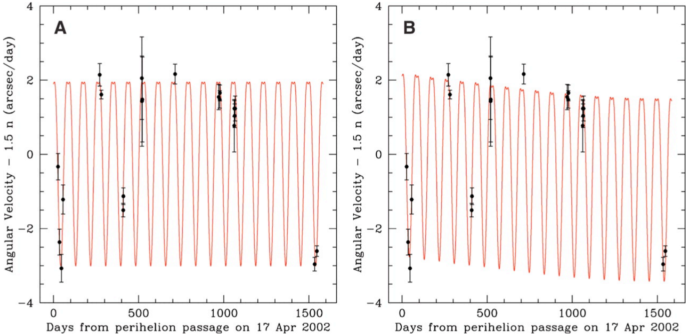
- Is Mercury at the exact equilibrium or not ? Existence of proper (free) libration motions
- Toy model : harmonic oscillator + dissipation

$$\ddot{x} - h^2 \dot{x} + \nu^2 x = 0 \quad \rightarrow \quad x(t) = A(t) \cos(\nu' t + \phi)$$

- Peale (2005) : dissipation over periods of 10^5 to 10^6 years
- 3 dissipation mechanisms
  - Tidal dissipation
  - Viscous core-mantle coupling (dominant)
  - Recent excitation mechanism : impact of a small body
    Collisions could not explain a significant free libration
- Mercury is very, very close to the Cassini's equilibrium

Margot's results : radar data

Radar data compared with two models over 4.5 years
A : at the exact Cassini's state (period of 88 days)
B : with a free libration (period of 88 days and 12 years)

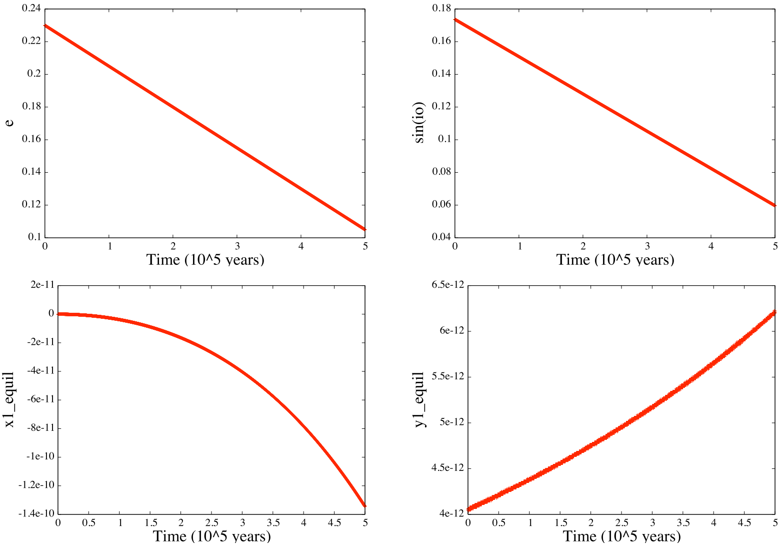


Margot et al (2007), Science

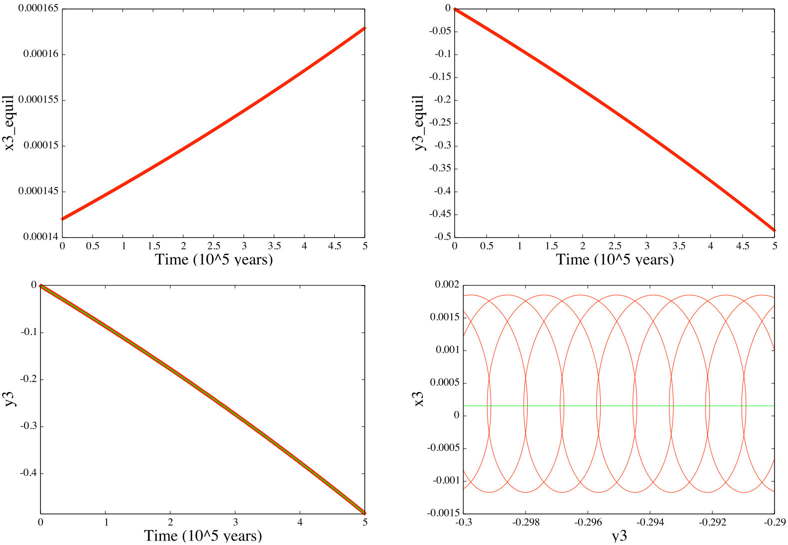
Very long periods

- The (mean) orbit of Mercury is not keplerian
- Introduction of a secular motion of i_o, e_o, h_o and ω_o
- Periods of 10^5 years
- Idea of Peale (1976, 2006) : adiabaticity on the (σ_1, Λ_1)
- Slow evolution with time for the stable equilibrium (captured)
- Averaging process on proper angles (on periods of 10^3 years)
- D'Hoedt & Lemaitre (2008)
  - Generalization to two degrees of freedom
    Adiabatic model : (σ_1, Λ_1) and (σ_3, Λ_3)
  - Confirmation of the behavior for the (2-degree of freedom) equilibrium for long periods of time

Adiabaticity of (σ_1, Λ_1)



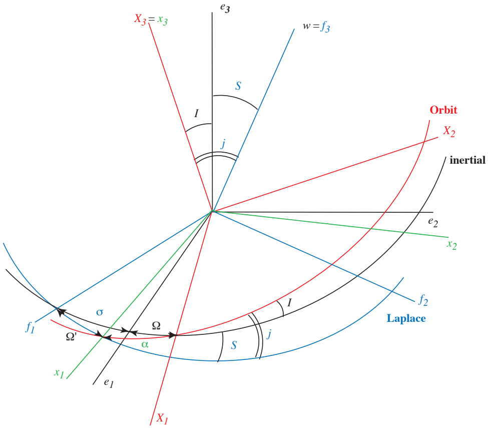
Adiabaticity of (σ_3, Λ_3)



Laplace plane

- Cassini's equilibrium : function of i_o and of the precession rate
- Calculated with respect to a specific plane : Laplace plane
- Ideal Laplace plane = the plane about which the orbital inclination remains constant throughout a precessional cycle.
- Instantaneous Laplace plane : the plane about which variations in inclination are minimized.
- Dependence on the interval of time, the chosen approach, the set of ephemeris or synthetic theory
- Dependence on the goal (academic or practical)
- D'Hoedt et al (2009), ASR

Positions of the Laplace planes



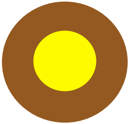
- σ : the longitude of the ascending node of the Laplace plane on the inertial plane
- S : the inclination of the Laplace plane on the inertial plane.
- Ω' : the longitude of the ascending node of the orbital plane on the Laplace plane
- j : the inclination of the orbital plane on the Laplace plane.

Unicity

- 4 papers
  - Peale (2006) : numerical fit to ephemerides
  - Yseboodt & Margot (2006) : secular theory + numerical fit
  - Rambaux & Bois (2004) : principles but no values
  - D'Hoedt et al (2009) : Henrard's simple formulation
- Y&M : a unique instantaneous Laplace plane
  20000 years, in intervals of 2000 years (JPL DE408)
- Namur : an infinity of instantaneous Laplace planes, *best one*
  interval of 6000 years (JPL DE406)
- Comparable results :

angle	Y&M	Namur	P
S	3.3°	2.7°	
j	5.33°	7.5°	8.6°

Liquid core



- Mercury is not a rigid body
- Old question treated by Peale
  - 1976, Nature Does Mercury have a molten core ?
  - 1981, Icarus Measurement accuracies required for the determination of a Mercurian liquid core
  - 1997, LPI Characterizing the core of Mercury
- Existence of a molten or fluid core : influence on I_3
- I_3 has to be C or C_m : simple introduction of two layers
- Peale et al (2007 and 2009) : viscous core
  - Solidarity core-mantle only for very slow motions
  - Slow motions or long periodic terms : $I_3 = C$ (rigid planet)
  - Fast motions or short periodic terms : $I_3 = C_m$ (only the mantle)

Simple two layers model

$$\mathcal{H} = \mathcal{H}(\sigma_1, \Lambda_1, \sigma_2, \Lambda_2, \sigma_3, \Lambda_3, l_o, \Lambda_0)$$

- First order averaging over the fast variable l_o :
  $\bar{\mathcal{H}} = \bar{\mathcal{H}}(\bar{\sigma}_1, \bar{\Lambda}_1, -, -, \bar{\sigma}_3, \bar{\Lambda}_3)$ - No wobble
- 2 proper periods : $T_1 = 15.8573$ years, $T_3 = 1065.08$ years
- Third or fourth order averaging over the fast variable l_o results of Sandrine's PhD : too small changes
- Margot + Peale : new set of data with a fluid core hypothesis
  $C_m = 0.579\, C$, $C = 0.34$, $J_2 = 6\, 10^{-5}$, $C_{22} = 10^{-5}$
- 2 proper periods : $T_1 = 12.055$ years, $T_3 = 615.69$ years

Lie triangle

$$\mathcal{H}(\sigma_1, \Lambda_1, \sigma_2, \Lambda_2, \sigma_3, \Lambda_3, l_o, \Lambda_0) \to \bar{\mathcal{H}}(\bar{\sigma}_1, \bar{\Lambda}_1, \bar{\sigma}_2, \bar{\Lambda}_2, \bar{\sigma}_3, \bar{\Lambda}_3)$$

- Canonical transformation, order by order, Lie triangle
- $\mathcal{H} = \sum_{i=0}^{n} H_i^0 \frac{\epsilon^i}{i!}$ and $\bar{\mathcal{H}} = \sum_{i=0}^{n} H_0^i \frac{\epsilon^i}{i!}$
- H_i^0 are data and H_0^i are results :

$$\begin{array}{cccc} H_0^0 & & & \\ H_1^0 & H_0^1 & & \\ H_2^0 & H_1^1 & H_0^2 & \\ H_3^0 & H_2^1 & H_1^2 & H_0^3 \end{array}$$

- Homological equation : $H_0^n = H_1^{n-1} + \left(H_0^{n-1}; W_1\right)$
- Recurrence formulae : $H_j^n = H_{j+1}^{n-1} + \sum_{i=0}^{j} \binom{j}{i} \left(H_{j-i}^{n-1}; W_{1+i}\right)$
- W_i is the ith generator

Short periodic terms

Inverse algorithm : Deprit (1969) and Henrard (1970)

- Introduction of cartesian coordinates :
  $(\sigma_1, \Lambda_1) \rightarrow (x_1, y_1)$ and $(\sigma_3, \Lambda_3) \rightarrow (x_3, y_3)$
- $f(x_1, x_3, y_1, y_3) =$
  $$f(\bar{x}_1, \bar{x}_3, \bar{y}_1, \bar{y}_3) + \sum_{i=1}^{\text{order}} \frac{\epsilon^i}{i!} \left(f(x_1, x_3, y_1, y_3); W_i\right)_{(\bar{x}_1, \bar{x}_3, \bar{y}_1, \bar{y}_3)}$$
  Any function f(non averaged variables) can be expressed as a function of the averaged solution through an expansion using the generators W_i
- In particular : $f = x_i$ or $f = y_i$.
- $(\bar{x}_1, \bar{x}_3, \bar{y}_1, \bar{y}_3)$ evaluated at the equilibrium of the averaged model
  Peale's results about the proximity of Mercury to the Cassini's state
- Keplerian case : $\mathrm{var}_o = \mathrm{var}_o^\star + \mathcal{F}_{\mathrm{var}}(l_o)$

Non Keplerian case

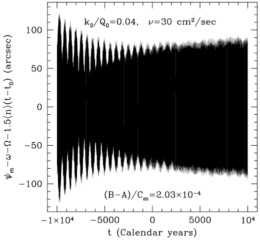
- Short periodic planetary perturbations : VSOP, IMCCE (courtesy of J.L. Simon)
- Orbital elements of Mercury : a_o, e_o, i_o, g_o, h_o, l_o
- Validity of more than 100 years
- Introduction of the mean longitudes of all the planets

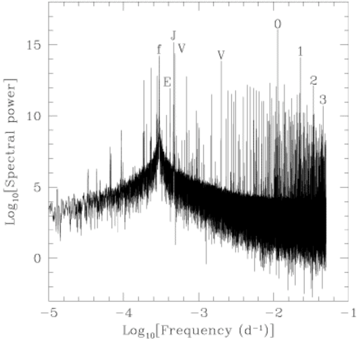
$$\mathrm{var}_o = \mathrm{var}_o^\star + \mathcal{F}_{\mathrm{var}}(l_o, l_V, l_E, l_{Ma}, l_J, l_S, l_U, l_N),$$

- $H = -\frac{\mu^2}{2L_o^2} + n_J\Lambda_J + n_V\Lambda_V + n_S\Lambda_S + n_E\Lambda_E + \dot{\varpi}_o G_o + \dot{\Omega}_o H_o$
  $+\frac{\Lambda_1^2}{2C_m} + V_G(l_o, \varpi_o, \Omega_o, e_o, a_o, i_o, \sigma_1, \sigma_3, L_o, \Lambda_1, \Lambda_3, l_V, l_E, l_J, l_S)$

Results of Peale et al (2007) on σ_1

- Numerical integration - One degree of freedom : (σ_1, Λ_1)
- Complete two layers model : core - mantle dissociated
- Using JPL DE408 (20 000 years) for planetary contributions
- Damping factor : tidal effect - Elimination of proper (free) frequencies in the final spectrum





Short periodic terms on σ_1

Comparison between Namur and Peale et al (2007) :
$C_{22} = 1.5\ 10^{-5}$ (period of $\sigma_1 \simeq 9$ years)

angle combination	Period (years)	Amplitude (rad)	Relative amplitude
NAMUR			
Mercury (l_o)	0.24084	$0.197285\ 10^{-3}$	1
Jupiter (λ_J)	11.86200	$0.643367\ 10^{-4}$	0.326110
Mercury ($2l_o$)	0.12042	$0.219964\ 10^{-4}$	0.111496
Venus ($2l_o - 5\lambda_V$)	5.66608	$0.210918\ 10^{-4}$	0.106910
Jupiter ($2\lambda_J$)	5.93100	$0.811086\ 10^{-5}$	0.041112
Saturn ($2\lambda_S$)	14.7285	$0.597894\ 10^{-5}$	0.030306
Earth ($l_o - 4\lambda_E$)	6.57966	$0.347122\ 10^{-5}$	0.017595
PEALE			
Mercury ($\lambda_M - \varpi = l_o$)	0.24084	1	1
Venus ($2l_o - 5\lambda_V + 3\varpi$)	5.66608	0.1427	0.1289
Mercury ($2(\lambda_M - \varpi) = 2l_o$)	0.12042	0.1028	0.1115
Jupiter (λ_J)	11.86200	not listed ($\simeq 0.04$)	0.0571
Jupiter ($2\lambda_J - 2\varpi$)	5.93100	0.3483	0.0509
Saturn ($2\lambda_S$)	14.7285	not listed ($\simeq 0.02$)	0.0138
Earth ($l_o - 4\lambda_E$)	6.57966	not listed ($\simeq 0.01$)	0.0239

Dufey et al (2008), CM&DA

Explanations

- Comparisons with SONYR (Rambaux & Bois) : encouraging results
- Especially using a forced analytical orbital motion

Angle combination	Period (years) SONYR	Amplitude (rad) SONYR	Relative amplitude SONYR	Relative amplitude NAMUR
Mercury (l_o)	0.24084	$0.201135\ 10^{-3}$	1	1
Jupiter (λ_J)	11.86200	$0.633015\ 10^{-4}$	0.314721	0.326110
Mercury ($2l_o$)	0.12042	$0.195272\ 10^{-4}$	0.097085	0.111496
Venus ($2l_o - 5\lambda_V$)	5.66608	$0.211462\ 10^{-4}$	0.105134	0.106910
Jupiter ($2\lambda_J$)	5.93100	$0.813315\ 10^{-5}$	0.040436	0.041112
Saturn ($2\lambda_S$)	14.7285	$0.596094\ 10^{-5}$	0.029365	0.030306
Earth ($l_o - 4\lambda_E$)	6.57966	$0.348243\ 10^{-5}$	0.017314	0.017595

- Differences with SONYR (full N-Body integration) : OK
- Main differences with Peale et al (2007) : no planetary perturbations on Mercury's mean anomaly
- $l_o = n_o t + l_o^0$ and not $l_o = l_o(l_V, l_E, l_J, l_S)$
- New results of Peale et al (2009): in agreement with Namur

Short periodic terms on σ_1

Coefficients : $\frac{C_m}{C} = 0.579$ and $C_{22} = 1.0\ 10^{-5}$
$T_{\sigma_1} = 12.055$ years and $T_{\sigma_3} = 615.69$ years
Comparisons with numerical integration and frequency analysis

N	l_o	l_V	l_E	l_J	l_S	ϖ_o	Period	Amplitude	Ratio	Phase at J2000
1	-	-	-	1	-	-1	11.862 y	43.712 as	1.2193	164.46°
2	1	-	-	-	-	-	87.970 d	35.849 as	1.0000	84.80°
3	2	-	-	-	-	-	43.985 d	3.754 as	0.1047	79.59°
4	2	-5	-	-	-	5	5.664 y	3.597 as	0.1003	166.85°
5	-	-	-	-	2	-2	14.729 y	1.568 as	0.0437	-85.96°
6	-	-	-	2	-	-2	5.931 y	1.379 as	0.0385	125.91°
7	1	-	-4	-	-	4	6.575 y	0.578 as	0.0161	-25.57°
8	3	-	-	-	-	-	29.323 d	0.386 as	0.0108	-105.62°
9	1	-	-	-2	-	2	91.692 d	0.201 as	0.0056	-58.69°
10	1	-	-	2	-	-2	84.537 d	0.191 as	0.0053	48.28°
11	-	-	-	2	-5	3	883.28 y	0.103 as	0.0029	-153.53°
12	2	-	-	-1	-	1	44.436 d	0.069 as	0.0019	-25.51°
13	2	-	-	1	-	-1	43.541 d	0.067 as	0.0019	4.71°
14	1	-	-	-1	-	1	89.793 d	0.044 as	0.0012	-17.94°
15	1	-	-	1	-	-1	86.217 d	0.043 as	0.0012	7.17°
16	2	-	-	-2	-	2	44.897 d	0.041 as	0.0011	-63.89°
17	2	-	-	2	-	-2	43.110 d	0.040 as	0.0011	43.07°

Dufey et al (2009), CM&DA - calculation of the phase

Short periodic terms on σ_3

Coefficients : $\frac{C_m}{C} = 0.579$ and $C_{22} = 1.0\ 10^{-5}$
$T_{\sigma_1} = 12.055$ years and $T_{\sigma_3} = 615.69$ years
Comparisons with numerical integration and frequency analysis

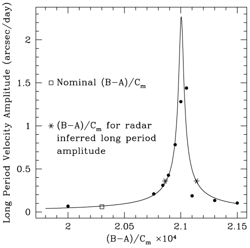
N	l_o	l_V	l_E	l_J	l_S	ϖ_o	Ω_o	Period	Amplitude	Ratio	Phase
1	-	-	-	-	-	*2*	*-2*	*63315 y*	*3.74 as*	*26.128*	*-56.84°*
2	1	-	-	-	-	-	-	87.970 d	143.06 mas	1.0000	84.80°
3	2	-	-	-	-	2	-2	43.985 d	67.31 mas	0.4705	-67.25°
4	-	-	-	2	-5	3	-	883.280 y	65.44 mas	0.4574	15.01°
5	3	-	-	-	-	2	-2	29.323 d	45.02 mas	0.3147	107.55°
6	4	-	-	-	-	2	-2	21.992 d	14.71 mas	0.1028	-77.66°
7	-	-	-	2	-	-2	-	5.931 y	14.64 mas	0.1023	-117.10°
8	2	-	-	-	-	-	-	43.985 d	12.91 mas	0.0902	-100.41°
9	1	-	-	-	-	2	-2	87.970 d	8.20 mas	0.0573	-62.04°
11	-	-	-	1	-	-1	-	11.862 y	7.83 mas	0.0547	-148.63°
10	2	-5	-	-	-	5	-	5.664 y	6.00 mas	0.0420	100.24°
12	-	-	-	-	2	-2	-	14.729 y	4.43 mas	0.0309	-157.27°
13	1	-	-4	-	-	4	-	6.575 y	1.93 mas	0.0135	-66.00°

- Precessional motion : a long-period perturbation
- Limit of the model : core - mantle dissociated

Dufey et al (2009), CM&DA - calculation of the phase

Resonances : 1) with Jupiter

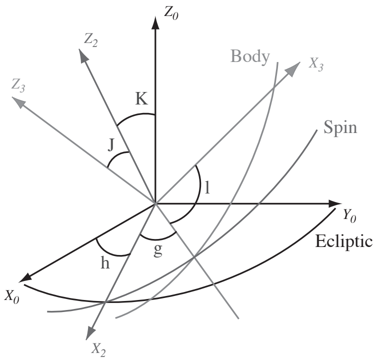
- Main short periodic term
  l_o with a period of 88 days
- As a function of C_{22} and of $\frac{C_m}{C}$:
  8 years $< T_{\sigma_1} <$ 16 years
- Present *best* value : $T_{\sigma_1} = 12.055$ years
- Other short periodic terms :
  l_J with a period of 11.86 years
- Critical value of C_{22} : exact resonance
- Potential commensurability mentioned
- Peale et al (2009), Icarus
- Complete study of this resonance
- Potential (small) influence : BepiColombo



Resonances : 2) with the Great Inequality

- $\frac{C_m}{C} = 0.579$ and $C_{22} = 1.0\ 10^{-5}$: $T_{\sigma_3} = 615.69$ years
- As a function of C_{22} and of $\frac{C_m}{C}$:
  400 years $< T_{\sigma_3} <$ 1100 years
- A possible value of C_{22} and for $\frac{C_m}{C} = 0.829$:
  $T_{\sigma_3} = 883.28$ years
- Exact 1:1 resonance with the Great Inequality :
  $\sigma_{25} = 2l_J - 5l_S$
- Complete study of this resonance, resonant angle
  $\alpha = \psi_3 - \sigma_{25}$
- Period of 10^8 years
- Variations of $K \simeq 0.32°$
- No influence on BepiColombo

Wobble : σ_2



- σ_2 canonical variable associated to $\Lambda_2 = \Lambda_3\,(1 - \cos J)$
- Angle between Spin and Body axes $J \simeq 0$
- Associated to a period of 589 years
- Uncoupled motion (first orders)
- Pole motion on Mercury's surface : a few meters
- Short periodic terms are negligible
- No resonance detected
- Undetectable for BepiColombo

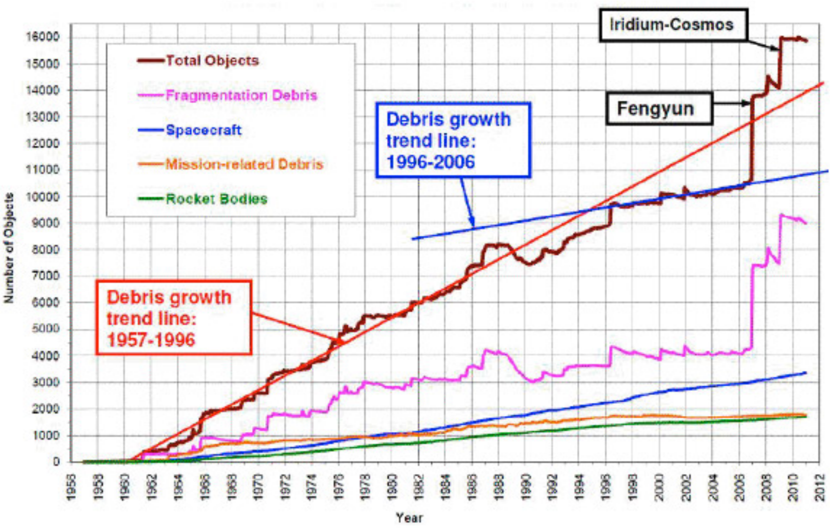
Conclusions

- Effective non-linear stability of the equilibrium (Sansottera, Lhotka, Lemaitre, MNRS 2015) : several physical parameters, Birkhoff normal form, Nekhoroshev stability theory
- Long and short periodic contributions analyzed
- Academic and practical views could be different and complementary
- Very promising features with core frequency
- Better use of the phases of the short periodic contributions
- Waiting for data : Messenger + radar
- Juice mission : Galilean satellites
- Many open mathematical problems

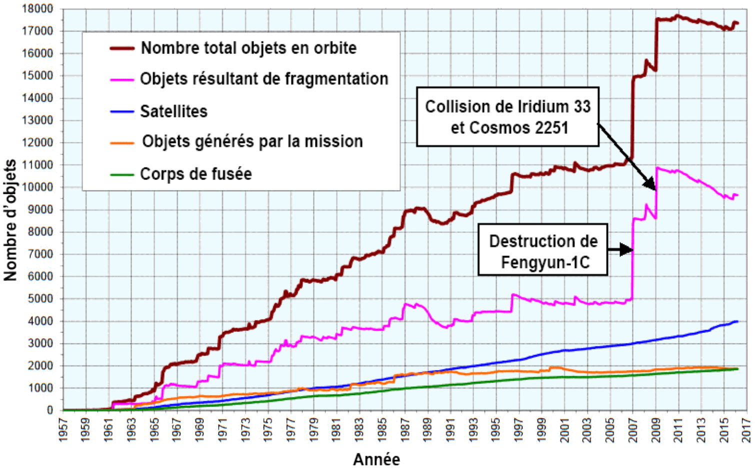
Gravitational resonances

- Resonance between the orbital motion of a satellite and the rotation of the primary
- Primary is not anymore a point mass
- Mainly for artificial satellites
- Short lifetimes, permanent control and re-orbitation, real time orbits
- Space debris act as natural bodies: abandoned, for long time, no control, perturbed
- Explosion or collision : mass, spin, not necessary known
- More "interesting" objects for CM

Number of debris

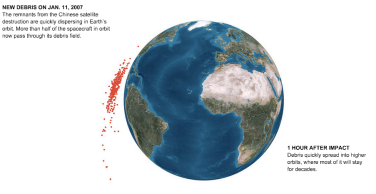


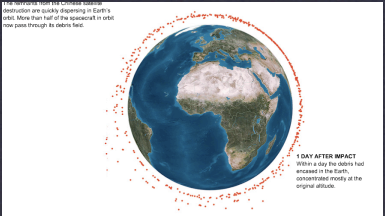
Number of debris

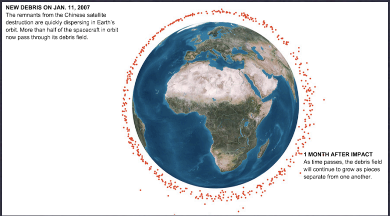


Chinese explosion : Fengyun 2007

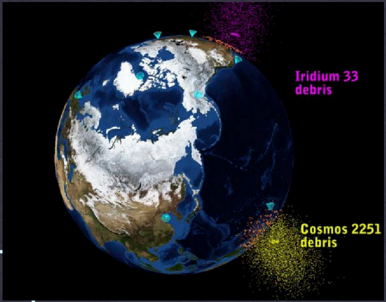
New York Times
Explosion of the chinese satellite
Fengyun FY-1C on January 11, 2007







Collision



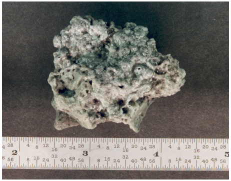
- Iridium 33 (active American telecommunication satellite)
- Cosmos 2251 (non active military Russian satellite)
- Date : February 10, 2009
- Speed : 11.7 km/second

What are Orbital Space Debris?

Definition

Orbital debris refers to material on orbit resulting from space missions but no longer serving any function.

- Launch vehicle upper stages
- Abandoned satellites
- Lens caps
- Momentum flywheels
- Core of nuclear reactors
- Objects breakup
- Paint flakes
- Solid-fuel fragments



Current debris population

- There are about 18 000 objects larger than 10 cm
  TLE Catalogue
- About 350 000 objects larger than 1 cm
- More than 3×10^8 objects larger than 1 mm

Catalogued objects (NASA)

- 6 % Operational spacecrafts
- 24% Non-operational spacecrafts
- 17% Upper stages of rockets
- 13% Mission related debris
- 40% Debris mostly generated by explosions & collisions

Computer generated images

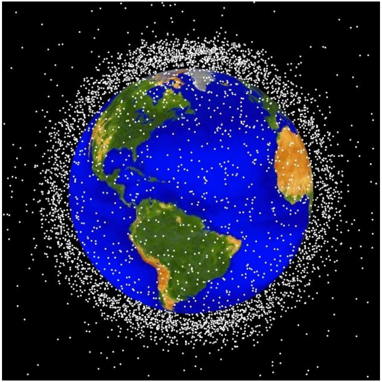


Figure: LEO image

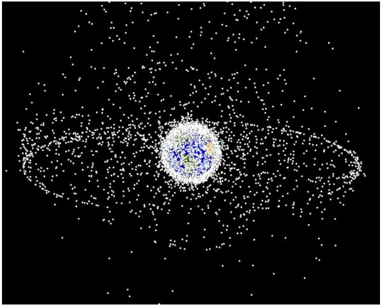
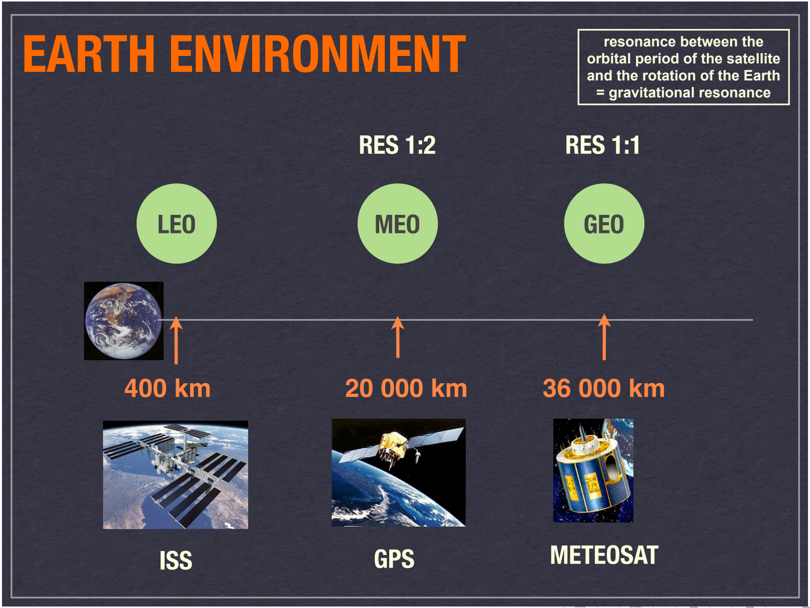


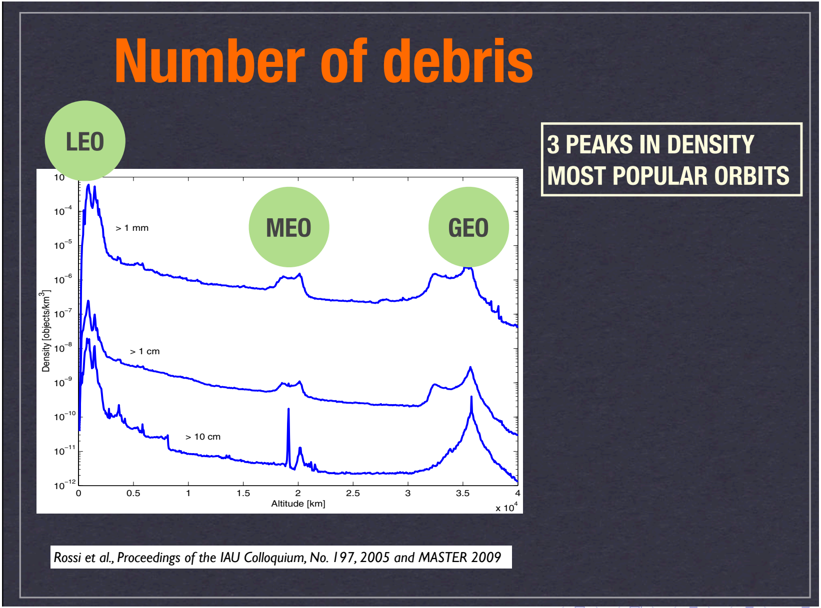
Figure: GEO image

LEO-MEO-GEO



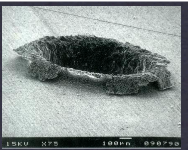
EARTH ENVIRONMENT
resonance between the orbital period of the satellite and the rotation of the Earth = gravitational resonance
RES 1:2
RES 1:1
LEO
MEO
GEO
400 km
20 000 km
36 000 km
ISS
GPS
METEOSAT

Rossi et al (2005)



Problematic situation

Situation and solutions



Size (r)	Characteristics	Protection	Number
r < 0.01 cm	cumulative effects surface erosion	not necessary	
0.01 < r < 1 cm	significant damages perforation	armor plating	170 000 000 objects
1 < r < 10 cm	important damages	no solution	670 000 objects
r > 10 cm	catastrophic events catalogued (TLE)	manoeuvres	< 20 000 objects

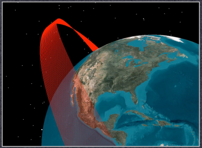
Analogy with natural bodies

Natural and artificial objects

Natural	Artificial	Debris
existing orbits	chosen orbits	existing orbits
no control	control	no control
long times	short times	long times
model and observations	huge numerical integrations	model and observations
stability	precision	stability

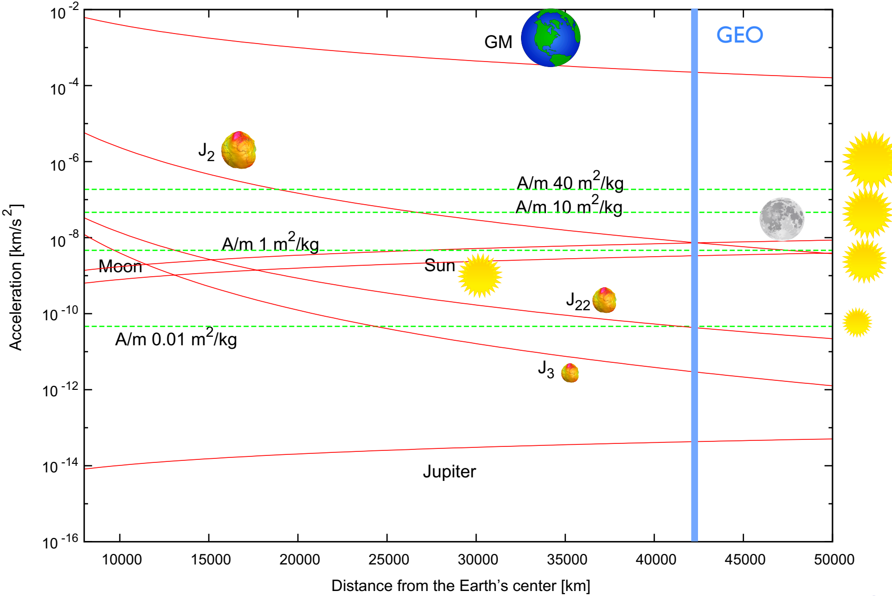
Long term dynamics

ESTIMATION OF LIFETIMES FOR USUAL OBJECTS



300 km	1 month
400 km	1 year
500 km	10 years
700 km	50 years
900 km	1 century
1200 km	1 millennium

The forces for MEO and GEO



First contribution : forces

Dynamics of a debris

= Keplerian orbit around the Earth

+ rotation of the Earth

+ shape of the Earth (geopotential - J_2)

+ third body perturbations (Moon and Sun)

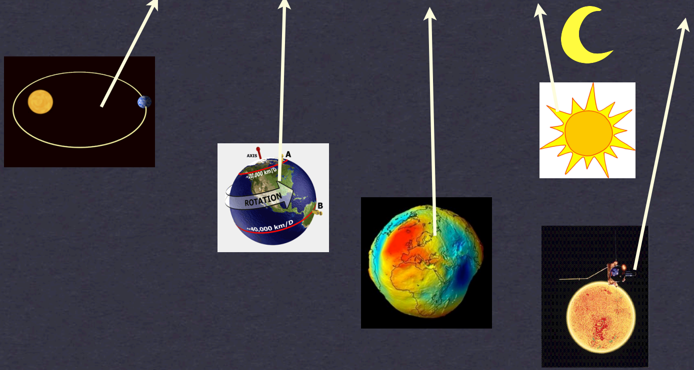
+ solar radiation pressure

+ shadowing effects

+ atmospheric drag (LEO) : cleaner

Hamiltonian formalism

$$H_{deb}(v,\Lambda,r,\theta) = H_{kep}(v,r) + H_{rot}(\Lambda) + H_{geo}(r,\theta) + H_{3b}(r) + H_{srp}(r)$$



The geopotential

$$U(\boldsymbol{r}) = \mu \int_V \frac{\rho(\boldsymbol{r_p})}{\|\boldsymbol{r} - \boldsymbol{r_p}\|}\, dV\,, \qquad \mu = G\, m_\oplus$$

$$\begin{aligned} x &= r \cos\phi \cos\lambda & \qquad x_p &= r_p \cos\phi_p \cos\lambda_p \\ y &= r \cos\phi \sin\lambda & y_p &= r_p \cos\phi_p \sin\lambda_p \\ z &= r \sin\phi & z_p &= r_p \sin\phi_p \end{aligned}$$

$$U(r,\lambda,\phi) = -\frac{\mu}{r} \sum_{n=0}^{\infty} \sum_{m=0}^{n} \left(\frac{R_e}{r}\right)^n \mathcal{P}_n^m(\sin\phi)(C_{nm} \cos\, m\lambda + S_{nm} \sin\, m\lambda)$$

R_e : the equatorial Earth's radius

$$C_{nm} = \frac{2-\delta_{0m}}{M_\oplus} \frac{(n-m)!}{(n+m)!} \int_V \left(\frac{r_p}{R_e}\right)^n \mathcal{P}_n^m(\sin\phi_p) \cos\,(m\lambda_p)\, \rho(\boldsymbol{r_p})\, dV$$

$$S_{nm} = \frac{2-\delta_{0m}}{M_\oplus} \frac{(n-m)!}{(n+m)!} \int_V \left(\frac{r_p}{R_e}\right)^n \mathcal{P}_n^m(\sin\phi_p) \sin\,(m\lambda_p)\, \rho(\boldsymbol{r_p})\, dV$$

The geopotential

$$J_2 = -C_{20} = \frac{2C - B - A}{2\, M_\oplus\, R_e^2} \qquad \text{and} \qquad C_{22} = \frac{B - A}{4\, M_\oplus\, R_e^2}$$

$$U(r,\lambda,\phi) = -\frac{\mu}{r} + \frac{\mu}{r} \sum_{n=2}^{\infty} \sum_{m=0}^{n} \left(\frac{R_e}{r}\right)^n \mathcal{P}_n^m(\sin\phi)\, J_{nm} \cos m(\lambda - \lambda_{nm})$$

$$C_{nm} = -J_{nm} \cos(m\lambda_{nm})$$

$$S_{nm} = -J_{nm} \sin(m\lambda_{nm})$$

$$J_{nm} = \sqrt{C_{nm}^2 + S_{nm}^2}$$

$$m\,\lambda_{nm} = \arctan\left(\frac{-S_{nm}}{-C_{nm}}\right) .$$

The geopotential: Kaula formulation

$$U = -\frac{\mu}{r} - \sum_{n=2}^{\infty} \sum_{m=0}^{n} \sum_{p=0}^{n} \sum_{q=-\infty}^{+\infty} \frac{\mu}{a} \left(\frac{R_e}{a} \right)^n F_{nmp}(i)\, G_{npq}(e)\, S_{nmpq}(\Omega, \omega, M, \theta)$$

$$\begin{aligned} S_{nmpq}(\Omega, \omega, M, \theta) &= \begin{bmatrix} +C_{nm} \\ -S_{nm} \end{bmatrix}_{n-m\,\text{odd}}^{n-m\,\text{even}} \cos \Theta_{nmpq}(\Omega, \omega, M, \theta) \\ &+ \begin{bmatrix} +S_{nm} \\ +C_{nm} \end{bmatrix}_{n-m\,\text{odd}}^{n-m\,\text{even}} \sin \Theta_{nmpq}(\Omega, \omega, M, \theta) \end{aligned}$$

Kaula gravitational argument, θ the sidereal time :

$$\Theta_{nmpq}(\Omega, \omega, M, \theta) = (n-2p)\,\omega + (n-2p+q)\,M + m(\Omega - \theta)$$

The luni-solar perturbations

The acceleration :

$$\ddot{\mathbf{r}} = -\mu_i \left(\frac{\mathbf{r} - \mathbf{r_i}}{\|\mathbf{r} - \mathbf{r_i}\|^3} + \frac{\mathbf{r_i}}{\|\mathbf{r_i}\|^3} \right) .$$

The potential (i=1 for the Sun, i=2 for the Moon):

$$\mathcal{R}_i = \mu_i \left(\frac{1}{\|\mathbf{r} - \mathbf{r_i}\|} - \frac{\langle \mathbf{r} \, . \, \mathbf{r_i} \rangle}{\|\mathbf{r_i}\|^3} \right) .$$

$$\mathcal{R}_i = \frac{\mu_i}{r_i} \sum_{n \geq 2} \left(\frac{r}{r_i} \right)^n \mathcal{P}_n(\cos\psi)$$

r_i the geocentric distance
ψ the geocentric angle between the third body and the satellite
$\mathcal{P}_n$ the Legendre polynomial of degree n.

The luni-solar perturbations

- The three components (x, y, z) of the position vector $\boldsymbol{r}$ expressed in Keplerian elements $(a, e, i, \Omega, \omega, f)$
- The Cartesian coordinates X_i, Y_i and Z_i of the unit vector pointing towards the third body.
- Usual developments of f and $\frac{r}{a}$ in series of e, $\sin\frac{i}{2}$ and M

$$\mathcal{R}_i = \frac{\mu_i}{r_i} \sum_{n=2}^{+\infty} \sum_{k,l,j_1,j_2,j_3} \left(\frac{a}{r_i}\right)^n A^{(n)}_{k,l,j_1,j_2,j_3}(X_i, Y_i, Z_i)\, e^{|k|+2j_2} \left(\sin\frac{i}{2}\right)^{|l|+2j_3} \cos\Phi$$

$$\Phi = j_1\,\lambda + j_2\,\varpi + j_3\,\Omega, \quad \lambda = M + \omega + \Omega, \quad \varpi = \omega + M$$

Poincaré variables

Delaunay canonical momenta associated with λ, ϖ and Ω :

$$L = \sqrt{\mu\, a}, \qquad G = \sqrt{\mu\, a(1 - e^2)}\,, \qquad H = \sqrt{\mu\, a(1 - e^2)}\,\cos i$$

Non singular Delaunay elements, keeping L and λ :

$$\begin{aligned} P &= L - G & \qquad p &= -\omega - \Omega \\ Q &= G - H & \qquad q &= -\Omega \end{aligned}$$

Poincaré variables :

$$\begin{aligned} x_1 &= \sqrt{2P}\,\sin p & \qquad x_4 &= \sqrt{2P}\,\cos p \\ x_2 &= \sqrt{2Q}\,\sin q & \qquad x_5 &= \sqrt{2Q}\,\cos q \\ x_3 &= \lambda = M + \Omega + \omega & \qquad x_6 &= L \end{aligned}$$

Dimensionless Poincaré variables

$$U = \sqrt{\frac{2P}{L}} \qquad V = \sqrt{\frac{2Q}{L}}$$

$$e = U\left(1 - \frac{U^2}{4}\right)^{\frac{1}{2}} = U - \frac{1}{8}U^3 - \frac{1}{128}U^5 + \mathcal{O}(U^7)$$

$$2\,\sin\frac{i}{2} = V\left[1 - \frac{U^2}{2}\right]^{-\frac{1}{2}} = V + \frac{1}{4}\,VU^2 + \frac{3}{32}\,VU^4 + \mathcal{O}(U^6)$$

Non canonical dimensionless cartesian coordinates

$$\xi_1 = U\,\sin p \qquad \eta_1 = U\,\cos p$$
$$\xi_2 = V\,\sin q \qquad \eta_2 = V\,\cos q$$

Hamiltonian

$$\begin{aligned}
\mathcal{H}_{pot} &= \mathcal{H}_{2b} + \dot{\theta}\,\Lambda + \sum_{n=2}^{n_{max}} \mathcal{R}_{pot}^{(n)} + \sum_{i=1}^{2} \mathcal{H}_i \\
&= -\frac{\mu^2}{2\,L^2} + \dot{\theta}\,\Lambda + \sum_{n=2}^{n_{max}} \frac{1}{L^{2n+2}} \sum_{j=1}^{N_n} \mathcal{A}_j^{(n)}(\xi_1,\eta_1,\xi_2,\eta_2)\,\mathcal{B}_j^{(n)}(\lambda,\theta) \\
&+ \sum_{i=1}^{2} \sum_{n=2}^{n_{max}} \frac{L^{2n}}{r_i^{n+1}} \sum_{j=1}^{N_n} \mathcal{C}_j^{(n)}(\xi_1,\eta_1,\xi_2,\eta_2,X_i,Y_i,Z_i)\,\mathcal{D}_j^{(n)}(\lambda)
\end{aligned}$$

Dynamical system

$$\dot{\xi}_i = \frac{1}{L}\frac{\partial \mathcal{H}}{\partial \eta_i} \qquad \dot{\eta}_i = -\frac{1}{L}\frac{\partial \mathcal{H}}{\partial \xi_i} \qquad i = 1,2$$

$$\dot{\lambda} = \frac{\partial \mathcal{H}}{\partial L} - \frac{1}{2L}\left[\sum_{i=1}^{2} \frac{\partial \mathcal{H}}{\partial \xi_i}\xi_i + \sum_{i=1}^{2} \frac{\partial \mathcal{H}}{\partial \eta_i}\eta_i\right] \qquad \dot{L} = -\frac{\partial \mathcal{H}}{\partial \lambda}$$

Semi-analytical averaged method

- Use of a series manipulator

λ	θ	ξ_1	η_1	ξ_2	η_2	L	X	Y	Z	r	$X_\odot$	$Y_\odot$	$Z_\odot$	$r_\odot$	Coefficient
cos (0	0)	(0	0	0	0	-6	0	0	0	0	0	0	0	0)	0.12386619D-04
cos (0	0)	(0	0	0	2	-6	0	0	0	0	0	0	0	0)	-0.18579928D-04
cos (0	0)	(0	0	0	4	-6	0	0	0	0	0	0	0	0)	0.46449822D-05

- Averaging process over the fast variable : λ
- Semi-analytical averaged solution

Number of terms

Perturbation	Number of terms			
n-order expansion				
$\xi_1^{i_1}\eta_1^{i_2}\xi_2^{i_3}\eta_2^{i_4}$ with $i_1+i_2+i_3+i_4 \leq n$	$n=2$	$n=4$	$n=6$	$n=8$
Geopotential				
$\mathcal{H}_{J_2}$	5	15	31	53
	(33)	(145)	(410)	(895)
External Body - Sun & Moon				
up to degree 2	27	86	197	390
	(205)	(836)	(2374)	(5480)
up to degree 3	73	250	611	1227
	(645)	(2642)	(7854)	(18380)

See also STELA (Deleflie - CNRS)

The geopotential: Kaula formulation

$$U = -\frac{\mu}{r} - \sum_{n=2}^{\infty} \sum_{m=0}^{n} \sum_{p=0}^{n} \sum_{q=-\infty}^{+\infty} \frac{\mu}{a} \left(\frac{R_e}{a} \right)^n F_{nmp}(i)\, G_{npq}(e)\, S_{nmpq}(\Omega, \omega, M, \theta)$$

$$\begin{aligned} S_{nmpq}(\Omega, \omega, M, \theta) &= \begin{bmatrix} +C_{nm} \\ -S_{nm} \end{bmatrix}_{n-m \text{ odd}}^{n-m \text{ even}} \cos \Theta_{nmpq}(\Omega, \omega, M, \theta) \\ &+ \begin{bmatrix} +S_{nm} \\ +C_{nm} \end{bmatrix}_{n-m \text{ odd}}^{n-m \text{ even}} \sin \Theta_{nmpq}(\Omega, \omega, M, \theta) \end{aligned}$$

Kaula gravitational argument, θ the sidereal time :

$$\Theta_{nmpq}(\Omega, \omega, M, \theta) = (n-2p)\,\omega + (n-2p+q)\,M + m(\Omega - \theta)$$

Gravitational resonances : resonances with the Earth rotation

- $\frac{P_\oplus}{P_{obj}} = \frac{q_1}{q_2}$
- $P_\oplus$: Earth's rotational period : $2\pi/n_\oplus = 1$ day $(n_\oplus = \dot{\theta})$
- P_{obj} : body orbital period : $2\pi/n = P_{obj}$ day $(n = \dot{M})$
- 1/1 for GEO and 2/1 for MEO
- $\Theta_{nmpq}(\Omega, \omega, M, \theta) = (n-2p)\,\omega + (n-2p+q)\,M + m(\Omega-\theta)$
- $\dot{\Theta}_{nmpq}(\dot{\Omega}, \dot{\omega}, \dot{M}, \dot{\theta}) = (n-2p)\,\dot{\omega} + (n-2p+q)\,\dot{M} + m(\dot{\Omega}-\dot{\theta}) \simeq 0$
- $q = 0$: $\frac{\dot{M}}{\dot{\theta}} \simeq \frac{\dot{\lambda}}{\dot{\theta}} \simeq \frac{q_1}{q_2}$
- Resonant Hamiltonian $\mathcal{H}_{J_{22}}$

Geostationary model of resonance

- Cartesian Hamiltonian coordinates for e, i, ϖ, Ω : ξ_i and η_i
- $\mathcal{H} = \mathcal{H}_{J_{22}}(\xi_1, \eta_1, \xi_2, \eta_2, \Lambda, \lambda, L, \theta) + \dot{\theta}\,\Lambda$
- Resonant angle : $\sigma = \lambda - \theta$
- Corrected momentum : $L' = L, \qquad \theta' = \theta, \qquad \Lambda' = \Lambda + L$
- $\mathcal{H} = \mathcal{H}_{J_{22}}\left(\xi_1, \eta_1, \xi_2, \eta_2, \sigma, L', \theta\right) + \dot{\theta}\,\left(\Lambda' - L'\right)$

Resonant averaging

$$\mathcal{H}_{J_{22}}\left(\xi_1, \eta_1, \xi_2, \eta_2, L, \Lambda, \theta, \lambda\right)$$
$$\downarrow$$
$$\mathcal{H}_{J_{22}}\left(\xi_1, \eta_1, \xi_2, \eta_2, L', \Lambda', \theta', \sigma\right)$$
$$\downarrow$$
$$\overline{\mathcal{H}}_{J_{22}}\left(\bar{\xi}_1, \bar{\eta}_1, \bar{\xi}_2, \bar{\eta}_2, \bar{L}', \bar{\Lambda}', -, \bar{\sigma}\right)$$

Resonant averaged hamiltonian

Perturbation	Number of terms			
n-order expansion				
$\xi_1^{i_1}\eta_1^{i_2}\xi_2^{i_3}\eta_2^{i_4}$ with $i_1+i_2+i_3+i_4 \leq n$	$n=2$	$n=4$	$n=6$	$n=8$
Resonant disturbing function				
$\mathcal{H}_{J_{22}} = \mathcal{H}_{C_{22}} + \mathcal{H}_{S_{22}}$	10	40	104	206
	(94)	(468)	(1392)	(3178)

σ	θ	ξ_1	η_1	ξ_2	η_2	L	X	Y	Z	r	$X_\odot$	$Y_\odot$	$Z_\odot$	$r_\odot$	Coefficient
cos (2	0)	(0	0	0	0	-6	0	0	0	0	0	0	0	0)	0.1077767255D-06
cos (2	0)	(0	0	0	0	-6	0	0	0	0	0	0	0	0)	0.1080907167D-06
sin (2	0)	(0	0	0	0	-6	0	0	0	0	0	0	0	0)	-0.6204881922D-07

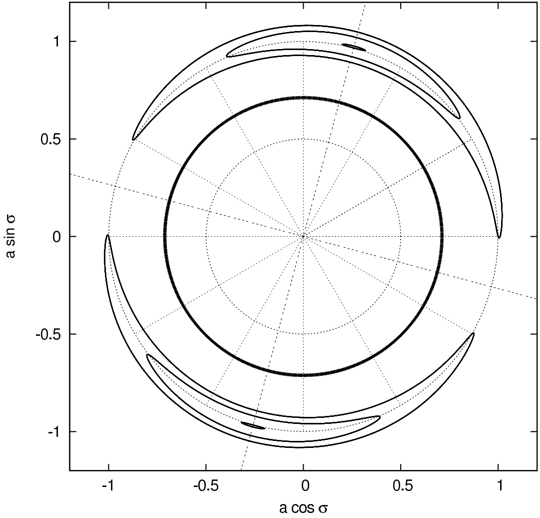
Simple resonant model

- $\mathcal{H}(L, \sigma, \Lambda) = -\frac{\mu^2}{2L^2} + \dot{\theta}(\Lambda - L) + \frac{1}{L^6}\left[\alpha_1 \cos 2\sigma + \alpha_2 \sin 2\sigma\right]$
- $\alpha_1 \simeq 0.1077 \times 10^{-6}, \qquad \alpha_2 \simeq -0.6204 \times 10^{-7}$
- Equilibria : $\frac{\partial \mathcal{H}}{\partial L} = 0 = \frac{\partial \mathcal{H}}{\partial \sigma}$
- Two stable equilibria $(\sigma^*_{11}, L^*_{11})$, $(\sigma^*_{12}, L^*_{12})$
- Two unstable equilibria $(\sigma^*_{21}, L^*_{21})$, $(\sigma^*_{22}, L^*_{22})$ are found to

$$\sigma^*_{11} = \lambda^* \qquad \sigma^*_{12} = \lambda^* + \pi$$
$$\sigma^*_{21} = \lambda^* + \frac{\pi}{2} \qquad \sigma^*_{22} = \lambda^* + \frac{3\pi}{2},$$

- $L^*_{11} = L^*_{12} = 0.99999971, \qquad L^*_{21} = L^*_{22} = 1.00000029,$
- $L = 1$ corresponds to 42 164 km.
- $\lambda^* \simeq 75.07°$

Resonant phase space



Resonant period

- $x = \sqrt{2L}\cos\sigma$, $y = \sqrt{2L}\sin\sigma$ and consequently x^*, y^*.
- Taylor series around (x^*, y^*)
- $X = (x - x^*), \qquad Y = (y - y^*)$
- $\mathcal{H}^*(X, Y, \Lambda) = \dot{\theta}\,\Lambda + \frac{1}{2}(aX^2 + 2bXY + cY^2) + \cdots$
- Rotation : $X = \ p\cos\Psi + q\sin\Psi$ and $Y = -p\sin\Psi + q\cos\Psi$
- Choice of Ψ : $(a - c)\sin 2\Psi + 2b\cos 2\Psi = 0$
- $\mathcal{H}^*(p, q, \Lambda) = \dot{\theta}\,\Lambda + \frac{1}{2}\left[A\,p^2 + C\,q^2\right]$
- Scaling : $p = \alpha\,p'$ and $q = \frac{1}{\alpha}\,q'$ by $A\,\alpha^2 = \dfrac{C}{\alpha^2}$,
- $\mathcal{H}(J, \phi, \Lambda) = \dot{\theta}\,\Lambda + \sqrt{AC}\,J$
- Action-angle (J, ϕ) : $p' = \sqrt{2J}\cos\phi\,, \quad q' = \sqrt{2J}\cos\phi\,.$
- $\nu_f = \frac{\partial\mathcal{H}}{\partial J} = \sqrt{AC} = 7.674 \times 10^{-3}/d$, period of 818.7 days.

Resonant motion

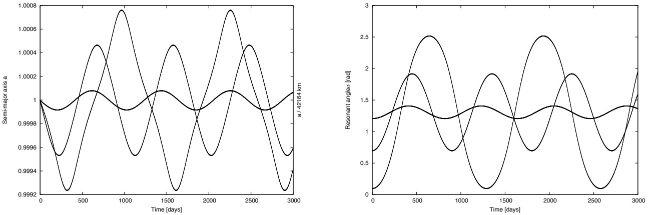


Fig. 6. Semi-major axis a [left] and resonant angle $\sigma = \lambda - \theta$ [right] of several geosynchronous space debris [$a_0 = 42164\ km$, $e_0 = 0$, $i_0 = 0$] the initial longitude of which are $\lambda_0 = 5°, 35°, 75°$.

Resonant motion

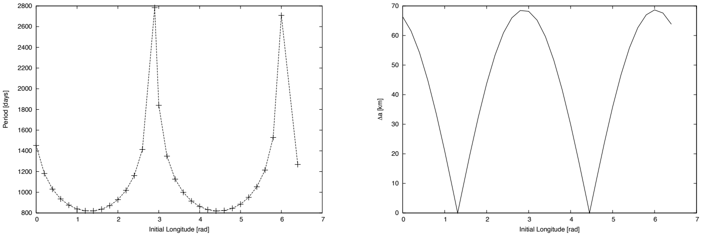


Fig. 7. Libration periods of 32 virtual space debris the initial longitude λ_0 of which varied from 0 to 2π.

Width of the resonant zone

- Hamiltonian level curve corresponding to one of the unstable equilibria L_u and σ_u

$$\mathcal{H}(L_u, \sigma_u, \Lambda) = -\frac{\mu^2}{2L^2} + \dot{\theta}(\Lambda - L) + \frac{1}{L^6}\left[\alpha_1 \cos 2\sigma + \alpha_2 \sin 2\sigma\right]$$

- Maxima and minima of this "banana curve", corresponding to the stable equilibria
- Quadratic approximation about L_u : the width Δ of the resonant zone

$$\Delta = \sqrt{\frac{\gamma^2 + 8\delta\beta}{\beta^2}} \quad \delta = \frac{\alpha_1}{L_u^6 \cos 2\sigma_u} \quad \beta = -\frac{3}{2}\frac{\mu^2}{L_u^4} \quad \gamma = \frac{\mu^2}{L_u^3} - \dot{\theta}$$

- The numerical value is of the order of 69 km.

Generalization

- Similar approach : Rossi on MEO (resonance 2:1) CM&DA
- Paper of Celletti and Gales : On the Dynamics of Space Debris: 1:1 and 2:1 Resonances (JNS) 2014
- Very complete paper :

Celest Mech Dyn Astr (2015) 123:203–222
DOI 10.1007/s10569-015-9636-1

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ORIGINAL ARTICLE

Dynamical investigation of minor resonances for space debris

Alessandra Celletti[1] · Cătălin Galeş[2]

Resonant motion

Table 2 Value of the semimajor axis corresponding to several resonances

$j : \ell$	a (km)	$j : \ell$	a (km)
1:1	42164.2	4:3	34805.8
2:1	26561.8	5:1	14419.9
3:1	20270.4	5:2	22890.2
3:2	32177.3	5:3	29994.7
4:1	16732.9	5:4	36336

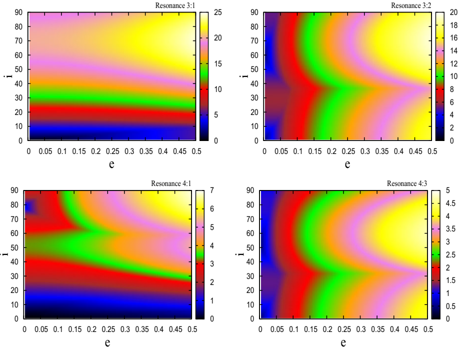
Resonant motion

Table 3 Terms whose sum provides the expression of $R_{earth}^{res\ j:\ell}$ up to the order N

$j:\ell$	N	Terms
3:1	4	$\mathcal{T}_{330\text{-}2}, \mathcal{T}_{3310}, \mathcal{T}_{3322}, \mathcal{T}_{431\text{-}1}, \mathcal{T}_{4321}$
3:2	4	$\mathcal{T}_{330\text{-}1}, \mathcal{T}_{3311}, \mathcal{T}_{430\text{-}2}, \mathcal{T}_{4310}, \mathcal{T}_{4322}$
4:1	6	$\mathcal{T}_{441\text{-}1}, \mathcal{T}_{4421}, \mathcal{T}_{541\text{-}2}, \mathcal{T}_{5420}, \mathcal{T}_{5432}, \mathcal{T}_{642\text{-}1}, \mathcal{T}_{6431}$
4:3	5	$\mathcal{T}_{440\text{-}1}, \mathcal{T}_{4411}, \mathcal{T}_{540\text{-}2}, \mathcal{T}_{5410}, \mathcal{T}_{5422}$
5:1	6	$\mathcal{T}_{551\text{-}2}, \mathcal{T}_{5520}, \mathcal{T}_{5532}, \mathcal{T}_{652\text{-}1}, \mathcal{T}_{6531}$
5:2	6	$\mathcal{T}_{551\text{-}1}, \mathcal{T}_{5521}, \mathcal{T}_{651\text{-}2}, \mathcal{T}_{6520}, \mathcal{T}_{6532}$
5:3	6	$\mathcal{T}_{550\text{-}2}, \mathcal{T}_{5510}, \mathcal{T}_{5522}, \mathcal{T}_{651\text{-}1}, \mathcal{T}_{6521}$
5:4	6	$\mathcal{T}_{550\text{-}1}, \mathcal{T}_{5511}, \mathcal{T}_{650\text{-}2}, \mathcal{T}_{6510}, \mathcal{T}_{6522}$

Resonant motion

Fig. 2 The amplitude of the resonances for different values of the eccentricity (within 0 and 0.5 on the x axis) and the inclination (within $0°$ and $90°$ on the y axis) for $\omega = 0°$, $\Omega = 0°$; the *color bar* provides the measure of the amplitude in kilometers. In order from *top left* to *bottom right*: 3:1, 3:2, 4:1, 4:3, 5:1, 5:2, 5:3, 5:4



Solar Radiation pressure

- Solar radiation pressure is a quite complicated force with different components
- *Theory of Orbit determination* : Milani and Gronchi - ch 14
- *New solar Radiation Pressure Force Model for navigation* : McMahon and Scheeres - 2010
- Direct radiation pressure acceleration
- Starting point : simplified models

Scheeres and Rosengren : Averaged model, based on e and angular momentum

Long-term Dynamics of HAMR Objects in HEO

Aaron Rosengren*, Daniel Scheeres†
University of Colorado at Boulder, Boulder, CO 80309

Gachet, Celletti, Pucacco, Efthymiopoulos : Complete perturbation theory with planetary motion

Celest Mech Dyn Astr (2017) 128:149–181
DOI 10.1007/s10569-016-9746-4
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ORIGINAL ARTICLE

Geostationary secular dynamics revisited: application to high area-to-mass ratio objects

Fabien Gachet[1] · Alessandra Celletti[1] · Giuseppe Pucacco[3] · Christos Efthymiopoulos[2]

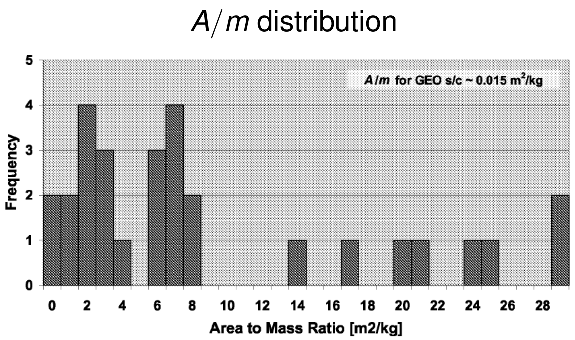
Direct radiation pressure acceleration

The acceleration due to the direct radiation pressure can be written in the form:

$$\mathbf{a_{rp}} = C_r\, P_r \left[\frac{a_\odot}{\|\mathbf{r} - \mathbf{r}_\odot\|} \right]^2 \frac{A}{m} \frac{\mathbf{r} - \mathbf{r}_\odot}{\|\mathbf{r} - \mathbf{r}_\odot\|},$$

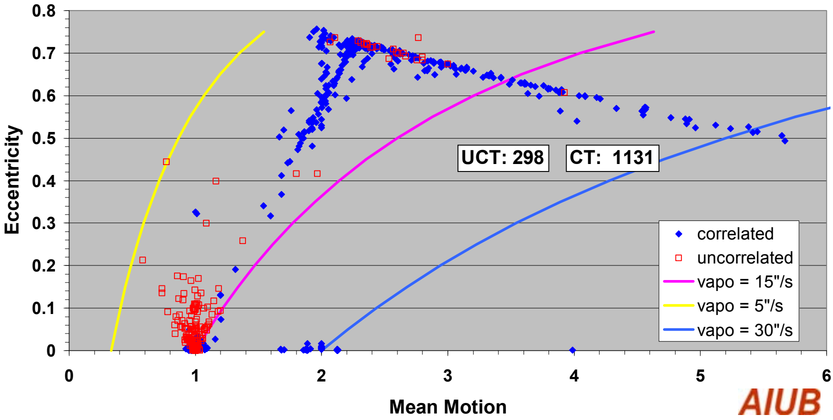
- C_r is the non-dimensional reflectivity coefficient ($0 < C_r < 2$),
- $P_r = 4.56 \cdot 10^{-6}\ N/m^2$ is the radiation pressure per unit of mass for an object located at a distance of $a_\odot = 1\ AU$,
- $\mathbf{r}$ is the geocentric position of the space debris; $\mathbf{r}_\odot$ is the geocentric position of the Sun,
- A is the exposed area to the Sun of the space debris,
- m is the mass of the space debris.

Perturbations & A/m distribution



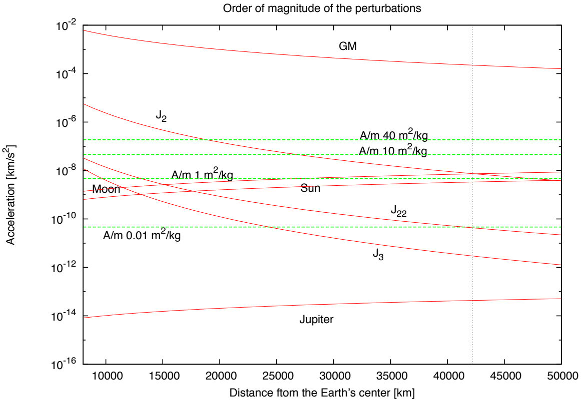
Object	A/m $\mathrm{m^2/kg}$
Lageos 1 and 2	0.0007
Starlette	0.001
GPS (Block II)	0.02
Moon	$1.3 \cdot 10^{-10}$
Space debris	$0 < A/m <$?

GEO debris with very high eccentricity



Schildknecht et al, 2010

Order of magnitude of radiation pressure



Chao 2009

Hamiltonian formulation

$$\mathcal{H}(\mathbf{v}, \mathbf{r}) = \mathcal{H}_{kepl}(\mathbf{v}, \mathbf{r}) + \mathcal{H}_{srp}(\mathbf{r})$$

fixed inertial equatorial geocentric frame

$\mathbf{r}$	$=$	geocentric position of the satellite
$\mathbf{v}$	$=$	velocity of the satellite
$\mathcal{H}_{kepl}(\mathbf{v}, \mathbf{r})$	$=$	attraction of the Earth
$\mathcal{H}_{srp}(\mathbf{r})$	$=$	direct solar radiation pressure potential

$$\mathcal{H}_{kepl} = \frac{\|\mathbf{v}\|^2}{2} - \frac{\mu}{\|\mathbf{r}\|}$$

$$\mathcal{H}_{srp} = -C_r \, \frac{1}{\|\mathbf{r} - \mathbf{r}_\odot\|} \, P_r \, \frac{A}{m} \, a_\odot^2$$

$\mu = \mathcal{G}M_\oplus$, $C_r \simeq 1$, $\mathbf{r}_\odot$ position of the Sun, $P_r = 4.56 \times 10^{-6}\ N/m^2$, A/m area-to-mass ratio, $a_\odot = 1$ AU.

Polynômes de Legendre : first order

The toy model

$$\mathcal{H} = -\frac{\mu^2}{2L^2} + C_r\, P_r\, \frac{A}{m}\, r\, \bar{r}_\odot\, \cos(\phi)$$

ϕ the angle between $\mathbf{r}$ and $\mathbf{r}_\odot$, $\quad L = \sqrt{\mu a}$, $\quad \bar{r}_\odot = \frac{r_\odot}{a_\odot}$.

$$\begin{aligned}\mathcal{H} &= -\frac{\mu^2}{2L^2} + C_r\, P_r\, \frac{A}{m}\, a\,(u\,\xi + v\,\eta) \\ &= H(L, G, H, M, \omega, \Omega, r_\odot)\end{aligned}$$

Debris orbital motion : $u = \cos E - e$ and $v = \sin E\, \sqrt{1 - e^2}$.

Debris orbit orientation and Sun orbital motion :

$$\begin{aligned}\xi &= \xi_1\, \bar{r}_{\odot,1} + \xi_2\, \bar{r}_{\odot,2} + \xi_3\, \bar{r}_{\odot,3} \\ \eta &= \eta_1\, \bar{r}_{\odot,1} + \eta_2\, \bar{r}_{\odot,2} + \eta_3\, \bar{r}_{\odot,3}\end{aligned}$$

$$\begin{aligned}\xi_1 &= \cos\Omega\ \cos\omega - \sin\Omega\ \cos i\ \sin\omega & \eta_1 &= -\cos\Omega\ \sin\omega - \sin\Omega\ \cos i\ \cos\omega \\ \xi_2 &= \sin\Omega\ \cos\omega + \cos\Omega\ \cos i\ \sin\omega & \eta_2 &= -\sin\Omega\ \sin\omega + \cos\Omega\ \cos i\ \cos\omega \\ \xi_3 &= \sin i\ \sin\omega & \eta_3 &= \sin i\ \cos\omega\end{aligned}$$

Averaging over the short periods : 1 day

Periods : 1 day (Orbital motion E) and 1 year (Sun $\overline{r}_{\odot,i}$)
Averaging over the fast variable (M the mean anomaly) :

$$\begin{aligned}\overline{\mathcal{H}} &= \frac{1}{2\pi}\int_0^{2\pi} \mathcal{H}\, dM \\ &= -\frac{\mu^2}{2\overline{L}^2} + \frac{1}{2\pi}\, C_r\, P_r\, \frac{A}{m}\, \overline{a} \int_0^{2\pi} (u\,\xi + v\,\eta)\, dM\end{aligned}$$

$dM = (1 - e\cos E)\, dE$

$$\begin{aligned}\overline{\mathcal{H}} &= -\frac{\mu^2}{2\overline{L}^2} - \frac{3}{2}\, C_r\, P_r\, \frac{A}{m}\, \frac{\overline{L}^2}{\mu}\, \overline{e}\, \xi \\ &= \overline{\mathcal{H}}(\overline{L}, \overline{G}, \overline{H}, -, \overline{\varpi}, \overline{\Omega}, r_\odot)\end{aligned}$$

The development

$$\overline{\mathcal{H}} = -\frac{\mu^2}{2L^2} - \frac{3}{2}\, C_r\, P_r\, \frac{A}{m}\, \frac{L^2}{\mu}\, e\, \xi$$

Poincaré variables :

$$\begin{aligned} p &= -\varpi & P &= L - G \\ q &= -\Omega & Q &= G - H \\ x_1 &= \sqrt{2P}\, \sin p & y_1 &= \sqrt{2P}\, \cos p \\ x_2 &= \sqrt{2Q}\, \sin q & y_2 &= \sqrt{2Q}\, \cos q \end{aligned}$$

Approximations : $e \simeq \sqrt{\frac{2P}{L}}$, $\cos^2 \frac{i}{2} = 1 - \frac{Q}{2L}$, $\sin \frac{i}{2} \simeq \sqrt{\frac{Q}{2L}}$

Circular orbit for the Sun (obliquity ϵ)

$$\begin{aligned} \bar{r}_{\odot,1} &= \cos \lambda_\odot \\ \bar{r}_{\odot,2} &= \sin \lambda_\odot \cos \epsilon \\ \bar{r}_{\odot,3} &= \sin \lambda_\odot \sin \epsilon \end{aligned}$$

with $\lambda_\odot = n_\odot t + \lambda_{\odot,0}$.

The truncated Hamiltonian in e and i

$$\begin{aligned}\mathcal{H} &= \mathcal{H}(x_1, y_1, x_2, y_2, \lambda_\odot) \\ &\simeq -n_\odot\,\kappa\,\bar{r}_{\odot,1}\,(x_1\,R_2 + y_1\,R_1) \\ &+ \;n_\odot\,\kappa\,\bar{r}_{\odot,2}\,(x_1\,R_3 + y_1\,R_2) \\ &+ \;n_\odot\,\kappa\,\bar{r}_{\odot,3}\,(x_1\,R_5 - y_1\,R_4)\end{aligned}$$

$\kappa = \frac{3}{2}\,C_r\,P_r\,\frac{A}{m}\,\frac{a}{\sqrt{L}}$

$R_i(x_2, y_2)$ are second degree polynomials in x_2 and y_2.

Dynamical system associated :

$$\begin{aligned}\dot{x}_1 &= \frac{\partial \mathcal{H}}{\partial y_1} & \qquad \dot{y}_1 &= -\frac{\partial \mathcal{H}}{\partial x_1} \\ \dot{x}_2 &= \frac{\partial \mathcal{H}}{\partial y_2} & \qquad \dot{y}_2 &= -\frac{\partial \mathcal{H}}{\partial x_2}.\end{aligned}$$

The eccentricity - pericenter motion : x_1 and y_1

$$\boxed{x_2 = 0 = y_2}$$

$$\begin{aligned}
\dot{x}_1 &= -n_\odot \kappa \, \bar{r}_{\odot,1} \\
\dot{y}_1 &= -n_\odot \kappa \, \bar{r}_{\odot,2}
\end{aligned}$$

Solution explicitly given by

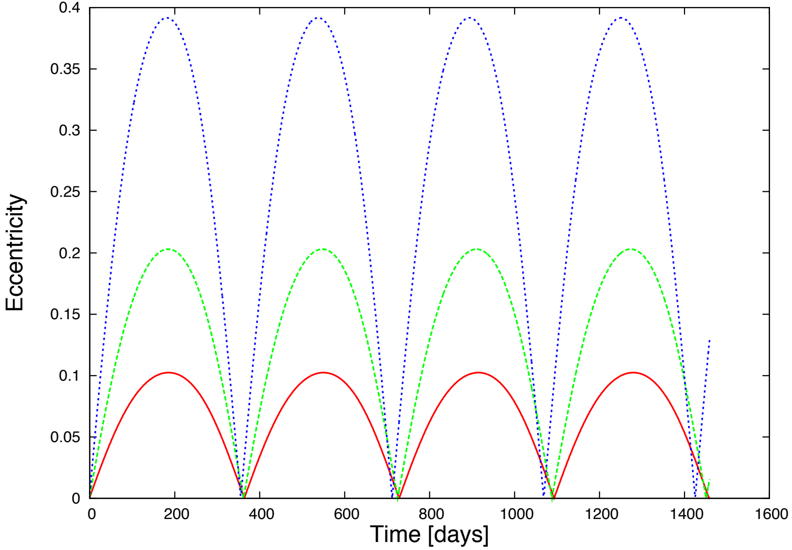
$$\begin{aligned}
x_1 &= -\kappa \, \sin \lambda_\odot + C_x &&= -\kappa \, (\sin \lambda_\odot - D_x) \\
y_1 &= \kappa \, \cos \lambda_\odot \, \cos \epsilon + C_y &&= \kappa \, (\cos \lambda_\odot \, \cos \epsilon + D_y).
\end{aligned}$$

e and ϖ : a periodic motion (1 year)
κ increases, e_{max} increases

Explanation of the behavior of GEO space debris (high e)

The eccentricity - pericenter motion : 1 year

$A/m = 5\ m^2/kg$ $\quad A/m = 10\ m^2/kg$ $\quad A/m = 20\ m^2/kg$



$x_2 \neq 0 \neq y_2$

$$\mathcal{H} = \mathcal{H}(x_1(\lambda_\odot), y_1(\lambda_\odot), R_i(x_2, y_2), \lambda_\odot)$$

Averaged equations over $\lambda_\odot$: system of mean linear equations

$$\begin{aligned} \dot{\bar{x}}_2 &= \nu\, \bar{y}_2 - \rho \\ \dot{\bar{y}}_2 &= -\nu\, \bar{x}_2 \end{aligned}$$

$\nu = n_\odot\ \kappa^2\ \cos\epsilon\ \frac{1}{2L}, \quad \rho = n_\odot\ \kappa^2\ \sin\epsilon\ \frac{1}{2\sqrt{L}}$

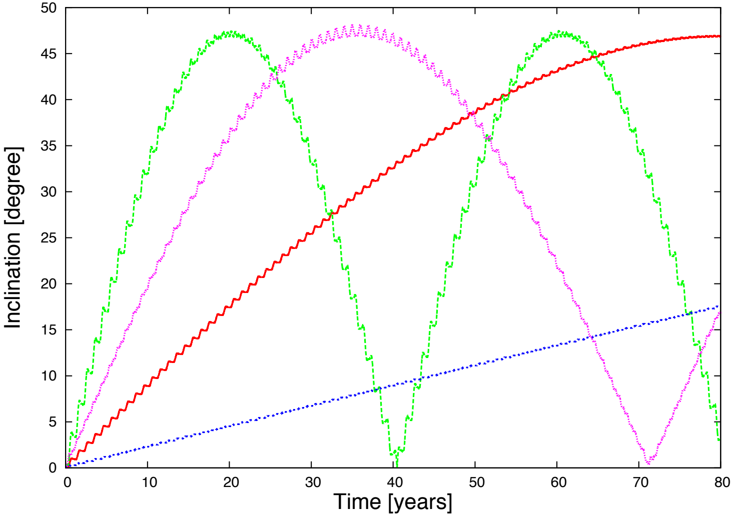
Solution : $\left\{ \begin{aligned} \bar{x}_2 &= \mathcal{A}\sin\psi \\ \bar{y}_2 &= \mathcal{A}\cos\psi - \frac{\rho}{\nu} = \mathcal{A}\cos\psi - \tan\epsilon\sqrt{L} \end{aligned} \right.$

$\psi = \nu\, t + \psi_0$

i and Ω : a periodic motion (dozens of years) with $i_{max} \simeq 2\epsilon$
κ increases, ν increases and the period decreases.

The inclination - node motion : dozens of years

$A/m = 5\ m^2/kg$ $A/m = 10\ m^2/kg$ $A/m = 20\ m^2/kg$ $A/m = 40\ m^2/kg$



The inclination and eccentricity combined motion

Back to the averaging process

$$\mathcal{K} = n_\odot \, \Lambda_\odot - n_\odot \, \kappa^2 \, f_0(x_2, y_2) - n_\odot \, \kappa^2 \, f_1(x_2, y_2, \lambda_\odot)$$

$$\begin{aligned} f_0(x_2, y_2) &= \frac{1}{2} \left(R_1 \cos\epsilon + R_3 \cos\epsilon + R_5 \sin\epsilon \right) \\ f_1(x_2, y_2, \lambda_\odot) &= g_1 \, \cos\lambda_\odot + g_2 \, \sin\lambda_\odot + g_3 \, \cos 2\lambda_\odot + g_4 \, \sin 2\lambda_\odot \end{aligned}$$

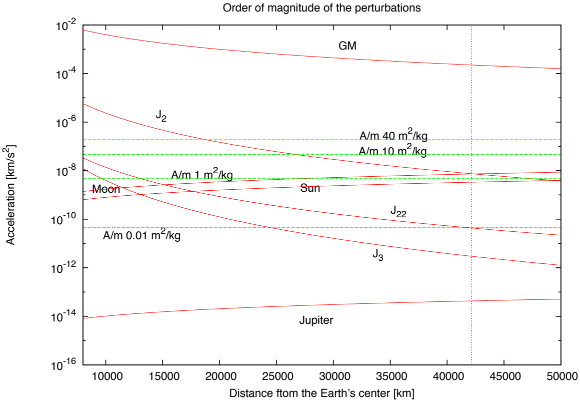
with $g_i = g_i(x_2, y_2)$ and $R_i = R_i(x_2, y_2)$.

The homological equation : $\boxed{\bar{\mathcal{H}}_1 = \mathcal{H}_1 + \{\mathcal{H}_0; \mathcal{W}\} = \mathcal{H}_1 - \frac{\partial \mathcal{H}_0}{\partial \Lambda_\odot} \, \frac{\partial \mathcal{W}}{\partial \lambda_\odot}}$

$$\mathcal{W} = -\kappa^2 \left(g_1 \, \sin\lambda_\odot - g_2 \, \cos\lambda_\odot + \frac{1}{2} \, g_3 \, \sin 2\lambda_\odot - \frac{1}{2} \, g_4 \, \cos 2\lambda_\odot \right)$$

$$x_2 = \bar{x}_2 + \frac{\partial \mathcal{W}}{\partial y_2}(\lambda_\odot) \qquad y_2 = \bar{y}_2 - \frac{\partial \mathcal{W}}{\partial x_2}(\lambda_\odot)$$

Order of magnitude of radiation pressure



Other perturbations

J_2

$$H_{J_2}(\vec{r}) = \frac{\mu}{r} J_2 \left(\frac{r_\oplus}{r}\right)^2 P_2 (\sin \phi_{sat})$$
$$= \frac{\mu}{r} J_2 \left(\frac{r_\oplus}{r}\right)^2 \frac{1}{2} \left(3 \left(\frac{z}{r}\right)^2 - 1\right)$$

where ϕ_{sat} represents the latitude of the satellite, and consequently $\sin \phi_{sat} = z/r$.

SRP second order

$$H_{SRP}(\vec{r}, \vec{r}_\odot) = -C_r P_r \frac{A}{m} a_\odot^2 \frac{1}{||\vec{r} - \vec{r}_\odot||}$$
$$\simeq -C_r P_r \frac{A}{m} a_\odot^2 \sum_{n=1}^{n=2} \left(\frac{r}{a_\odot}\right)^n P_n(\cos \phi)$$

Third body : Sun on a circular orbit

$$
\begin{aligned}
H_{3bS}(\vec{r},\vec{r}_\odot) &= -\mu_\odot \frac{1}{||\vec{r}-\vec{r}_\odot||} + \mu_\odot \frac{\vec{r}\cdot\vec{r}_\odot}{||\vec{r}_\odot||^3} \\
&\simeq -\frac{\mu_\odot}{a_\odot}\sum_{n\geq 0}\left(\frac{r}{a_\odot}\right)^n P_n(\cos\phi) + \mu_\odot \frac{r a_\odot \cos(\phi)}{a_\odot^3} \\
&\simeq -\frac{\mu_\odot}{a_\odot}(1+\left(\frac{r}{a_\odot}\right)^2 P_2(\cos\phi)),
\end{aligned}
$$

where $\mu_\odot = GM_\odot$ with $M_\odot$ the mass of the Sun.

Third body : Moon on a circular orbit

$$
H_{3bM}(\vec{r},\vec{r}_{☾}) = -\frac{\mu_{☾}}{a_{☾}}(1+\sum_{n\geq 2}\left(\frac{r}{a_{☾}}\right)^n P_n(\cos\phi_M))
$$

where $\mu_{☾} = GM_{☾}$ with $M_{☾}$ the mass of the Moon, and ϕ_M the angle between the satellite and the Moon

The Sun contributions

$$
\begin{aligned}
& H_{SRP}(\vec{r},\vec{r}_\odot) + H_{3bS}(\vec{r},\vec{r}_\odot) \\
\simeq \;& H_{SRP_1}(\vec{r},\vec{r}_\odot) + H_{SRP_2}(\vec{r},\vec{r}_\odot) + H_{3bS}(\vec{r},\vec{r}_\odot) \\
\simeq \;& C_r P_r \frac{A}{m} a_\odot \, r \cos(\phi) \\
+ \;& \left[C_r P_r \frac{A}{m} a_\odot - \frac{\mu_\odot}{a_\odot} \right] \left(\frac{r}{a_\odot} \right)^2 P_2(\cos\phi)
\end{aligned}
$$

Averaging over daily period :

$$
\begin{aligned}
\overline{H}(x_1,y_1,x_2,y_2) = \;& \overline{H}_{kepler} + \overline{H}_{J_2}(x_1,y_1,x_2,y_2) \\
+ \;& \overline{H}_{SRP_1}(x_1,y_1,x_2,y_2,\vec{r}_\odot) \\
+ \;& \overline{H}_{SRP_2+3bS}(x_1,y_1,x_2,y_2,\vec{r}_\odot) \\
+ \;& \overline{H}_{3bM}(x_1,y_1,x_2,y_2,\vec{r}_{☾})
\end{aligned}
$$

Averaging results

$$\overline{H}_{J_2} = C_p\, P + C_q\, Q = \frac{C_p}{2}(x_1^2 + y_1^2) + \frac{C_q}{2}(x_2^2 + y_2^2)$$

$$\overline{H}_{SRP_1} = -\frac{3}{2}\, C_r P_r\, \frac{A}{m}\, a\, e\, \xi$$

$$\overline{H}_{SRP_2+3bS} = -\left[C_r P_r \frac{A}{m} a_\odot - \frac{\mu_\odot}{a_\odot}\right] \frac{3a^2}{4a_\odot^2} w^2 = -\beta\, \frac{3a^2}{4a_\odot^2} w^2$$

$$\overline{H}_{3bM} = \frac{\mu_☾}{a_☾}\, \frac{3a^2}{4a_☾^2} w_M^2$$

$$w = -\sin q\, \sin i\, \vec{r}_{\odot,1} - \cos q\, \sin i\, \vec{r}_{\odot,2} + \cos i\, \vec{r}_{\odot,3}$$
$$w_M = -\sin q\, \sin i\, \vec{r}_{☾,1} - \cos q\, \sin i\, \vec{r}_{☾,2} + \cos i\, \vec{r}_{☾,3}$$

Short periodic motion : Kepler + J2 + SRP1

$$\begin{aligned} \dot{x}_1(t) &= -C_2\, y_1 - n_\odot\, k\, r_{\odot,1}, \\ \dot{y}_1(t) &= C_2\, x_1 - n_\odot\, k\, r_{\odot,2}, \end{aligned}$$

$C_2 = \frac{3}{2}\sqrt{\frac{\mu}{a^3}} J_2 \frac{r_\oplus^2}{a^2}$

$$\begin{aligned} x_1(t) &= C_x + \frac{k \sin(n_\odot t + \lambda_{\odot,0})}{1 - eta^2} [\eta \cos\epsilon + 1], \\ y_1(t) &= C_y + \frac{k \cos(n_\odot t + \lambda_{\odot,0})}{1 - \eta^2} [\cos\epsilon + \eta], \end{aligned}$$

Long periodic motion

$$
\begin{aligned}
\dot{x}_2(t) &= C_q\, y_2 - n_\odot k \left[r_{\odot,1}\left(\frac{x_1 x_2}{2L}\right) - r_{\odot,2}\left(\frac{-2x_1 y_2}{2L} + \frac{y_1 x_2}{2L}\right) - r_{\odot,3}\left(\frac{x_1}{\sqrt{L}}\right. \right. \\
&+ \frac{\partial \bar{H}_{SRP_2+3bS}}{\partial y_2} + \frac{\partial \bar{H}_{3bM}}{\partial y_2} \\
\dot{y}_2(t) &= -C_q\, x_2 + n_\odot k \left[r_{\odot,1}\left(\frac{-2x_2 y_1}{2L} + \frac{x_1 y_2}{2L}\right) - r_{\odot,2}\left(\frac{y_1 y_2}{2L}\right) - r_{\odot,3}\left(- \right. \right. \\
&- \frac{\partial \bar{H}_{SRP_2+3bS}}{\partial x_2} - \frac{\partial \bar{H}_{3bM}}{\partial x_2}.
\end{aligned}
$$

Averaging over the motion of the Sun and of the Moon

$$
\begin{aligned}
\dot{x_2}(t) &= d_1 \; y_2 + d_3, \\
\dot{y_2}(t) &= -d_2 \; x_2,
\end{aligned}
$$

$$
\begin{aligned}
d_1 &= n_\odot \, \frac{k^2}{4L} \, \cos\epsilon + \frac{C_q}{2} - \delta - \delta \, \cos^2\epsilon - \gamma - \gamma \, \cos^2\epsilon_M, \\
d_2 &= n_\odot \, \frac{k^2}{4L} \, \cos\epsilon + \frac{C_q}{2} - 2\,\delta \, \cos^2\epsilon - 2\,\gamma \, \cos^2\epsilon_M, \\
d_3 &= -n_\odot \, \frac{k^2}{2\sqrt{L}} \, \sin\epsilon + 2\,\delta \, \sqrt{L} \, \sin^2\epsilon + 2\,\gamma \, \sqrt{L} \, \sin^2\epsilon_M,
\end{aligned}
$$

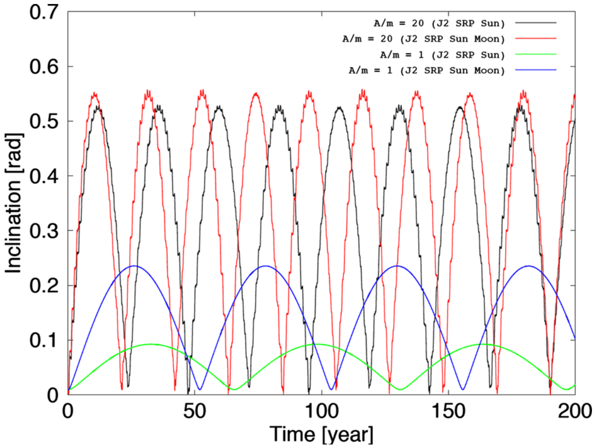
where $\delta = \beta \, \frac{3a^2}{16 \, L \, a_\odot^2}$ and $\gamma = -\frac{\mu_☾}{a_☾} \, \frac{3a^2}{16 \, L \, a_☾^2}$.

We write the corresponding solution for $x_2(t)$ and $y_2(t)$:

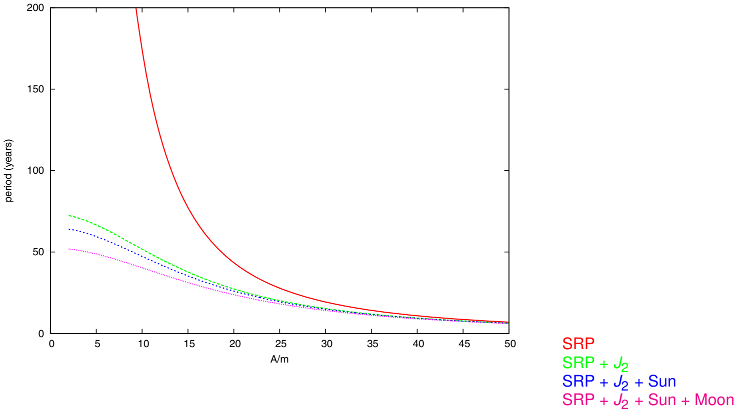
$$
\begin{aligned}
x_2(t) &= \mathcal{D} \, \sin(\sqrt{d_1 d_2} \; t - \psi), \\
y_2(t) &= \mathcal{D} \, \sqrt{\frac{d_2}{d_1}} \, \cos(\sqrt{d_1 d_2} \; t - \psi) - \frac{d_3}{d_1},
\end{aligned}
$$

Eccentricity and inclination motions

Introduction of J_2, Sun and Moon in the description (Casanova)

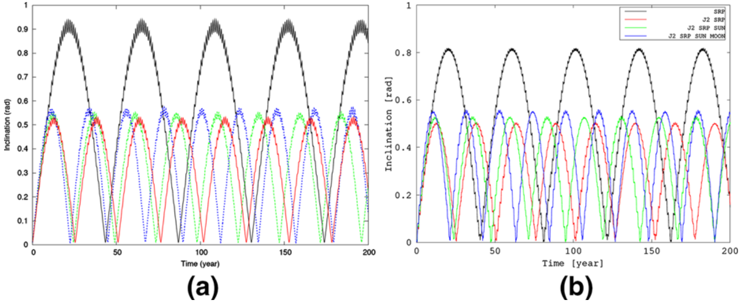


Inclination motion



Inclination motion : results

A/M = 20 m^2/kg - comparison with numerical integration



Mathematical work

- Presence of mathematical challenges
- Model of resonance + perturbations + averaging
- Comparisons between several models of atmosphere ($<$ 1000 km)
- Research for stability zones (chaos) : churchyard or concentration orbits
- Use of the right integrator : symplectic
- Yarkovsky effect on space debris : negligible over 200 years
- Presence of secondary resonance, affecting the semi-major axis (period of 13 000 years)