Regularization of discontinuous dynamical systems.

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Abstract

Discontinuous dynamical systems model many phenomena in control theory, in mechanical friction and impacts, in hysteresis in electrical circuits and plasticity, etc...

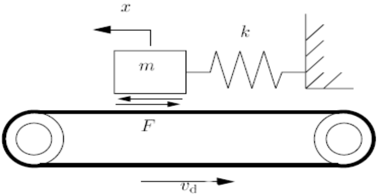
In these systems the phase space is divided into several regions where the system takes different forms. **Vector fields with jump discontinuities at the edges of these regions -the switching manifolds- are usually named Filippov Systems**. The question is if one can define "properly" a flow on the edges.

A regularization of a Filippov system is an embedding of it in a set of parametric regular systems in such a manner that the discontinuous one will be their limit, in a sense that defines a flow on the switching manifold.

But different regularization techniques can lead to different definitions of the edge solutions. We present different examples of Filippov systems with regularizations which produces qualitative different behaviour: a single degree dry friction oscillator, a grazing-sliding saddle-node bifurcation and the different effect of linear and nonlinear regularization of periodically forced oscillators.

A dry friction oscillator

Let us consider a mass m attached to a spring with a constant of recovery K. The mass is on a driving belt with constant velocity v_d.



If x denotes the displacement of m with respect to the equilibrium position of the spring K, on m act two forces: a force of resistance of the spring $-Kx$ (assuming the spring linear), and a friction force between the mass and the belt.
If we start from the equilibrium position $x = 0$, the mass will begin to move in stick with the belt (stick phase) at velocity v_d till the recovery force of the spring $-Kx$ compensate the static friction force and produce on m a damped harmonic motion (slip phase) until that, by energy dissipation, the mass will be once more in sticking with the belt, and so on.

So the equations are divided according to whether or not the relative speed between the mass and the belt, $v_r = \dot{x} - v_d$ is zero, in two phases:

- Stick phase ($v_r = 0$), the equations are:

$$m\ddot{x} = -Kx + \mathcal{F}_s(x),$$

  where the friction static force is $\mathcal{F}_s(x) = \min(|Kx|, F_s) sgn(Kx)$, and F_s is its maximum value.
  Note that if $|Kx| < F_s$, then $\ddot{x} = 0$ and $\dot{x} = v_d$, ie, m moves in sticking with the belt until the force of the spring recovery reaches F_s. From this moment on, m begins to oscillate on the belt. But now it enters into a state where $v_r \neq 0$ and there the frictional force depends on v_r. The system is now in slip phase.

- Slip phase ($v_r \neq 0$), the equations of motion are

$$m\ddot{x} = -Kx + \mathcal{F}_d(v_r),$$

  where $\mathcal{F}_d(v_r)$, represents the dynamic friction which has opposite sign to v_r.

Now we consider two models of friction related to two different types of $\mathcal{F}_d(v_r)$.

- *Coulomb model*. This model assumes that the dynamic friction is constant and equal to the static friction.
- *Stiction model*. This model assumes that static friction is greater than the dynamic friction. When the spring reaches the value of static friction, the frictional force falls instantaneously to a strictly less value.

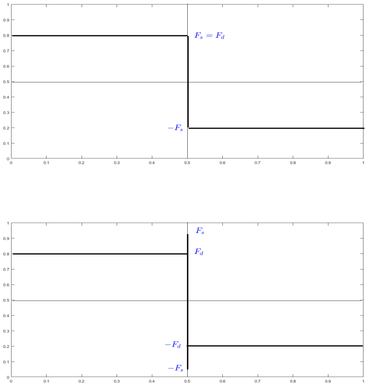


Figure: Coulomb and Stiction models of friction

The Coulomb model

If for simplicity we take $m = K = v_d = 1$, the equations of motion for the Coulomb model are:

$$\begin{aligned} \ddot{x} &= -x + \min(|x|, F_s) sign(x), \ \ v_r = 0 (stick) \\ \ddot{x} &= -x - F_s sign(v_r), \ \ v_r \neq 0 (slip) \end{aligned}$$

The change $y \to y - 1$ transforms the switching manifold in $y = 0$. Then we have: A stick system on $y = 0$:

$$\left.\begin{aligned} \dot{x} &= y + 1 \\ \dot{y} &= -x + \min(|x|, F_s) sgn(x), \end{aligned}\right\} y = 0 \quad (\dot{y} = 0)$$

and a discontinuous slip system $Z = (X, Y)$ on $y \neq 0$

$$\left.\begin{aligned} \dot{x} &= y + 1 \\ \dot{y} &= -x - F_s, \end{aligned}\right\} y > 0, \quad \mathrm{X}^+(x, y)$$

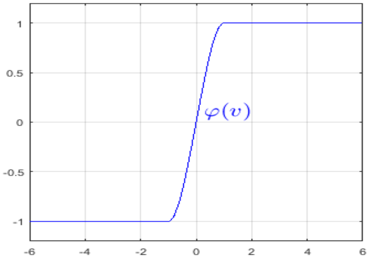
$$\left.\begin{aligned} \dot{x} &= y + 1 \\ \dot{y} &= -x + F_s, \end{aligned}\right\} y < 0, \quad \mathrm{X}^-(x, y)$$

The Sotomayor-Teixeira regularization Z_ϵ of $Z = (X^+, X^-)$ is defined by :

$$Z_\epsilon(x,y) = \frac{X^+(x,y) + X^-(x,y)}{2} + \varphi(\frac{y}{\epsilon})\frac{X^+(x,y) - X^-(x,y)}{2}, \qquad (1)$$

where φ is any increasing smooth function with:

$$\varphi(v) = -1, \text{ for } v \leq -1, \quad \varphi(v) = 1, \text{ for } v \geq 1.$$



Note that in $|y| \geq \epsilon$ the regularized vector field don't change the fields X^+, X^-.
Then, in our case, the Sotomayor-Teixeira regularized system $Z_\epsilon(x,y)$ will be

$$\left.\begin{array}{rcl} \dot{x} & = & 1 + y \\ \dot{y} & = & -x - \varphi(\frac{y}{\epsilon})F_s, \end{array}\right\} \qquad (2)$$

,

Now we consider the strechted varible $v = \frac{y}{\epsilon}$ and the system transforms to:

$$\left.\begin{array}{rcl} \dot{x} & = & 1 + \epsilon v \\ \epsilon \dot{v} & = & -x - \varphi(v) F_s, \end{array}\right\}$$

This is a singular perturbed system. The curve $x + \varphi(v)F_s = 0$, named the critical curve(manifold in the many dimensional case), is the limit as $\epsilon \to 0$ of a especial(s) solution(s)(slow manifold(s)) $v = m(x;\epsilon), |x| < F_s$ which attracts ϵ-exponentially a wide range of the phase space, as next figure shows.

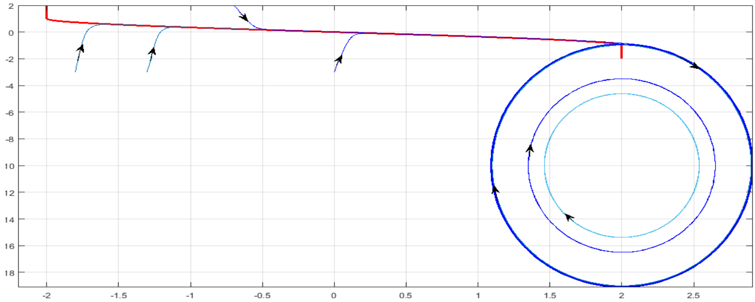
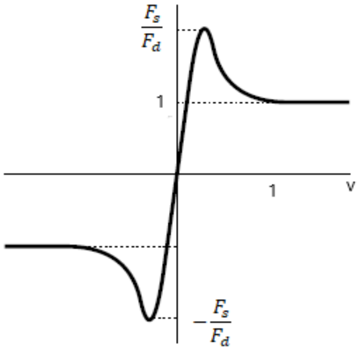


Figure: the slow manifold in red. $F_s = 2$

But $y \equiv \epsilon v = \epsilon m(x;\epsilon), |x| < F_s$ is a solution of system (2) that exponentially attracts any solution in its neighbourhood. Then we can consider the flow inside it and obtain the regular equation $\dot{x} = 1 + \epsilon m(x;\epsilon)$. And as $\epsilon \to 0$ the flow of this system tends to $\dot{x} = 1, y = 0$, which is the behaviour of the stick phase model.

But for the stiction model, the Sotomayor-Teixeira regularization is the same as we did, and don't provides its stick-behaviour. So we should perform other type of regularization, namely, other class of funtion $\varphi(v)$, with shape like



The Sotomayor-Teixeira regularization of a fold

For simplicity consider the discontinuos system $Z = (X, Y)$, where

$$X^+(x,y) = \begin{pmatrix} 1 \\ 2x \end{pmatrix}, y > 0 \tag{3}$$

and

$$X^-(x,y) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, y < 0. \tag{4}$$

The Sotomayor-Teixeira regularization Z_ϵ of this vector field will be

$$\begin{array}{rcl} \dot{x} & = & \frac{1}{2}(1+\varphi(\frac{y}{\epsilon})) \\ \dot{y} & = & \frac{1+2x}{2} + \frac{1}{2}\varphi(\frac{y}{\epsilon})(2x-1). \end{array} \tag{5}$$

with the change of variable $y = \epsilon v$ we transform it in a singular perturbed system, which is called the Slow system:

$$\begin{array}{rcl} \dot{x} & = & \frac{1+\varphi(v)}{2} \\ \epsilon\dot{v} & = & \frac{1+2x}{2} + \frac{1}{2}\varphi(v)(2x-1), \quad (Slow \quad system) \end{array} \tag{6}$$

Changing the time $t = \epsilon\tau$ we get the so called fast system:

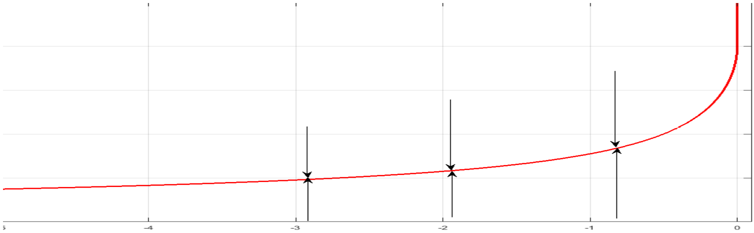
$$\begin{aligned} x' &= \epsilon\frac{1+\varphi(v)}{2} \\ v' &= \frac{1+2x}{2} + \frac{1}{2}\varphi(v)(2x-1), \quad (Fast \quad system). \end{aligned} \tag{7}$$

Note that for $\epsilon > 0$ the two systems are equivalent, but the fast system is regular and for $\epsilon = 0$

$$\begin{aligned} x' &= 0 \\ v' &= \frac{1+2x}{2} + \frac{1}{2}\varphi(v)(2x-1). \end{aligned} \tag{8}$$

and has a curve (manifold) of critical points (the *slow manifold*)

$$\Lambda_0 = \{(x, v),\ \varphi(v) = \frac{1+2x}{1-2x},\ x \leq 0\}. \tag{9}$$



.

Moreover, Λ_0 is a normally hyperbolic attracting manifold of critical points for $x<0$ Then in any compact subset of the region $x<0$, we can apply Fenichel theory which ensures the existence of a normally hyperbolic attracting invariant manifold Λ_ϵ for ϵ small enough of system (7) (and (6))
But Λ_0 bends and looses its hyperbolic character when $x\to 0$. To track Λ_ϵ beyond $x=0$ we match (7) with its transformation by the change $x=\epsilon^{\frac{2}{3}} \quad \eta = 1+\epsilon^{\frac{1}{3}}u$:

$$\begin{aligned} \dot\eta &= 1+O(\epsilon^{\frac{2}{3}}) \\ \dot u &= 2\eta - \frac{\varphi''(1)}{4}u^2 + O(\epsilon^{\frac{1}{3}}), \end{aligned} \tag{10}$$

then we can continue Λ_ϵ, which near the fold has the expression:

$$x=\epsilon^{\frac{2}{3}}\eta_0\left(\frac{y-\epsilon}{\epsilon^{\frac{4}{3}}}\right)+\epsilon\eta_1\left(\frac{y-\epsilon}{\epsilon^{\frac{4}{3}}}\right)+\mathcal{O}(\epsilon^{\frac{4}{3}}), \tag{11}$$

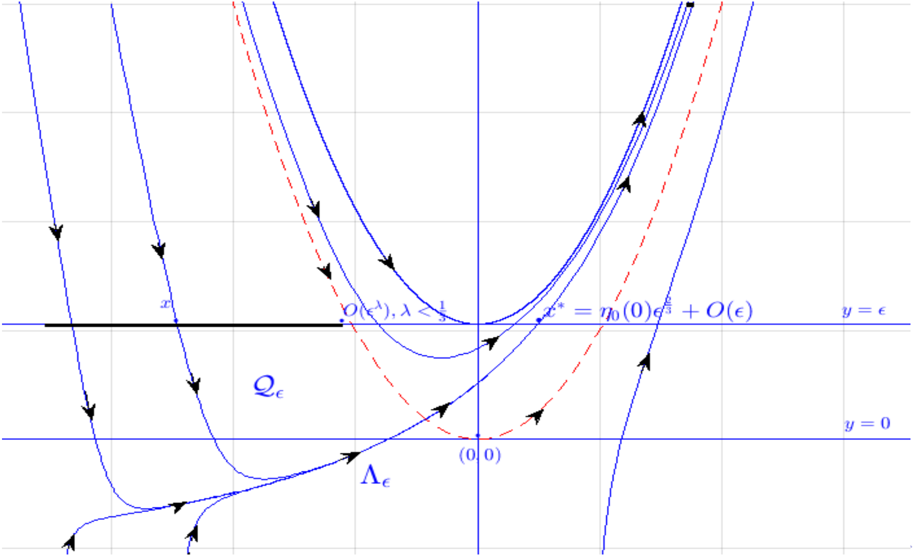
where $\eta_0(u)$ is the unique solution of the equation:

$$\frac{d\eta}{du}=\frac{1}{2\eta-\frac{\varphi''(1)}{4}u^2}, \tag{12}$$

satisfying $\eta(u)\sim \frac{\varphi''(1)}{8}u^2$ as $u\to-\infty$. Note that in $y=\epsilon$ $(v=1)$ we have:

$$x=\epsilon^{\frac{2}{3}}\eta_0(0)+\epsilon\eta_1(0)+\mathcal{O}(\epsilon^{\frac{4}{3}}) \tag{13}$$

In [1] we proved that the flow coming from intervals $[-L, -\epsilon^{\lambda}], 0 < \lambda < \frac{1}{3}$ is exponentially attracted by Λ_ϵ that collects they and hits $x > 0$ at $O(\epsilon^{\frac{2}{3}})$, as (13) states. The results are summarized in the next picture. (Note $-\epsilon^{\frac{1}{3}} < -\epsilon^{\frac{1}{2}}$)



The grazing-sliding saddle-node bifurcation

Consider Z_μ, a family of non-smooth planar systems having a grazing sliding bifurcation of a hyperbolic repelling periodic orbit Γ_μ of the vector field X^+_μ at $\mu = 0$.

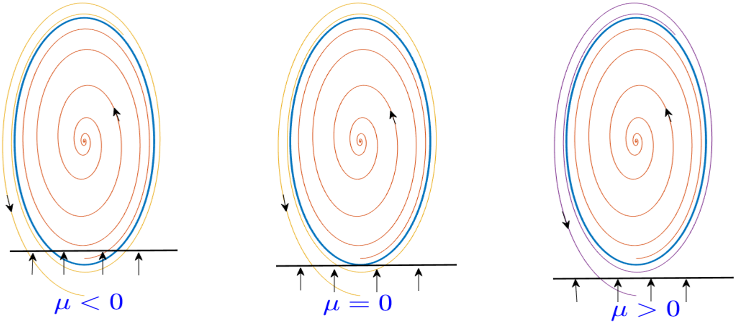
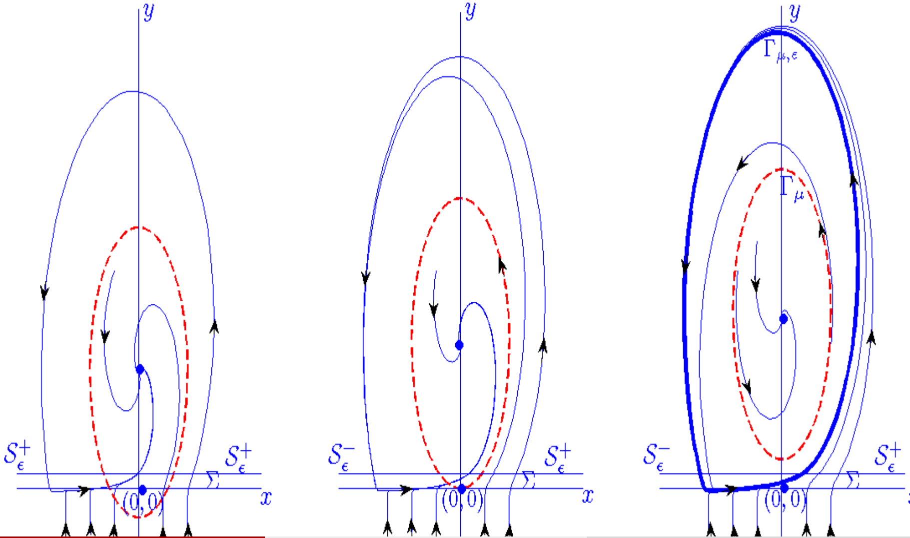


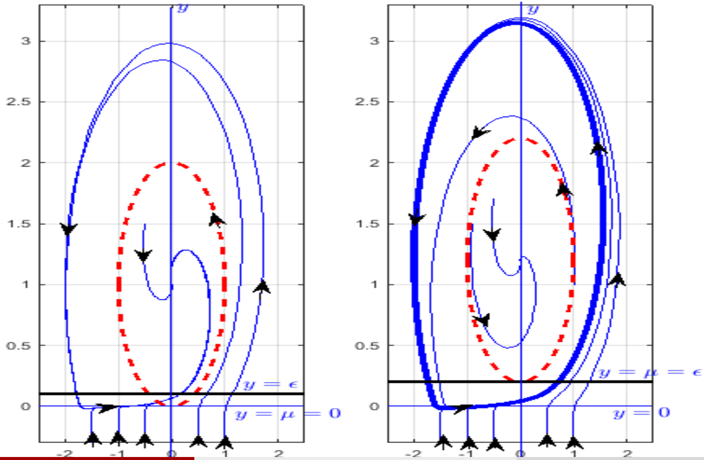
Figure: X^+_μ has a repelling periodic orbit Γ_μ and $X^- = (0, 1)$

Now we perform the Sotomayor Teixeira regularization to this family $Z_{\mu,\epsilon}$. In [1] we proved that if Γ_μ is repelling the regularized system $Z_{\mu,\epsilon}$ has a bifurcation of periodic orbits.

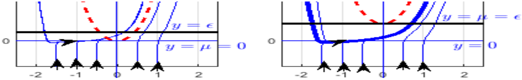


Let us see where can be the bifurcation. From the behaviour near the fold we conclude two facts:

- For $\mu = 0$ there is no periodic orbit. For $\mu = \epsilon$ there are, at least, one attracting periodic orbit besides the unstable.
- If an orbit of the regularized system enters to the vault of the periodic orbit, then it is trapped by the focus.



Then the bifurcation will take place when $0 < \mu < \epsilon$ and, more precisely, when the Fenichel solution(s) and the upper segment of the periodic orbit "collide" in $x > 0$ at some order.



Put $\delta = \epsilon - \mu$, and let x_δ^+, the intersection on $x > 0$ of the periodic orbit Γ_μ. Then as its tangency is a fold, we equate:

$$x_\delta^+ = \sqrt{\epsilon - \mu} + \ldots = \epsilon^{\frac{2}{3}}\eta_0(0) + \ldots \Rightarrow \mu = \epsilon - \epsilon^{\frac{4}{3}}\eta_0^2(0) + \mathcal{O}(\epsilon^{\frac{5}{3}}) \quad (14)$$

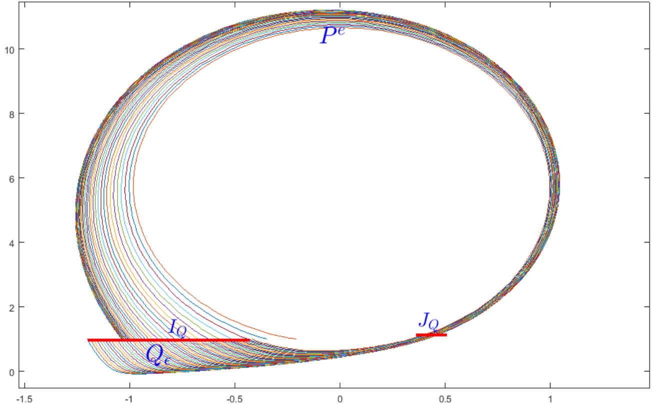
So the range of $\mu' s$ where the bifurcation will take place must be

$$\mu_1 := \epsilon - \epsilon^{\frac{4}{3}}\eta_0^2(0) - K\epsilon^{\frac{5}{3}} < \mu < \mu_2 := \epsilon - \epsilon^{\frac{4}{3}}\eta_0^2(0) + K\epsilon^{\frac{5}{3}} \quad (15)$$

with K as large as our conveniences.

Take this range of μ and let σ such that $1 > \sigma > \sqrt{1 - \frac{1}{\pi'(0)}}$, where $\pi'(0)$ is the derivative of the Poincaré map of Γ_0. Then it can be seen that the interval $J_{\mathcal{Q}} := [\sigma\epsilon^{\frac{2}{3}}, \sqrt{2}\epsilon^{\frac{2}{3}}]$, contains all the cuts, F_μ^+, x_δ^+, of the Fenichel solution and the intersection of Γ_μ, and also $(P^e)'' > 0$ there, where P^e is the map derived form the upper flow from $x > 0$ to $x < 0$.

Now define $\tau > \sqrt{1+\pi'(0)}$. Let $\mathcal{Q}_\epsilon$, the map derived from the regularized field $Z_{\mu,\epsilon}$ from $x<0$ to $x>0$. It can be seen this map sends the interval $I_{\mathcal{Q}} := [-\tau\epsilon^{\frac{2}{3}}, \mathcal{Q}_\epsilon^{-1}(\sigma\epsilon^{\frac{2}{3}})]$ to $J_{\mathcal{Q}}$. Note that $P^e(I_{\mathcal{Q}})$ not has to be all inside $J_{\mathcal{Q}}$ (when there is not periodic orbit, for instance), but is always inside $[-\tau\sqrt{\delta}, 0)$. Besides, $\mathcal{Q}_\epsilon'' < 0$ in $J_{\mathcal{Q}}$. Then the Poincaré map $P^e \circ \mathcal{Q}_\epsilon$ is convex, and the bifurcation is a saddle-node.

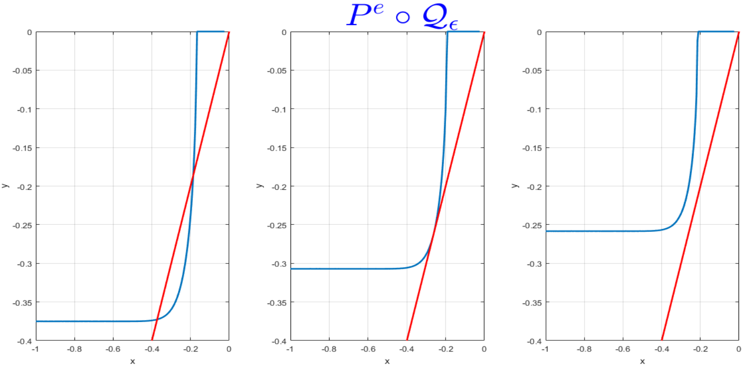


As an example, let's take the family of vector fields $Z_\mu = (X^+_\mu, X^-_\mu)$ where X^+ is given by

$$\left.\begin{array}{ll} \dot{x} & = f(x,y,\mu) = \quad -y+\mu+1+x(r-1) \\ \dot{y} & = g(x,y,\mu) = \quad x+(y-\mu-1)(r-1) \end{array}\right\} \quad r=\sqrt{x^2+(y-\mu-1)^2} \quad (16)$$

, and $X^- = (0,1)$.

Then the Poincaré map $P^e \circ \mathcal{Q}_\varepsilon$ defined in $[-1,0]$ and for $\varepsilon = .05$ and $\mu_{1,2,3} = \varepsilon - (.5, .5623, .6)\varepsilon^{\frac{4}{3}}$ has two, one and zero fixed points.



Hysteresis

Let variables $x \in \mathbb{R}^{n-1}$ and $y \in \mathbb{R}$ satisfy the differential equation

$$\begin{aligned} \dot{x} &= f(x, y; u) \\ \dot{y} &= g(x, y; u) \end{aligned} \quad (17)$$

where f and g are smooth functions of x, y, u, and where u (the control) is given by

$$u = \mathrm{sign}(y) \; . \quad (18)$$

The values of the vector field either side of the switch $y = 0$ can be written as

$$f^{\pm}(x, y) = f(x, y; \pm 1) \qquad \text{and} \qquad g^{\pm}(x, y) = g(x, y; \pm 1) \; . \quad (19)$$

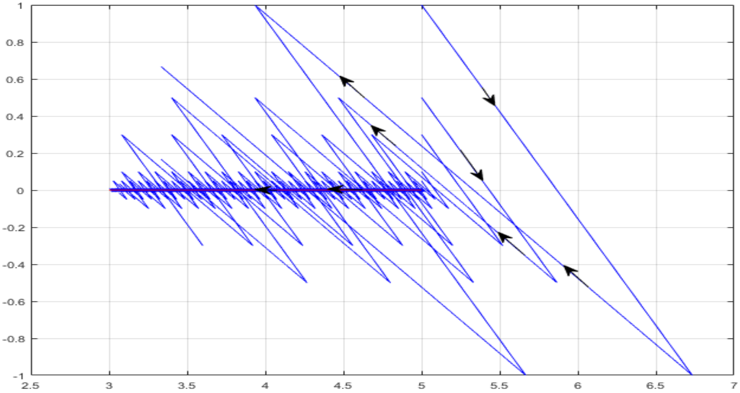
We now introduce hysterises as another way of regularizing discontinuous systems. In a 'negative' boundary layer we define an overlap in the non smooth system:

$$u \in \begin{cases} +1 & \text{if} \quad y > -\alpha \; , \\ [-1, +1] & \text{if} \quad |y| \leq \alpha \; , \\ -1 & \text{if} \quad y < +\alpha \end{cases} \quad (20)$$

and define a hysteretic process: A trajectory with $u = \mp 1$ switch to $u = \pm 1$ when it reaches $y = 0$, and so on.

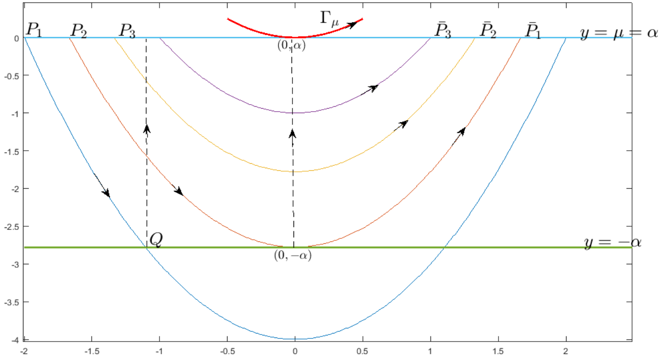
We can illustrate the method with a simple example. Consider the planar piecewise smooth system

$$\dot{x} = 0.3 + u^3 \qquad \dot{y} = -0.5 - u \qquad u = \mathrm{sign}(y)\ . \tag{21}$$

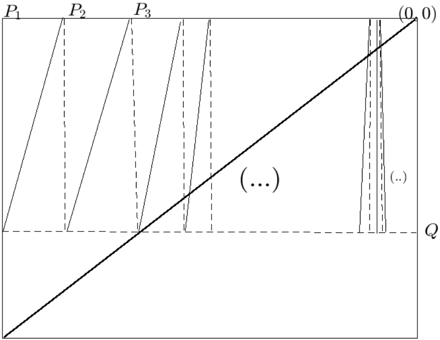


The hysteretic behaviour for the example for diminishing values of α. The line in red is the limit solution on the switching $y = 0$.

In [2] is proved that the hysteretic regularization and that of Sotomayor-Teixeira tend to the same flow in the hyperbolic regions of the switching manifold.
But they can produce quite different behaviours when the hyperbolicity is loosed.
That's the case of the regularization by hysteresis of the grazing sliding bifurcation, where appears chaotic phenomena. The following picture is a model of what happens when $\mu = \alpha$. (We suppose that all the tangencies at the lines $y = C$ are in $x = 0$)

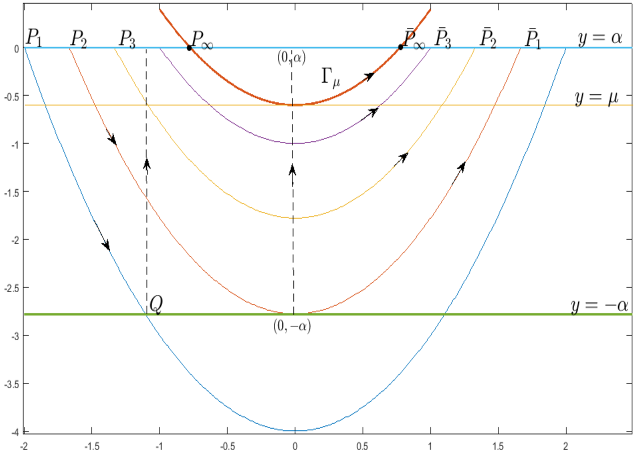


Now the key is the tangent orbit of the upper field in $y = -\alpha$. Denoting by $\bar{P}$ and P its intersections on $x > 0$ $y = \alpha$,and $x < 0$ $y = \alpha$, respectively, and supposing that the lower field is $X^{-} = (0, 1)$, a linear model of the discontinuities (in fact, are nonlinear as we will see) of the induced map on $x < 0$ onto itself will be

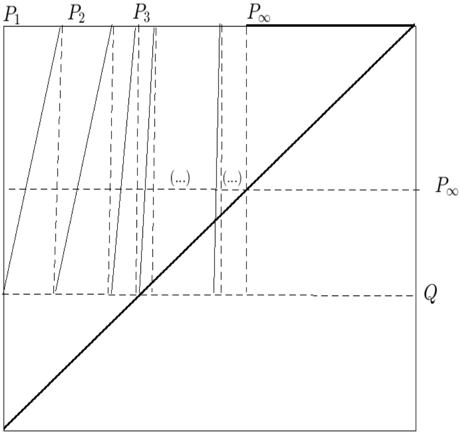


We see there's is jump accumulation near $x = 0$, and there will be infinitely many periodic orbits of all periods.

Now let's model the case $\mu < \alpha$

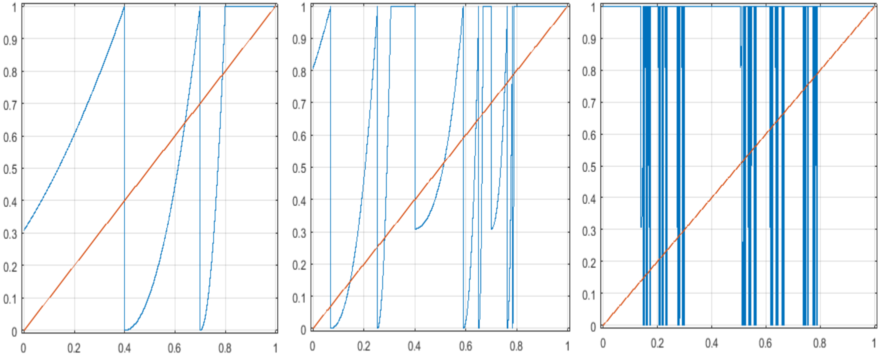


We see that for $x \geq P_\infty$,the map is constantly 0. Now the the map is



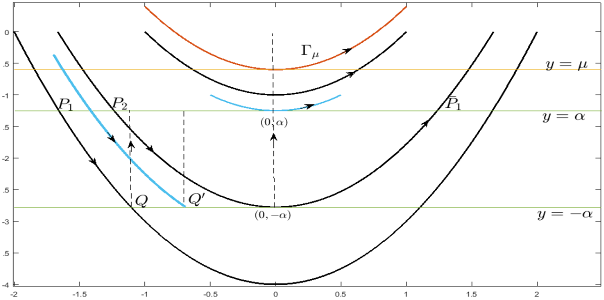
and the jumps accumulates at P_∞. This is a chaotic behaviour but the iterates of this type of maps tend to the map 0 almost everywhere.

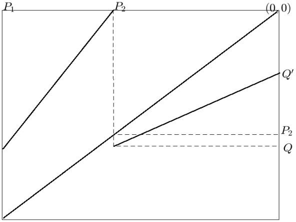
Consider an idealized model shown in the next picture



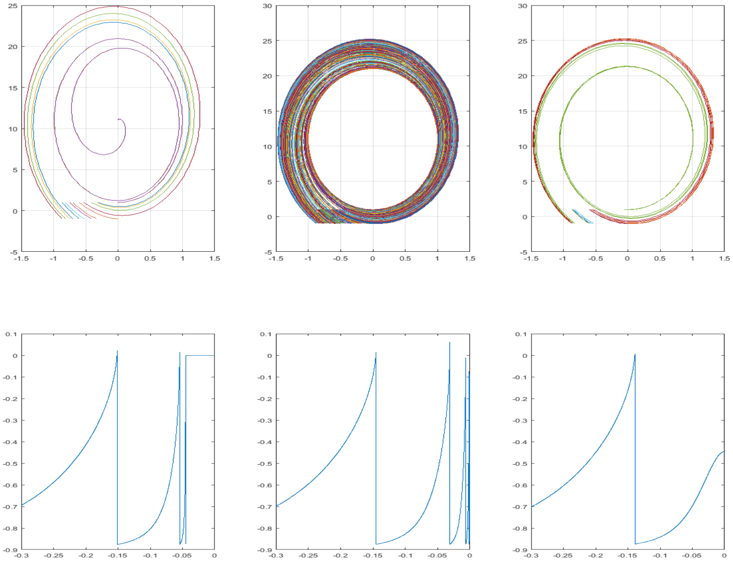
The model map T is in the first figure, while T^2 and T^7 are in the second and third figure respectively. One can see the intricate dynamics, but also we see how the $\mu(\{T^{-1}(1)\})$ is increasing to 1.

Finally the case $\mu > \alpha$. The curve in blue is the tangent of the upper field to $y = \alpha$. In the following picture we show the case that there is not another cut of the curve tangent to $y = -\alpha$ between P_2 and $\bar{P}_1$.





If we perform the hysteretic regularization of the system (16) we have the following pictures for trajectories and the Poincaré map.



Smoothing the hysteresis

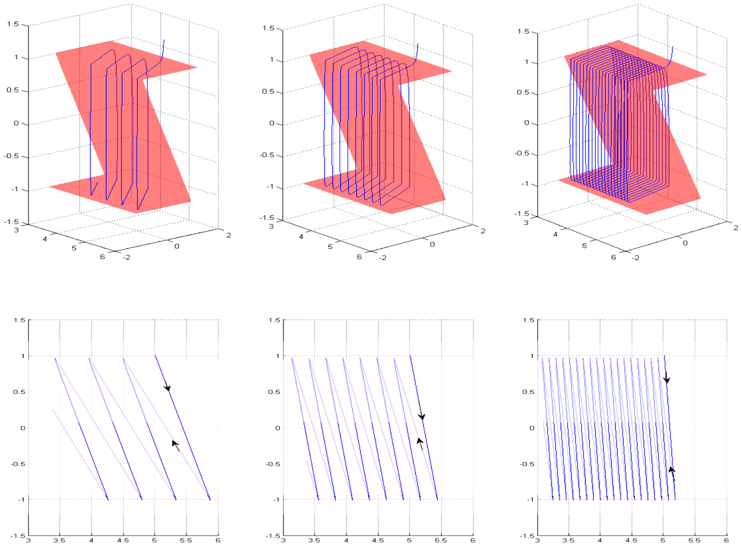
Of course, the hysteretic process we have defined is not smooth. In [2] we also faced the problem of "smoothing" the hysteresis. And this requires to embed the system in a higher dimension, where the control u is also a time dependent (fast)variable. Then the three dimensional system is:

$$\begin{aligned} \dot{x} &= f\,(x,y;u)\ , \\ \dot{y} &= g\,(x,y;u)\ , \\ \epsilon\dot{u} &= \varphi\left(\tfrac{y+\alpha u}{\epsilon}\right) - u\ , \end{aligned} \tag{22}$$

By the definition of φ, we have

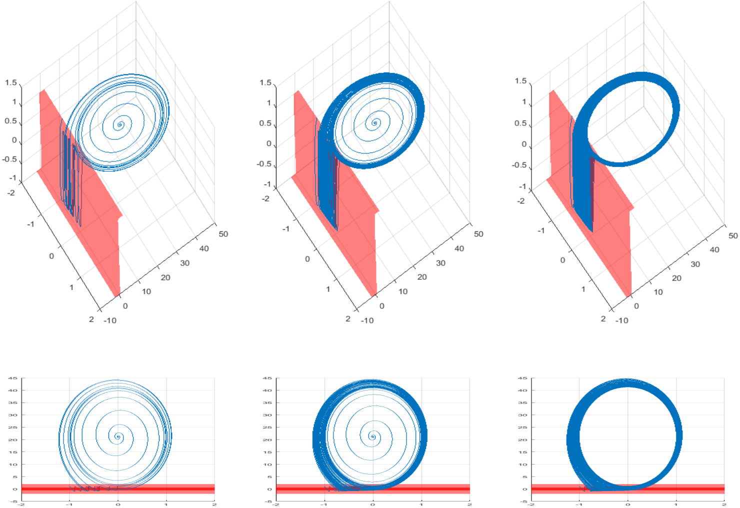
$$\lim_{\epsilon\to 0}\varphi\left(\tfrac{y+\alpha u}{\epsilon}\right) \in \Phi(y+\alpha u)\ . \tag{23}$$

Then we had embed a (x,y) problem with a parameter u, in the higher dimensional space (x,y,u), where u is now a fast variable that relaxes quickly to $u=\pm 1$.



Hysteretic relaxation of the example (7) for different values of α and his projection in the (x, y) plane.

Also we can embed the system (2) in a hysteretic relaxation. The picture shows the behaviour of a trajectory near bifurcation and the projections onto the (x, y) plane.



The aging phenomena

We have seen already how a regularization can produce different behaviour near the switching manifold. In [4] we are studying the regularization of an apparently simple equation where emerges a phenomena we call "aging phenomena". Consider the discontinuous damped periodically forced oscillator

$$\ddot{y} + a\dot{y} + y + f(t, \dot{y}) = 0, \tag{24}$$

where $a > 0$, $f(t, \dot{y}) = \sin(\frac{3\pi}{2}t)$ for $\dot{y} > 0$ and $f(t, \dot{y}) = \sin(\frac{\pi}{2}t)$ for $\dot{y} < 0$. Let the damping be small, say $a = 0.01$. Let $\lambda = sign(\dot{y})$. We perform two regularizations :

- linear
  $f(t, \lambda) = \frac{(1+\lambda)}{2}\sin(\frac{3\pi}{2}t) + \frac{(1-\lambda)}{2}\sin(\frac{\pi}{2}t)$
- nonlinear
  $f(t, \lambda) = \sin(\frac{(1+\lambda)}{2}(\frac{3\pi}{2}t) + \frac{(1-\lambda)}{2}(\frac{\pi}{2}t))$

In spite that this equation is integrable in each half plane, is very difficult to study it in the whole space. Nevertheless, for a=0, it can be proved that a periodic orbit of period $T = 8$ exist. Then if we consider $a = 0.01$ and $\lambda = \varphi(\frac{\dot{y}}{\epsilon})$, this periodic orbit persists in the linear regularization.

So if we put the equation in system form, we have:

$$\left.\begin{array}{ll} \dot{t} & = 1 \\ \dot{y} & = z \\ \dot{z} & = -az - y - \frac{(1+\lambda)}{2}\sin(\frac{3\pi}{2}t) - \frac{(1-\lambda)}{2}\sin(\frac{\pi}{2}t) \end{array}\right\} \lambda = \varphi(\frac{z}{\epsilon}) \qquad (25)$$

The next picture shows the flow trapped by this orbit

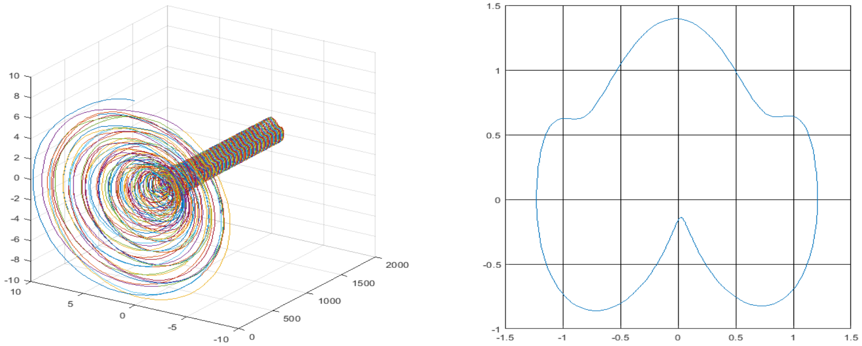
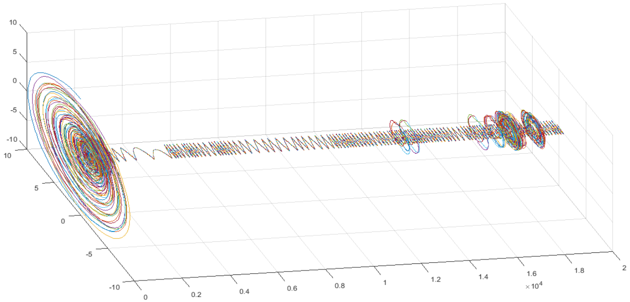


Figure: $\epsilon = 0.05$ and $(0, 0, 10)$ as initial condition.

But if with the same parameters we use the nonlinear regularization

$$\left.\begin{array}{rl} \dot{t} & = 1 \\ \dot{y} & = z \\ \dot{z} & = -az - y - sin(\frac{(1+\lambda)}{2}(\frac{3\pi}{2}t) + \frac{(1-\lambda)}{2}(\frac{\pi}{2}t)) \end{array}\right\} \lambda = \varphi(\frac{z}{\epsilon}) \qquad (26)$$



we obtain this picture, where we see the solution "crawling"(sliding) in $z = 0$. The solution has entered in the regularization zone, and then it take very long time to scape from it. Moreover, the more large is the time when flow enters, the more large time will be trapped. We say that system has aging.

This phenomena is very clear in two dimensions. Let the system

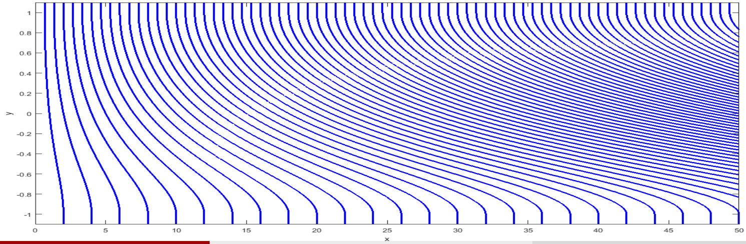
$$\left.\begin{aligned} \dot{t} &= 1 \\ \dot{z} &= -az - sin(\tfrac{(1+\lambda)}{2}(\tfrac{3\pi}{2}t) + \tfrac{(1-\lambda)}{2}(\tfrac{\pi}{2}t)) \end{aligned}\right\} \lambda = \varphi(\frac{z}{\epsilon}) \qquad (27)$$

that's the version two dimensional of the nonlinear regularization. To transform to a singularly perturbed system we perform the change $z = \frac{v}{\epsilon}$:

$$\left.\begin{aligned} \dot{t} &= 1 \\ \epsilon\dot{v} &= -a\epsilon v - sin(\tfrac{(1+\varphi(v))}{2}(\tfrac{3\pi}{2}t) + \tfrac{(1-\varphi(v))}{2}(\tfrac{\pi}{2}t)) \end{aligned}\right\} \qquad (28)$$

The crucial fact is the shape of the critical manifold

$$sin(\frac{(1+\varphi(v))}{2}(\frac{3\pi}{2}t) + \frac{(1-\varphi(v))}{2}(\frac{\pi}{2}t)) = 0 \qquad (29)$$



In this picture we see how the solution is forced to slide on a stable leave of the critical manifold, until enters in $y < -\epsilon$ (or $v =< -1$). Then oscillates but can't not cross to $y > \epsilon$ ($v > 1$). Finally the flow tends to a periodic orbit, but nothing to do with the periodic orbit of the linear regularization.
Note, however, that in the three dimensional case, the flow has chances to come back to $z > \epsilon$, but once achieved this, the flow, for large times, is again trapped in the sliding region, and will stay there for a time even larger that the precedent. Is the aging phenomena

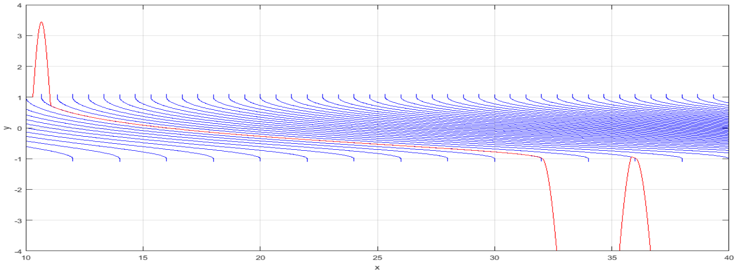


Figure: $\epsilon = 0.1$ and $(10.3, 1)$ as initial condition. The flow is scaled with $v = \frac{z}{\epsilon}$

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