Water-waves and reversible spatial dynamics

Gérard Iooss

IUF, Université de Nice, Laboratoire J.A.Dieudonné,

Parc Valrose, F-06108 Nice Cedex02

reproduction of the soliton observed by Scott Russell in 1834

References for travelling (math) water waves

Historical

Stokes 1847 2D periodic (travelling) 2D gravity waves (formal)

J.Boussinesq 1877 2D travelling and standing gravity waves

Nekrasov 1921, Levi-Civita 1925, Struik 1926 Thms periodic gravity waves

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using spatial dynamics

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E.Lombardi 1997 General theory applied to generalized 2D solitary waves with exp small oscillations at ∞

M.Groves, J.Toland, A.Champneys, B.Sandstede (Hamiltonian structure and multipulses)

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3D travelling waves using spatial dynamics

M.Groves, M.Haragus, A.Mielke 2001-2003. Thms on 3D travelling gravity-capillary waves localized in one direction

review paper F.Dias, G.Iooss. Handb. Math. Fluid Dyn., p.443 -499. S.Friedlander, D.Serre Eds. 2003

Other references for travelling water waves

3D

J.Reeder,M. Shinbrot 1981, W.Craig, D.Nicholls 2000 . existence of 3D periodic travelling gravity-capillary waves

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J.Reeder,M. Shinbrot 1981, W.Craig, D.Nicholls 2000 . existence of 3D periodic travelling gravity-capillary waves

Existence and regularity results of 3D waves without surface tension

G.Iooss, P.Plotnikov. Memoirs of A.M.S. 2009, No 940 (128p.) (Existence of diamond waves)

T.Alazard, G.Métivier. Com. Part. Diff. Equ., 34 (10-12) 1632 - 1704, 2009. (Regularity of diamond waves)

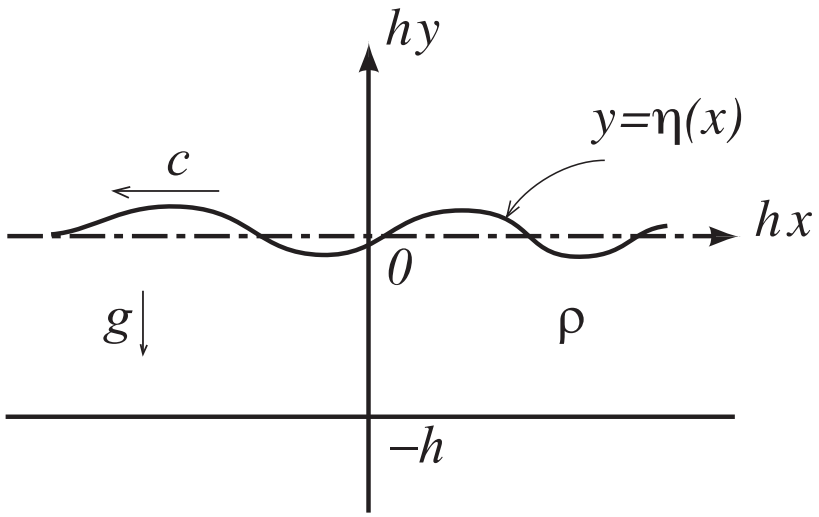
G.Iooss, P.Plotnikov. A.R.M.A. 200, 3 (2011), 789-880 (Existence of non symmetric 3-D travelling waves)

2D travelling waves in an infinitely deep fluid layer (cont. spec.)

G.I., P.Kirrmann, E.Lombardi, S.M.Sun 1996-2003 New types of solitary waves

M.Barrandon 2006 2D Solitary waves with polynomial decay- Benjamin-Ono asymptotics

The 2D Water-Wave problem



Dimensionless parameters: $\lambda = \frac{gh}{c^2}$, $b = \frac{T}{\rho h c^2}$

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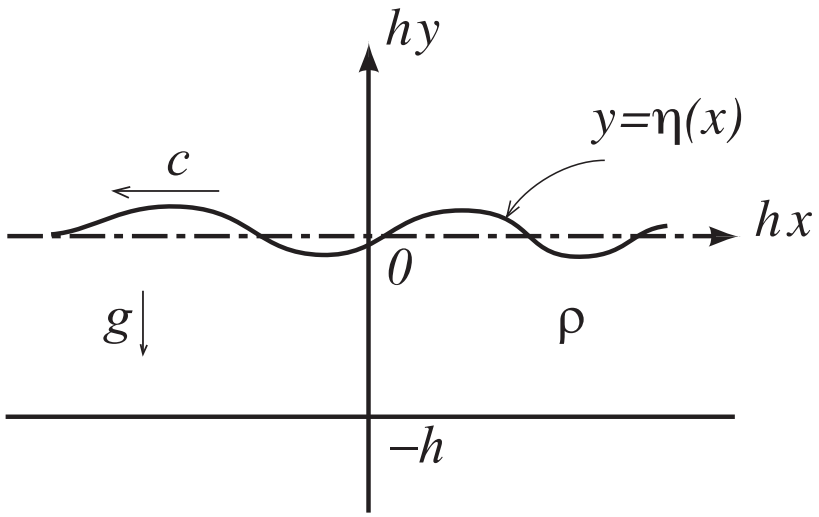

Dimensionless parameters: $\lambda = \frac{gh}{c^2}$, $b = \frac{T}{\rho h c^2}$

$$u = 1 + \frac{\partial \varphi}{\partial x} = \frac{\partial \psi}{\partial y}, \quad v = \frac{\partial \varphi}{\partial y} = -\frac{\partial \psi}{\partial x}$$

$$\begin{aligned} u\eta' - v &= 0 \text{ on } y = \eta(x) \\ 1/2(u^2 + v^2) + \lambda\eta - b\eta''(1 + \eta'^2)^{-3/2} &= 1/2 \text{ on } y = \eta(x) \end{aligned}$$

Complex potential: $w(x + iy) = \varkappa + i\psi$, $\varkappa = x + \varphi$

Change of variables

Levi-Civita variables: (hodograph transform)
Complex potential: $w(x+iy) = \varkappa + i\psi$, $\varkappa = x + \varphi(x,y)$, $\psi = \psi(x,y)$
Complex velocity: $w'(x+iy) = u - iv = e^{-i(\alpha+i\beta)}$
α, β harmonic in the strip $\varkappa \in \mathbb{R}, -1 < \psi < 0$

$$\tan(\alpha): \text{ slope of streamline, } \beta = \ln|\nabla\varkappa|$$

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On the free surface $\psi = 0$, $y = \eta(x) = Z(\varkappa)$, $\frac{dx}{d\varkappa} = e^{-\beta_0}\cos\alpha_0$

$$\frac{d\eta}{dx} = \tan\alpha_0, \quad \frac{dZ}{d\varkappa} = e^{-\beta_0}\sin\alpha_0$$

$$\text{curvature } \frac{\eta''}{(1+\eta'^2)^{3/2}} = \frac{d\alpha_0}{d\varkappa}e^{\beta_0}$$

Formulation as a spatial dynamical system

$$\frac{d\alpha_0}{d\varkappa} = b^{-1}\sinh\beta_0 + \lambda b^{-1}e^{-\beta_0}Z$$

$$\left.\begin{array}{l} \frac{\partial\alpha}{\partial\varkappa} = \frac{\partial\beta}{\partial\psi} \\ \frac{\partial\beta}{\partial\varkappa} = -\frac{\partial\alpha}{\partial\psi} \end{array}\right\} -1 < \psi < 0$$

$$Z(\varkappa) = \int_{-1}^{0}(e^{-\beta}\cos\alpha - 1)d\psi$$

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$$\frac{dU}{d\varkappa} = \mathbf{F}(U,\lambda) \text{ in } \mathcal{X} = \mathbb{R}\times[L^2(-1,0)]^2$$

$$[U(\varkappa)](\psi) = (\alpha_0(\varkappa), \alpha(\varkappa,\psi), \beta(\varkappa,\psi))\,,$$

$$\mathcal{Z} = \mathbb{R}\times[H^1(-1,0)]^2 \cap \{\alpha_{|_{\psi=0}} = \alpha_0, \alpha_{|_{\psi=-1}} = 0\}$$

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Reversibility symmetry $\mathbf{S} : \mathbf{S}U = (-\alpha_0, -\alpha, \beta),\ \mathbf{S}\mathbf{F}(U,\lambda) = -\mathbf{F}(\mathbf{S}U,\lambda)$

Linearized system for 2D waves

$$\frac{dU}{d\varkappa} = \mathbf{L}_{\lambda,b} U, \quad U \in \mathcal{Z}$$

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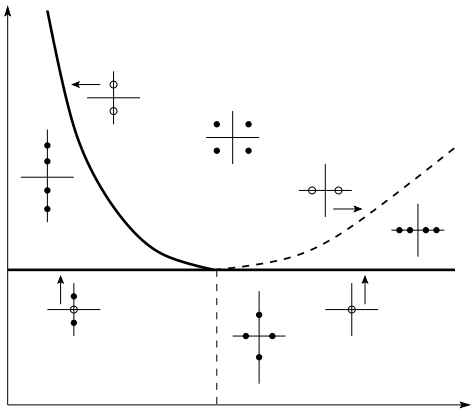
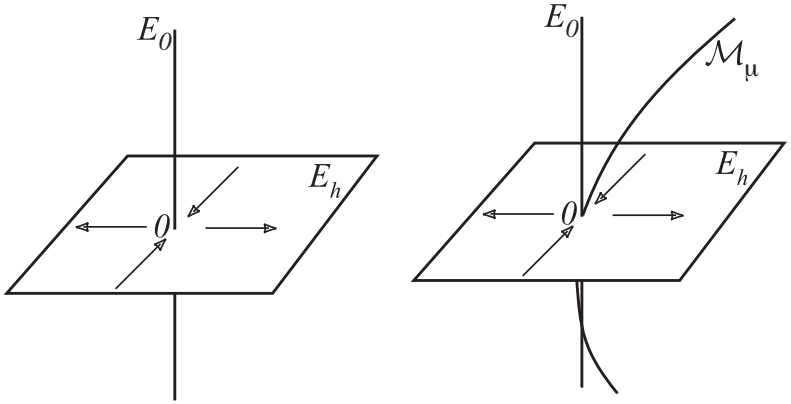
Spectrum of $\mathbf{L}_{\lambda,b}$: isolated eigenvalues ik of finite multiplicities such that $(\lambda + bk^2)\tanh k - k = 0$, for $k \neq 0$, and 0 only if $\lambda = 1$



Critical eigenvalues of $\mathbf{L}_{\lambda,b}$. Solid lines represent the bifurcation curves, solid and hollow dots represent simple and double eigenvalues, resp.

Reduction to a Center manifold

Pliss 1964, Kelley 1967, Lanford 1973, Henry 1981, Mielke 1988, Kirrmann 1991, Vanderbauwhede - Iooss 1992



left: linear case for $\mu = 0$, bounded solutions $\in E_0$, right: non linear case

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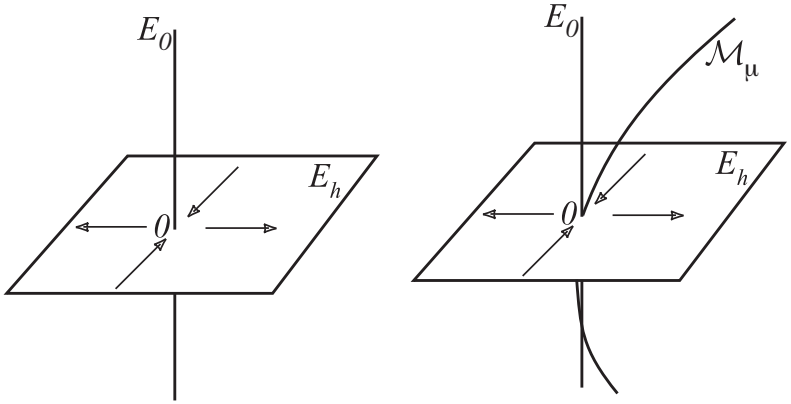

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$$\begin{aligned} \mathcal{M}_\mu &= \{U = U_0 + \mathbf{\Psi}(U_0, \mu),\ (U_0, \mu) \in \mathcal{O}_0\} \\ \mathbf{\Psi} &\in \mathcal{C}^k(\mathcal{O}_0, \mathcal{Z}_h),\ \mathcal{O}_0 \text{ neighb of } 0 \text{ in } E_0 \times \mathbb{R} \end{aligned}$$

$\mathcal{M}_\mu$ locally invariant and *contains all small bounded solutions*

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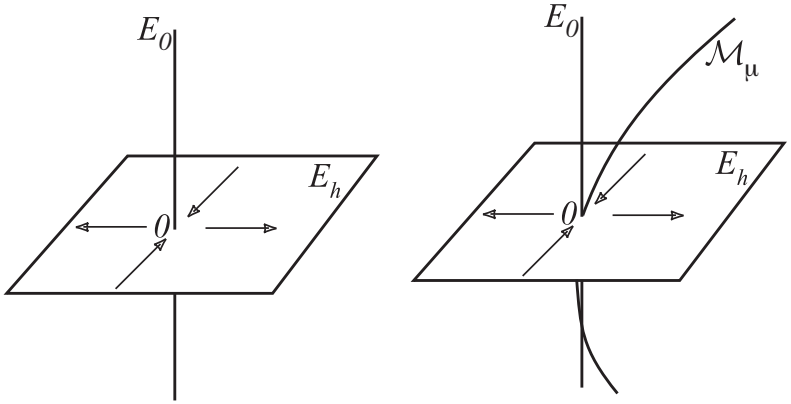

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Reduced system $\dfrac{dU_0}{d\varkappa} = \mathbf{L}_0 U_0 + \mathbf{R}_0(U_0, \mu) \in E_0$

Reversibility symmetry: $\mathbf{S}_0 = \mathbf{S}|_{E_0}$

Conditions for application of C. M. Thm

$$\mathcal{Z} \hookrightarrow \mathcal{Y} \hookrightarrow \mathcal{X} \quad \text{Banach spaces}$$

$$\begin{aligned} \frac{dU}{d\varkappa} &= \mathbf{L}U + \mathbf{R}(U,\mu)\ , \quad \mathbf{L} \in \mathcal{L}(\mathcal{Z},\mathcal{X}), \\ \mathbf{R}(\cdot,\cdot) &\in \mathcal{C}^k(\mathcal{Z}\times\mathbb{R}^p;\mathcal{Y}) \text{ in neighb. of } 0,\ \mathbf{R}(0,0)=0, \quad D_U\mathbf{R}(0,0)=0 \end{aligned}$$

Hypothesis on the spectrum of **L** :

$$\sigma = \sigma_+ \cup \sigma_0 \cup \sigma_-,\ \sigma_0 = \{\lambda \in \sigma; Re\sigma = 0\}$$

$$\sigma_+ = \{\lambda \in \sigma; Re\sigma > 0\}, \quad \sigma_- = \{\lambda \in \sigma; Re\sigma < 0\},$$

$$\exists\gamma > 0; \quad \inf_{\lambda\in\sigma_+} (Re\lambda) > \gamma, \quad \sup_{\lambda\in\sigma_-} (Re\lambda) < -\gamma$$

$\sigma_0 = \{$finite number of eigenvalues with finite multiplicities$\}$

Conditions for application of C. M. Thm (continued)

Define projection on the hyperbolic invariant subspaces $\mathcal{X}_h, \mathcal{Y}_h, \mathcal{Z}_h$:
$P_h = \mathbb{I} - P_0$ where P_0 is defined via Dunford integral
Hypothesis on the nonhomogeneous linear equation

$$\begin{aligned} \frac{dU_h}{d\varkappa} &= \mathbf{L}U_h + f(\varkappa), \quad f \in \mathcal{C}_\eta(\mathbb{R}, \mathcal{Y}_h) \text{ (may grow at infinity as } e^{\eta|\varkappa|}) \\ &\Rightarrow \exists! U_h = K_h f \in \mathcal{C}_\eta(\mathbb{R}, \mathcal{Z}_h) \text{ and } K_h \text{ is bounded} \end{aligned}$$

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Simpler assumption when $\mathcal{Z} \hookrightarrow \mathcal{Y} = \mathcal{X}$ are Hilbert spaces (Mielke 1988):
replace the above hypothesis by

$$||(i\omega\mathbb{I} - \mathbf{L})^{-1}||_{\mathcal{L}(\mathcal{X})} \leq \frac{c}{|\omega|} \text{ for } \omega \in \mathbb{R} \text{ large enough}$$

Proofs and references may be found in M.Haragus, G.Iooss. Local Bifurcations, center manifolds, and normal forms in infinite-dimensional dynamical systems. Springer UTX 2011

Normal form reduction

Poincaré, Birkhoff, Arnold, Belitskii, Elphick et al...
$p \geq 2$, $\exists$ polynomial $\Phi_\mu : E_0 \to E_0$, of degree p and a neighborhood $\mathcal{O}_0$ of 0 in $E_0 \times \mathbb{R}$, such that the local change of variable in E_0

$$U_0 = V_0 + \Phi_\mu(V_0)$$

transforms the reduced system into a new reversible system where $\mathbf{N}_\mu$ is a polynomial of degree p such that

$$\frac{dV_0}{d\varkappa} = \mathbf{L}_0 V_0 + \mathbf{N}_\mu(V_0) + \rho(V_0, \mu),$$

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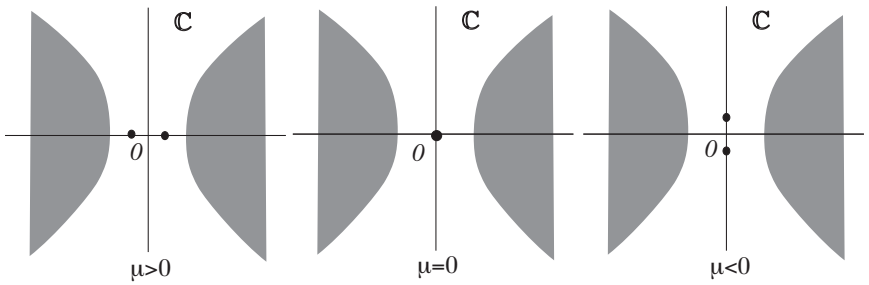
$$\begin{aligned}
\mathbf{N}_0(0) &= 0, \quad D_{V_0}\mathbf{N}_0(0) = 0 \\
e^{\mathbf{L}_0^* t}\mathbf{N}_\mu(V_0) &= \mathbf{N}_\mu(e^{\mathbf{L}_0^* t} V_0), \forall (t, V_0) \in \mathbb{R} \times E_0, \\
\rho(V_0, \mu) &= o(||V_0||^p).
\end{aligned}$$

dimension 2 center manifold for λ near 1, and $b > 1/3$



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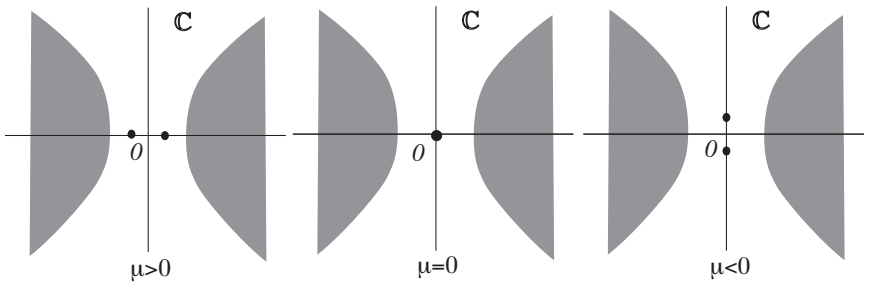

Reduced system on the center manifold $(\mu = \lambda - 1)$

$$\frac{dA}{d\varkappa} = B, \ \frac{dB}{d\varkappa} = P(A,\mu) + \rho(A,B,\mu)$$

$$P(A,\mu) = a\mu A - (3/2)aA^2 + h.o.t., \ \rho(A,B,\mu) = o\{(|A| + |B|)^p\}$$

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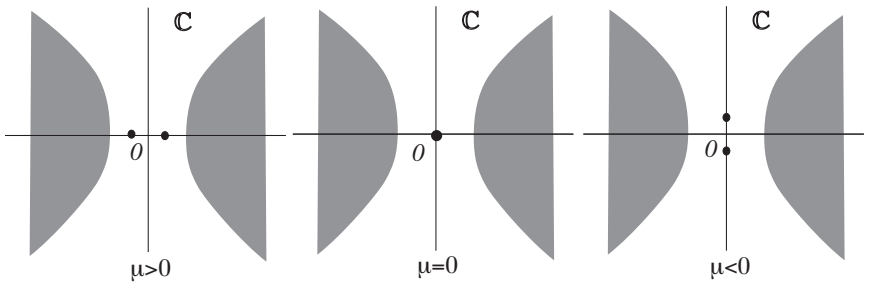

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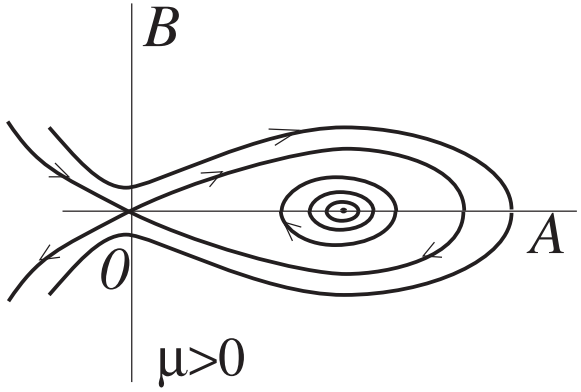
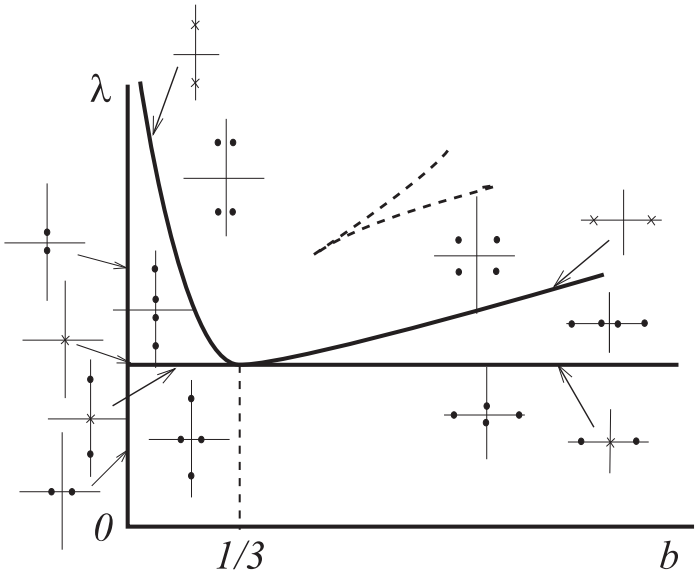
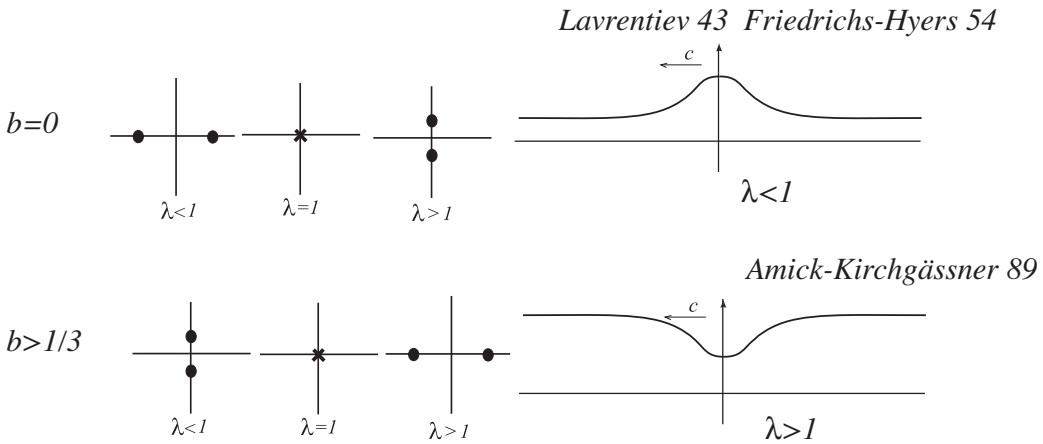


homoclinic curve = solitary wave

λ near 1, $b > 1/3$ or $b = 0$





dimension 4 center manifold for λ near 1 and $b < 1/3$

Normal form $0^2 i\omega$

$$\mathbf{L} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & i\omega & 0 \\ 0 & 0 & 0 & -i\omega \end{pmatrix}, V = (A, B, C, \overline{C})$$

$$\mathbf{N}(V) = \begin{pmatrix} AP_0(A, |C|^2) \\ BP_0(A, |C|^2) + P_1(A, |C|^2) \\ CP_2(A, |C|^2) \\ \overline{CP_2}(A, |C|^2) \end{pmatrix}$$

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reversible case $0^{2+} i\omega$:

$$\mathbf{S} \begin{pmatrix} A \\ B \\ C \\ \overline{C} \end{pmatrix} = \begin{pmatrix} A \\ -B \\ \overline{C} \\ C \end{pmatrix}, \mathbf{N}(V) = \begin{pmatrix} 0 \\ P(A, |C|^2) \\ iCQ(A, |C|^2) \\ -i\overline{C}Q(A, |C|^2) \end{pmatrix}, Q \text{ real valued}$$

dimension 4 center manifold for λ near 1 and $b < 1/3$

Truncated normal form at quadratic order (4-dim) $\mu = \lambda - 1$

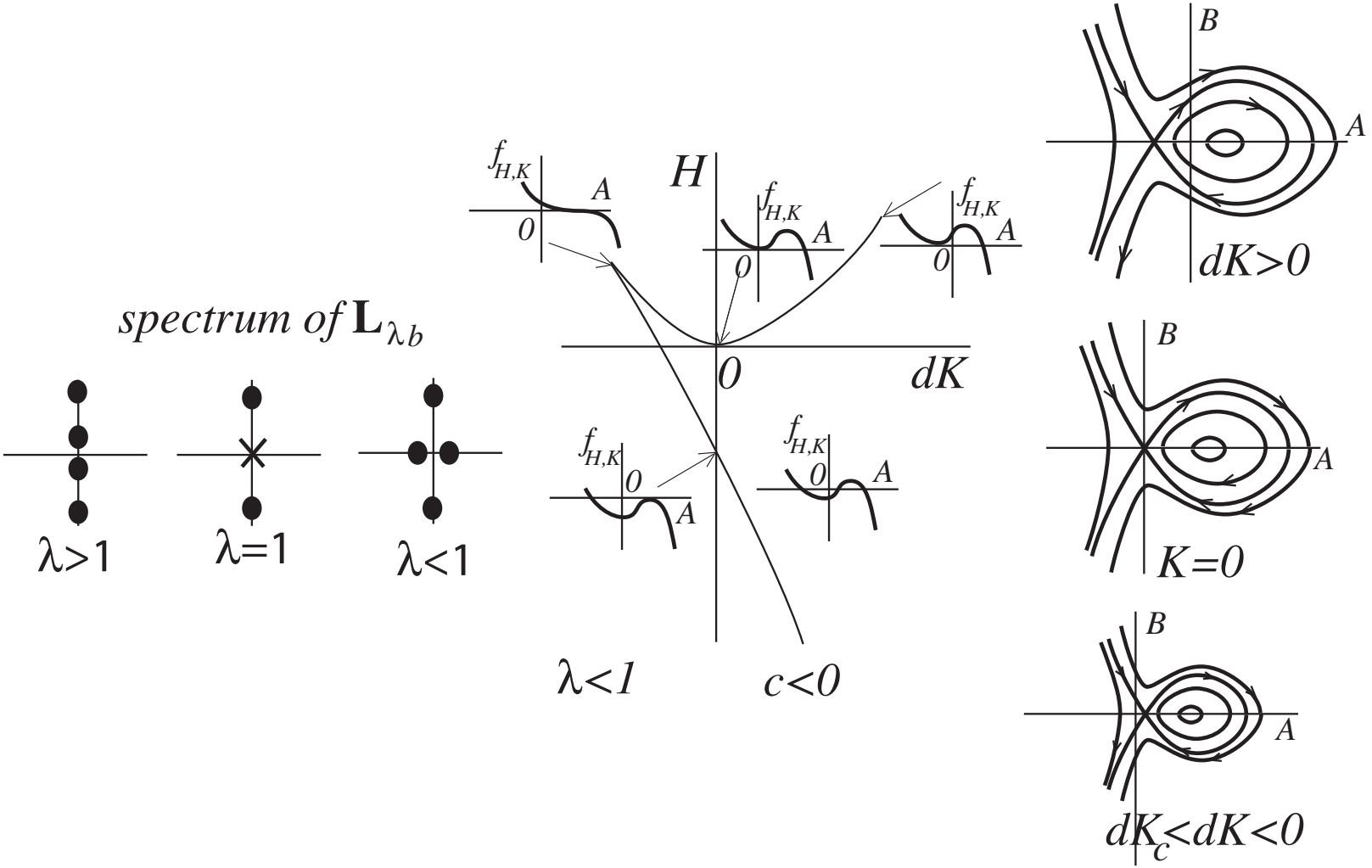
$$\begin{aligned} \frac{dA}{d\varkappa} &= B \\ \frac{dB}{d\varkappa} &= a\mu A + cA^2 + d|C|^2 \\ \frac{dC}{d\varkappa} &= i\omega C + iC(\gamma\mu + \delta A) \end{aligned}$$

coefficients $a, b, c, d, \gamma, \delta$ are real

2 first integrals K, H

$$\begin{aligned} K &= |C|^2, \\ \left(\frac{dA}{d\varkappa}\right)^2 &= B^2 = f_{H,K}(A) = \frac{2}{3}cA^3 + a\mu A^2 + 2dKA + H \end{aligned}$$

$\lambda < 1, b < 1/3$



For proofs on the full system, of no homoclinic to 0, and homocl. to exp. small per. orbits, see E.Lombardi, LNM 1741 (2000)

dimension 4 center manifold for $\lambda > C(b), b < 1/3$

looking for the normal form

$$\mathbf{L} = \begin{pmatrix} i\omega & 1 & 0 & 0 \\ 0 & i\omega & 0 & 0 \\ 0 & 0 & -i\omega & 1 \\ 0 & 0 & 0 & -i\omega \end{pmatrix}, \qquad \begin{aligned} V &= (A, B, \overline{A}, \overline{B}) \\ \mathbf{N}(V) &= (N_1, N_2, \overline{N_1}, \overline{N_2}) \end{aligned}$$

$$D\mathbf{N}(V)\mathbf{L}^* V = \mathbf{L}^*\mathbf{N}(V) \Rightarrow \begin{aligned} \mathcal{D}^* N_1 &= -i\omega N_1 \\ \mathcal{D}^* N_2 &= -i\omega N_2 + N_1 \end{aligned}$$

$$\mathcal{D}^* = -i\omega A\partial_A + (A - i\omega B)\partial_B + i\omega\overline{A}\partial_{\overline{A}} + (\overline{A} + i\omega\overline{B})\partial_{\overline{B}}$$

dimension 4 center manifold for $\lambda > C(b), b < 1/3$

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$$\mathcal{D}^* = -i\omega A\partial_A + (A - i\omega B)\partial_B + i\omega\overline{A}\partial_{\overline{A}} + (\overline{A} + i\omega\overline{B})\partial_{\overline{B}}$$

3 independent first integrals of $\mathcal{D}^* v = 0$:

$$v_1 = A\overline{A}, \quad v_2 = i(A\overline{B} - \overline{A}B), \quad v_3 = i\omega\frac{B}{A} + \ln A$$

dimension 4 center manifold for $\lambda > C(b), b < 1/3$

looking for the normal form

$$\mathbf{L} = \begin{pmatrix} i\omega & 1 & 0 & 0 \\ 0 & i\omega & 0 & 0 \\ 0 & 0 & -i\omega & 1 \\ 0 & 0 & 0 & -i\omega \end{pmatrix}, \qquad \begin{array}{r} V = (A, B, \overline{A}, \overline{B}) \\ \mathbf{N}(V) = (N_1, N_2, \overline{N_1}, \overline{N_2}) \end{array}$$

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$$\mathcal{D}^*\left(\frac{N_1}{A}\right) = 0 \Rightarrow N_1 = A\phi(v_1, v_2, v_3)$$

N_1 being polynomial in $(A, B, \overline{A}, \overline{B})$ implies that ϕ is a polynomial in its arguments, moreover independent of v_3

normal form - continued

$$N_1 = AP(v_1, v_2), \quad N_2 = BP(v_1, v_2) + AQ(v_1, v_2), \quad P, Q \text{ polynomials}$$

where $v_1 = A\overline{A}, \quad v_2 = i(A\overline{B} - \overline{A}B)$

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reversible case

$$\mathbf{S}\begin{pmatrix} A \\ B \\ \overline{A} \\ \overline{B} \end{pmatrix} = \begin{pmatrix} \overline{A} \\ -\overline{B} \\ A \\ -B \end{pmatrix}$$

$$\mathbf{N}(V) = \begin{pmatrix} iAP(|A|^2, i(A\overline{B} - \overline{A}B)) \\ iBP(|A|^2, i(A\overline{B} - \overline{A}B)) + AQ(\ldots) \\ -i\overline{A}P(|A|^2, i(A\overline{B} - \overline{A}B)) \\ -i\overline{B}P(|A|^2, i(A\overline{B} - \overline{A}B)) + \overline{A}Q(\ldots) \end{pmatrix}, \quad P, Q \text{ real valued}$$

dimension 4 center manifold for $\lambda > C(b), b < 1/3$

integration of the $(i\omega)^2$ reversible normal form

$$\begin{aligned}\dot{A} &= i\omega A + B + iAP(|A|^2, \frac{i}{2}(A\overline{B} - \overline{A}B), \mu) \\ \dot{B} &= i\omega B + iBP(|A|^2, \frac{i}{2}(A\overline{B} - \overline{A}B), \mu) + AQ(|A|^2, \frac{i}{2}(A\overline{B} - \overline{A}B), \mu)\end{aligned}$$

$O(2)$ symmetry: $R_\phi(A, B) = (Ae^{i\phi}, Be^{i\phi}), \quad R_\phi \mathbf{S} = \mathbf{S} R_{-\phi}$

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Integration of the normal form (1989)

First integrals:

$$\begin{aligned}K &= \frac{i}{2}(A\overline{B} - \overline{A}B) \\ H &= |B|^2 - \int_0^{|A|^2} Q(s, K, \mu) ds = |B|^2 - G(|A|^2, K, \mu)\end{aligned}$$

$$
\begin{aligned}
A &= r_0 e^{i(\omega\varkappa+\theta_0)}, \; B = r_1 e^{i(\omega\varkappa+\theta_1)} \\
u_0 &= r_0^2, \quad u_1 = r_1^2
\end{aligned}
$$

$$
\begin{aligned}
(\dot{u}_0)^2 &= 4 f_{H,K}(u_0,\mu), \quad f_{H,K}(u_0,\mu) = u_0[G(u_0,K,\mu)+H] - K^2 \\
u_1 &= G(u_0,K,\mu) + H \\
\dot{(\theta_1-\theta_0)} &= -\frac{K}{u_0 u_1}\frac{\partial}{\partial u_0} f_{H,K}(u_0,\mu)
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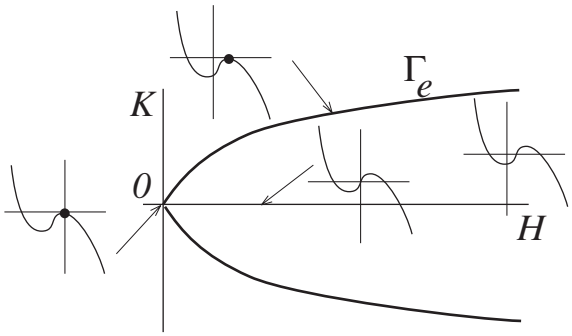
$$

$$
\begin{aligned}
Q(u_0,K,\mu) &= \alpha\mu + \beta u_0 + \gamma K + h.o.t. \\
f_{H,K}(u_0,\mu) &= (\beta/2)u_0^3 + (\alpha\mu+\gamma K)u_0^2 + H u_0 - K^2 + h.o.t.
\end{aligned}
$$

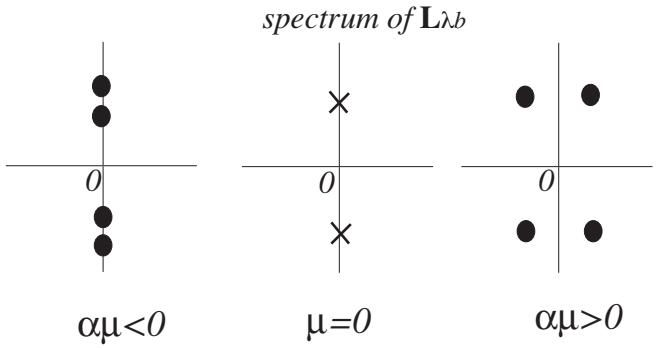
diagrams for reversible $i\omega^2$ normal form

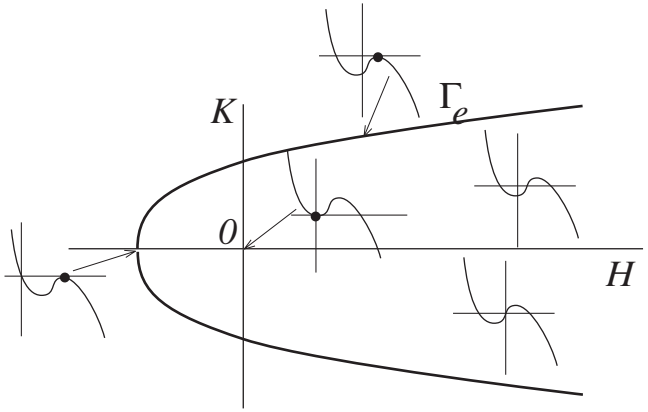
$$f_{H,K}(u_0,\mu) = (\beta/2)u_0^3 + (\alpha\mu + \gamma K)u_0^2 + Hu_0 - K^2 + h.o.t.$$

graphs of $f_{H,K}(.,\mu)$:

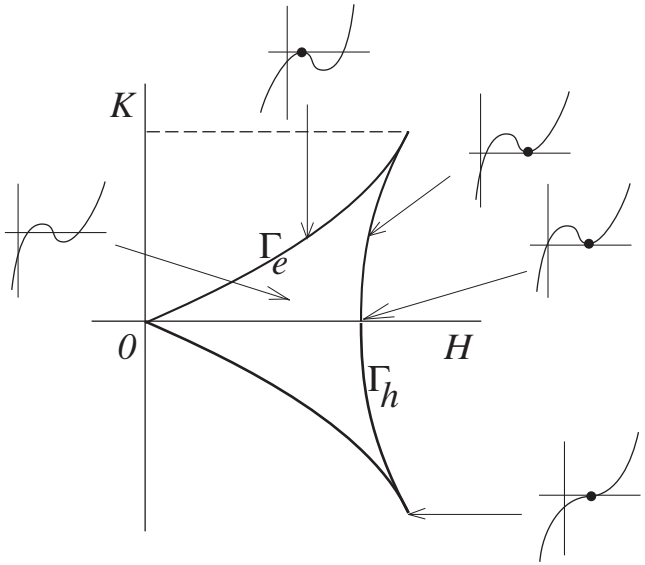


(i) $\alpha\mu<0,\ \beta<0$



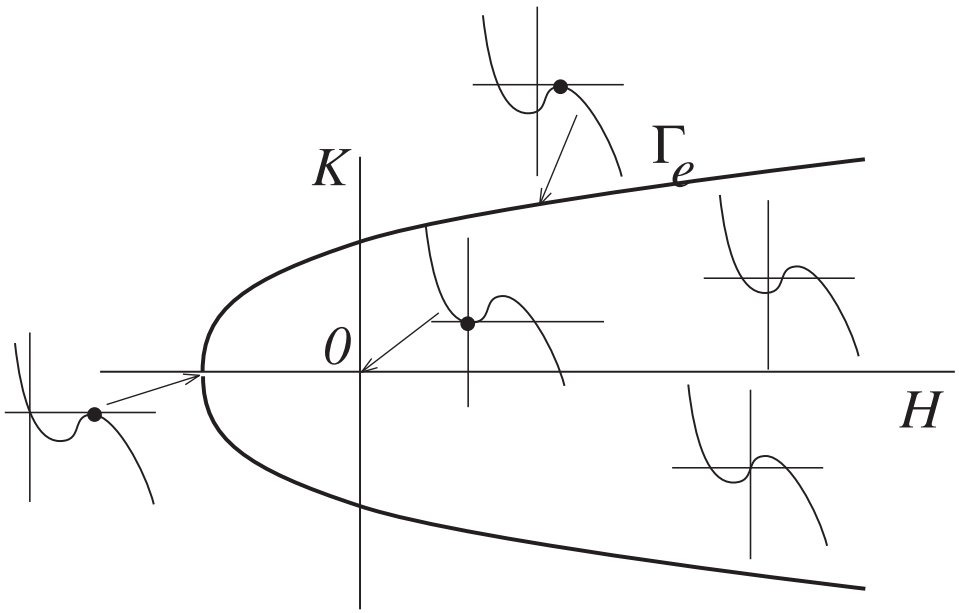


(iii) $\alpha\mu>0,\ \beta<0$

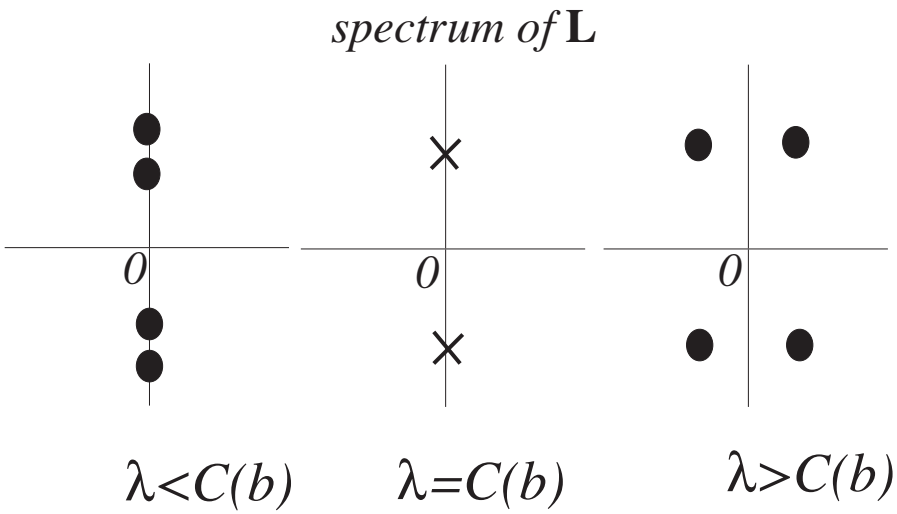


(ii) $\alpha\mu<0,\ \beta>0$

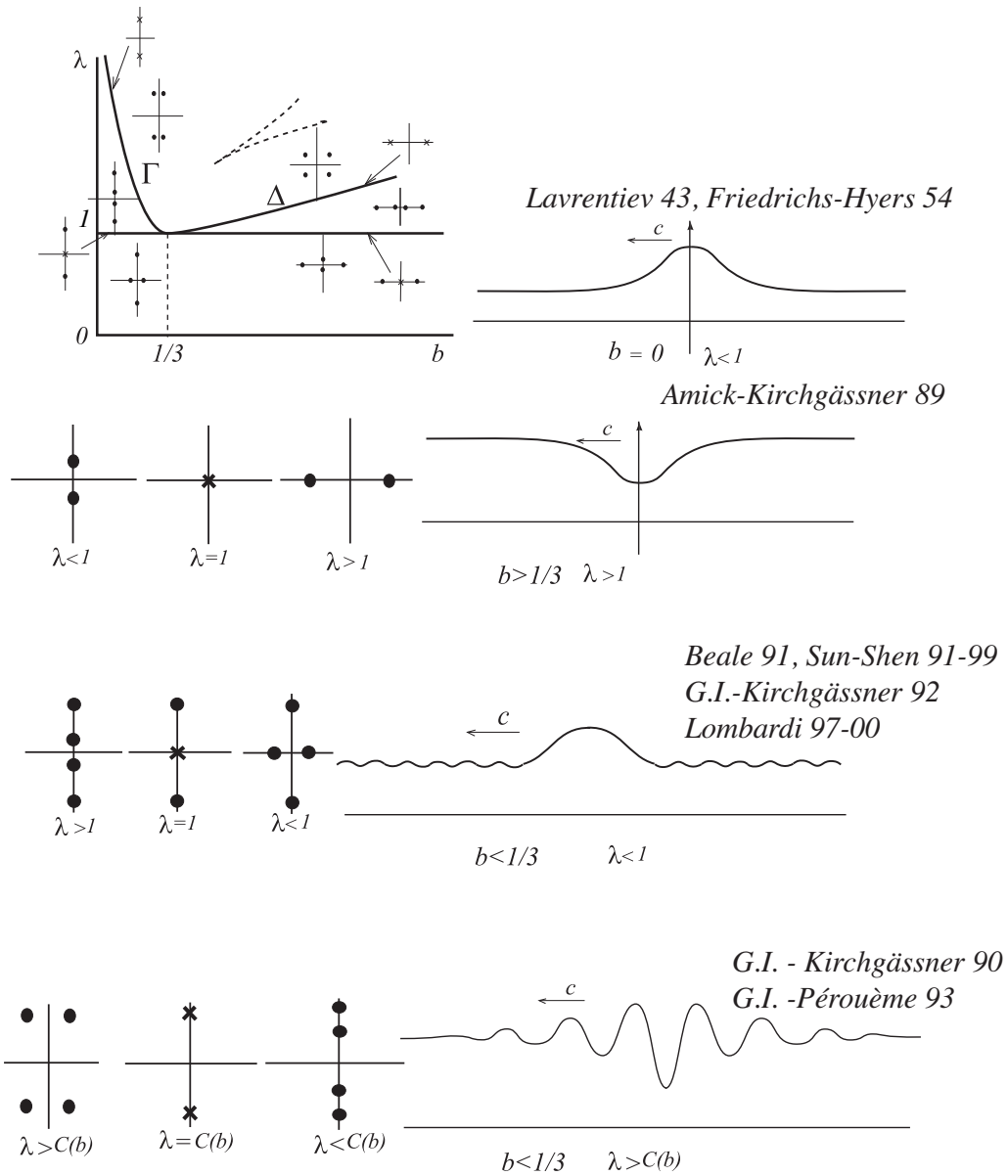
$\lambda > C(b), b < 1/3$



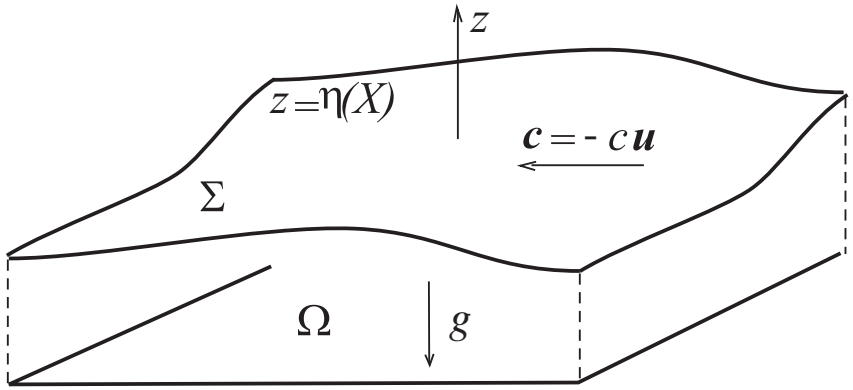
$\lambda > C(b)$, $\beta < 0$ *focusing case*



Solitary waves, depending on the spectrum I



The 3D travelling Water-Wave problem



$$\Delta\varphi = 0 \quad -1 < z < \eta(X), \quad \frac{\partial\varphi}{\partial z} = 0 \quad z = -1$$

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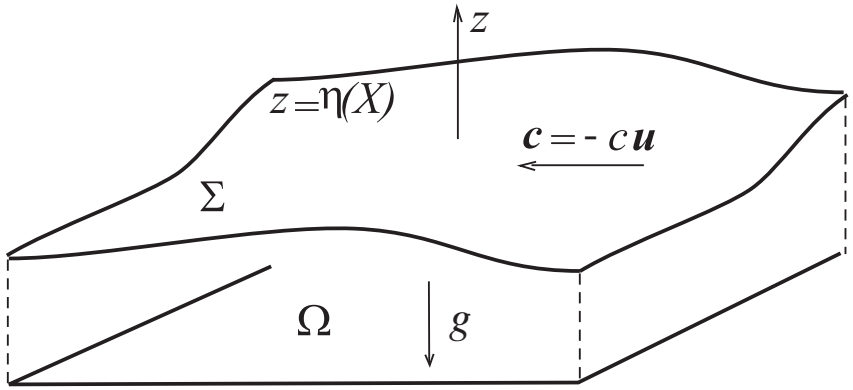

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Boundary conditions on $z = \eta(X)$

$$\nabla\eta \cdot (\mathbf{u} + \nabla_X\varphi) - \frac{\partial\varphi}{\partial z} = 0$$

$$\mathbf{u} \cdot \nabla\varphi + \frac{(\nabla\varphi)^2}{2} + \lambda\eta - b\nabla \cdot \left(\frac{\nabla\eta}{(1 + (\nabla\eta)^2)^{1/2}} \right) = 0$$

$$\lambda = gh/c^2, \; b = T/\rho hc^2$$

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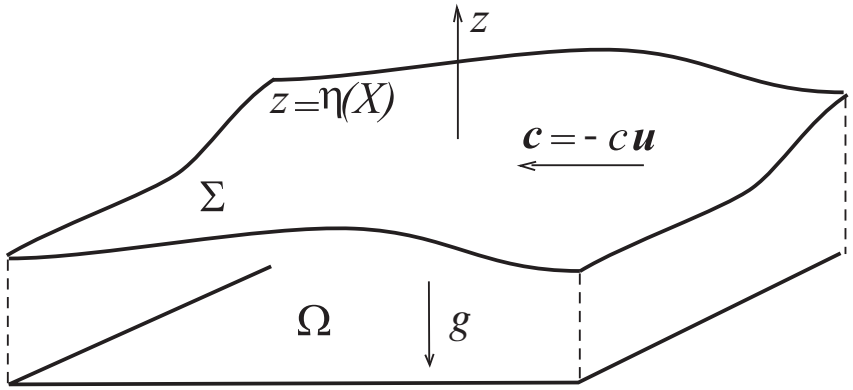

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Basic solution: (flat free surface) $\varphi = 0, \quad \eta = 0.$

Linearized operator for 3D waves

Analogous formulation as a spatial dynamical system

Eigenvalues ik of $\mathbf{L}_{\lambda,b}$:

$$(\lambda + b\chi_n^2)\chi_n \tanh \chi_n - k^2 = 0, \quad \chi_n^2 = k^2 + \frac{4\pi^2 n^2}{P^2}$$

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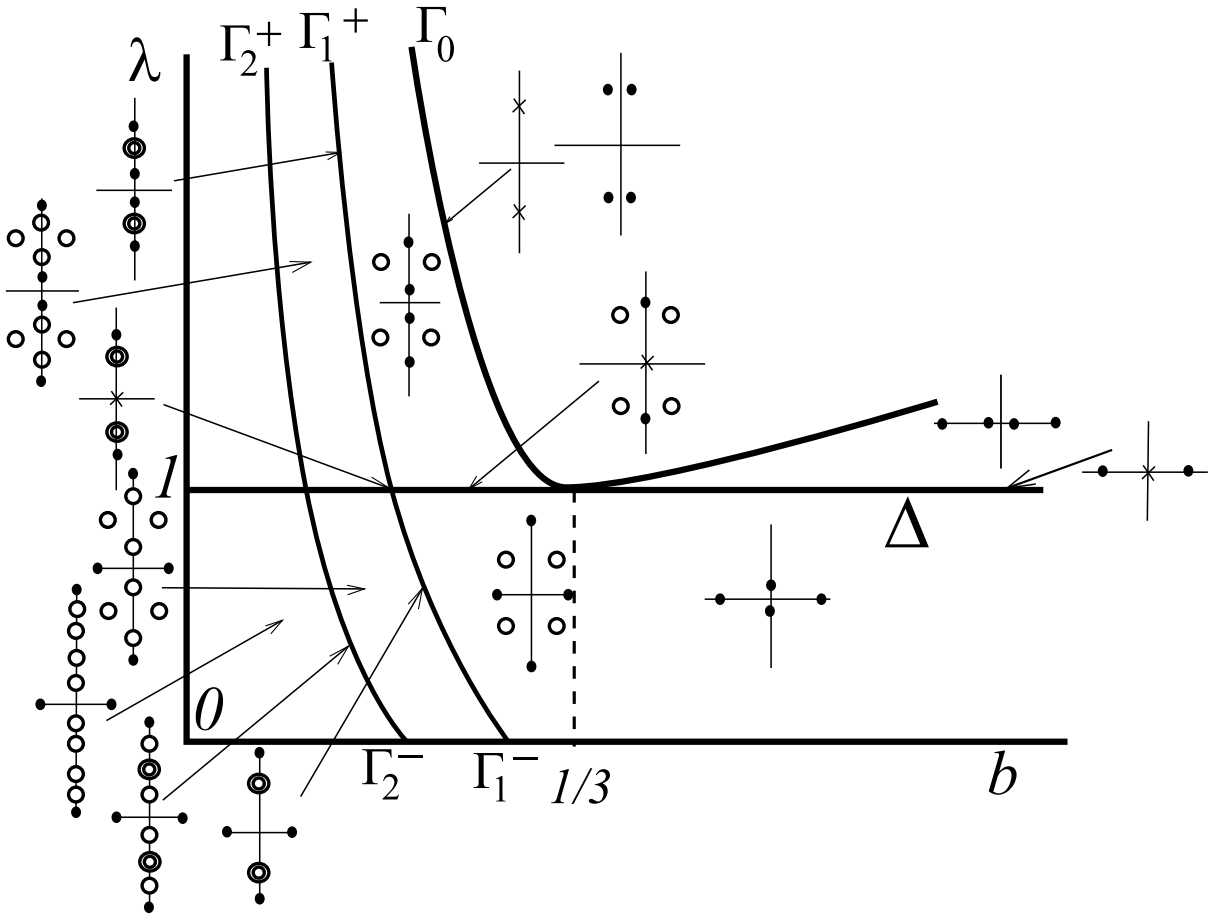
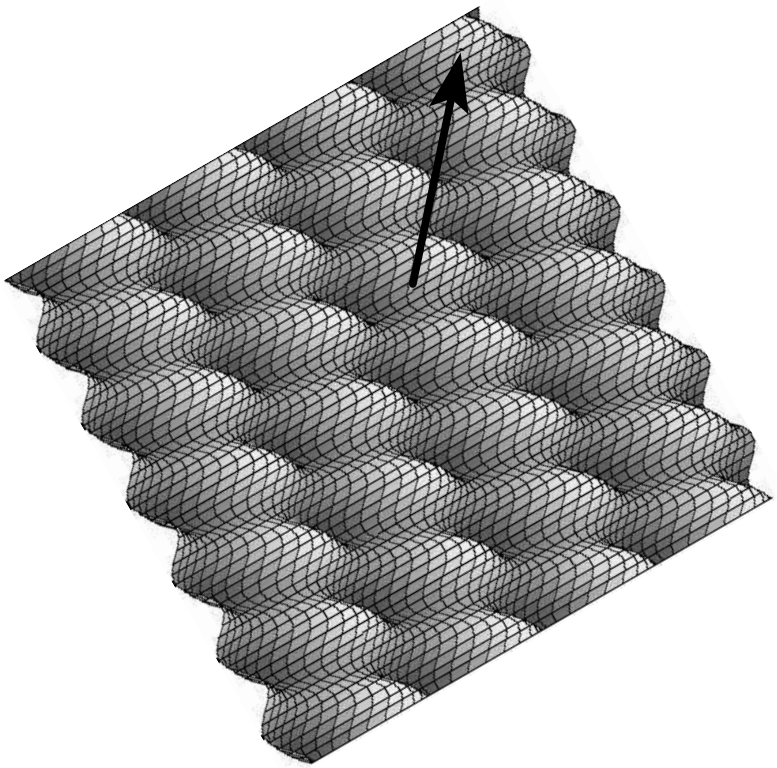
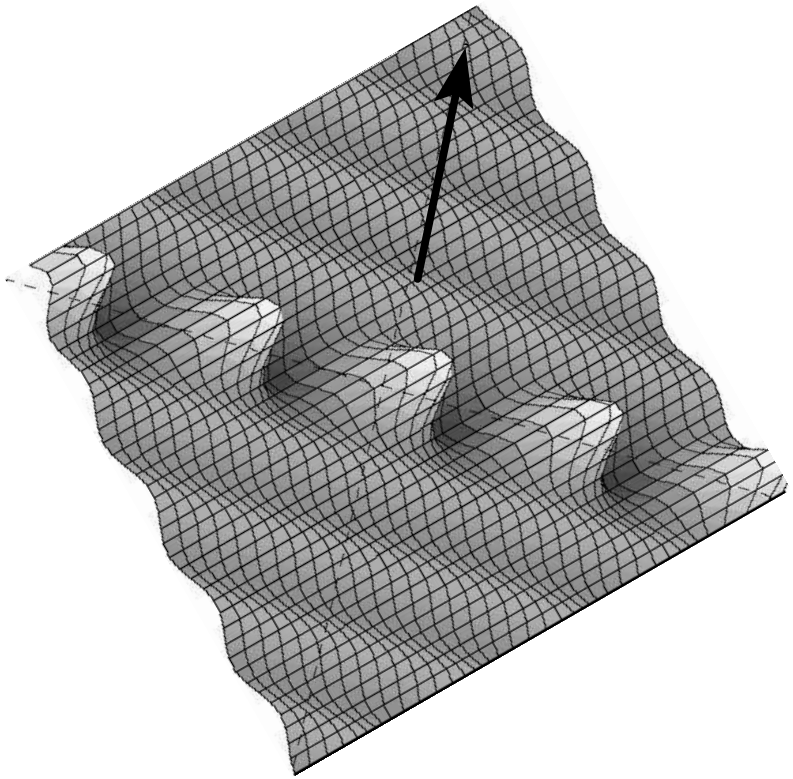
$$n = 0 \Rightarrow 2D \text{ case.}$$

P : wave length in the horizontal direction transverse to the wave propagation
When $b \longrightarrow 0$, the number of imaginary eigenvalues $\sigma = ik$ of $\mathbf{L}_{\lambda,b}$ tends to ∞ (with no accumulation)

Solitary waves, depending on the spectrum II







Three-dimensional water waves which have the profile of a periodic wave (left), and of a generalized solitary wave (right) in the direction of propagation, and are periodic in the perpendicular direction. The arrows indicate the direction of propagation.

Physical situations

Tsunami

$h = 4000\text{m}, \quad L = 100\text{km} \quad$ (wave length)

$\lambda_c = 1 \quad$ for $c \sim 195\text{m/s}$

$b = O(10^{-13}) \quad (\simeq$ no surface tension)

Solitary waves in a wave tank

$h = 10\text{cm}, \quad L = 250\text{cm}$

$\lambda_c = 1 \quad$ for $c \sim 98\text{cm/s}$

$b = O(10^{-4})$

Wind waves (generated by a storm)

$h = 1000\text{m}, \quad L = 150\text{m}$

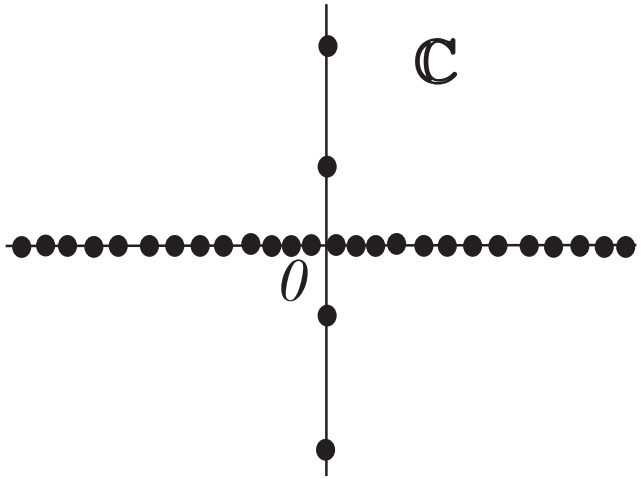
$\lambda_c = 1 \quad$ for $c \sim 100\text{m/s} \Rightarrow$ usually no solitary wave

$b = O(10^{-10})$

Limit $h \longrightarrow \infty$ for 2D travelling waves

$\lambda \to \infty$, $b \to 0$, $\lambda b = const$, length scale: $l = \frac{T}{\rho c^2} = hb$,
real eigenvalues σ satisfy

$$\sigma = \left\{\lambda b - \sigma^2\right\} \tan(\sigma/b)$$



eigenvalues accumulate on the real axis as $h \to \infty$

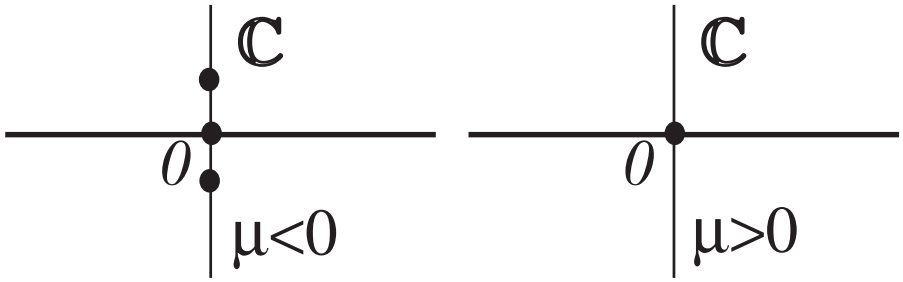
For $h = \infty$ the entire real axis $\in$ spectrum of $\mathbf{L}_0 \Rightarrow$ no gap with imaginary axis

Reduction to a center manifold inefficient

2D-Example with an infinitely deep layer - (Barrandon 2006)

Two superposed layers, bottom layer infinitely deep, large surface tension at the free surface (or rigid roof)

presence of an essential spectrum



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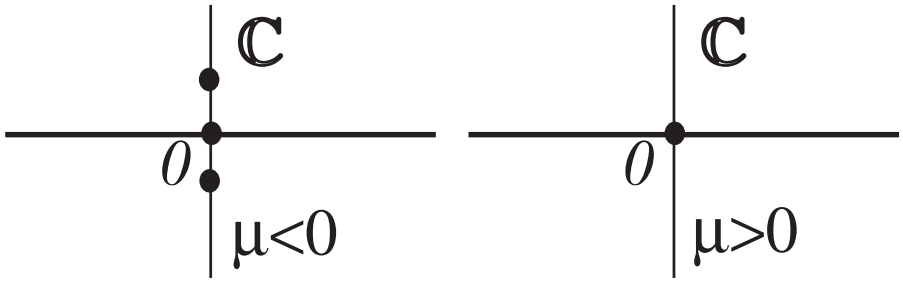

leading orders give the "Benjamin-Ono" equation

$$\mathcal{H}\left(\frac{dA}{dx}\right) = \mu A + bA^2$$

compare with $\frac{d^2A}{dx^2} = \mu A + bA^2$

similar dynamics, except asymptotics at infinity.
Open problem: passage to the limit $h \to \infty$