8th Winter School in Dynamical Systems

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Lecture Notes to The Course

Introduction to Smooth Ergodic Theory

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Deterministic Chaos.

Deterministic chaos – the appearance of "chaotic" motions in purely deterministic dynamical systems – is one of the most fundamental discoveries in the theory of dynamical systems in the second part of the last century.

It has been understood since the 1960s that a deterministic dynamical system can exhibit apparently stochastic behaviour. This is due to the fact that instability along typical trajectories of the system, which drives orbits apart, can coexist with compactness of the phase space, which forces them back together; the consequent unending dispersal and return of nearby trajectories is one of the hallmarks of chaos.

We shall only consider conservative dynamics, i.e., smooth systems on compact manifolds that preserve *smooth* measures (which is equivalent to volume; in particular, volume-preserving).

Is deterministic chaos prevalent? Of course, not every dynamical system exhibits instability; there are many systems whose behaviour is quite regular and not at all chaotic. Thus it is natural to ask which sort of behaviour prevails: is regularity the rule, and chaos the exception? Or is it the other way around? Perhaps there are different contexts in which either sort of behaviour is "typical". Many of the open problems regarding chaotic systems at the present time are related to this question.

In order to meaningfully address this issue, a number of things need to be made precise. What exactly do we mean by "chaos", and what sort of "instability" do we consider? What do we mean by a "typical" dynamical system, and what does it mean for one sort of behaviour to be the "rule", and the other the "exception"?

Intuitively, instability means that the behavior of orbits that start in a small neighborhood of a given one resembles that of the orbits in a small neighborhood of a hyperbolic fixed point. In other words the tangent space along the orbit $f^n(x)$ admits an invariant splitting

$$T_{f^n(x)}M = E^s(f^n(x)) \oplus E^u(f^n(x))$$

with contraction along the *stable subspace* E^s and expansion along the *unstable subspace* E^u.

One should distinguish *uniform* and *nonuniform* hyperbolicity. In the former case *every* trajectory is hyperbolic and the contraction and expansion rates are *uniform* in x on an invariant compact subset in the phase space (in particular, on the whole phase space). In the latter case the set of hyperbolic trajectories has *positive* (in particular, *full*) measure with respect to a smooth measure μ and the contraction and expansion rate *depend* on x. Thus nonuniformly hyperbolicity is a property of the system as well as of an invariant measure.

Nonuniformly hyperbolicity can also be expressed in more "practical" terms using the Lyapunov exponent of μ:

$$\chi(x,v) = \limsup_{n\to\infty} \frac{1}{n} \log \|df_x^n v\|,\ x \in M,\ v \in T_xM.$$

This means that for sufficiently large n,

$$\|df_x^n v\| \sim \exp(\chi(x,v) \pm \varepsilon)n.$$

If $\chi(x,v) > 0$, the differential asymptotically expends v with some exponential rate and if $\chi(x,v) < 0$, the differential asymptotically contracts v with some exponential rate.

Therefore, f is nonuniformly hyperbolic if for a.e. trajectory with respect to μ the Lyapunov exponent $\chi(x,v)$ is not equal to zero for **every** vector v; (μ is called a *hyperbolic measure*). In other words,

$$E^s(x) = \{v \in T_xM : \chi(x,v) < 0\},$$

$$E^u(x) = \{v \in T_xM : \chi(x,v) > 0\}.$$

Uniform Hyperbolicity (Anosov Systems)

A diffeomorphism f of a compact Riemannian manifold M is *Anosov* if for each $x \in M$ there is a continuous df-invariant decomposition of the tangent space $T_xM = E^s(x) \oplus E^u(x)$ and constants $c > 0$, $\lambda \in (0,1)$ s.t. for $x \in M$:

1. $\|d_x f^n v\| \le c\lambda^n \|v\|$ for $v \in E^s(x)$ and $n \ge 0$;

2. $\|d_x f^{-n}\| \le c\lambda^n \|v\|$ for $v \in E^u(x)$ and $n \ge 0$.

By the classical Hadamard–Perron theorem the sets

$$W^s(x) = \{y \in M : d(f^n(y), f^n(x)) \to 0, n \to \infty\},$$

$$W^u(x) = \{y \in M : d(f^n(y), f^n(x)) \to 0, n \to -\infty\},$$

are immersed smooth manifolds and $T_xW^{u,s}(x) = E^{u,s}(x)$. They form two invariant *stable* and *unstable foliations* with smooth leaves W^s and W^u. In general, the leaves of these foliations depend only continuously on x.

Example: A linear automorphism A of the torus T^2 with eigenvalues $\lambda > 1$ and $\lambda^{-1} < 1$. For $x \in T^2$ the stable and unstable subspaces $E^u(x)$ and $E^s(x)$ are the lines obtained by the translation of the eigenlines of A.

Stochastic Properties: An Anosov diffeomorphism f preserving a smooth measure is ergodic and for some $n > 0$ the map f^n is Bernoulli (in the sense of the probability theory).

The proof is based on the Hopf argument and uses a crucial property of stable and unstable foliations known as absolute continuity.

Absolute Continuity

Fix x and consider the family of local stable manifolds $\{V^s(w)\}$ for $w \in B(x,r)$. Let T^1 and T^2 be two transversals to this family. We define the *holonomy map* $\pi : T^1 \to T^2$ by $\pi(y) = T^2 \cap V^s(y)$. This map is a homeomorphism onto its image.

Given a submanifold W in M, we denote by ν_W the leaf-volume on W induced by the Riemannian metric to W.

Absolute Continuity Theorem.

The holonomy map π is absolutely continuous (with respect to the measures ν_{T^1} and ν_{T^2}) and the Jacobian $J^s(\pi)$ of the holonomy map is bounded from above and bounded away from zero.

Genericity: Anosov diffeomorphisms of class C^1 form an open set in the C^1 topology.

Anosov diffeomorphisms exist on tori and factors of compact nilpotent Lie groups (Smale). Existence of an Anosov diffeomorphism on a compact manifold imposes strong requirements on the topology (there are two continuous non-singular foliations, the action on the fundamental group is hyperbolic, etc.).

Basic Properties of Lyapunov Exponents

For any $x \in M$ and $v, w \in T_xM$:

1) $\chi(x, \alpha v) = \chi(x, v)$ for any $\alpha \neq 0$;

2) $\chi(x, v + w) \leq \max\{\chi(x, v), \chi(x, w)\}$;

3) $\chi(x, 0) = -\infty$.

It follows that for every x the function $\chi(\cdots, v)$ attaines finitely many distinct values:

$$\chi_1(x) < \cdots < \chi_{p(x)}(x), \quad p(x) \leq \dim M.$$

Each value $\chi_i(x)$ has its multiplicity $k_i(x)$ and the functions $\chi_i(x)$, $k_i(x)$ and $p(x)$ are (Borel) measurable f-invariant functions. Furthermore, there is a filtration

$$V_1(x) \subset \cdots \subset V_{p(x)}(x) = T_xM,$$

where each $V_i(x) = \{v \in T_xM : \chi(x, v) \leq \chi_i(x)\}$ is invariant under df and $\dim V_l(x) = \sum_{i=1}^{l} k_i(x)$.

Lyapunov–Perron Regularity

The point $x \in M$ is Lyapunov–Perron regular if and only if there exists a decomposition

$$T_x M = \bigoplus_{i=1}^{s(x)} E_i(x)$$

into subspaces $E_i(x)$ and numbers

$$\chi_1(x) < \cdots < \chi_{s(x)}(x)$$

such that:

1) $E_i(x)$ is invariant under $d_x f$, i.e., $d_x f E_i(x) = E_i(f(x))$ and depends (Borel) measurably on x; moreover, $\dim E_i(x) = k_i(x)$;

2) $V_l(x) = \bigoplus_{i=1}^{l} E_i(x)$, $l = 1, \ldots, k(x)$;

3) for $v \in E_i(x) \setminus \{0\}$,

$$\lim_{m \to \pm\infty} \frac{1}{m} \log \|d_x f^m v\| = \chi_i(x)$$

with uniform convergence on $\{v \in E_i(x) : \|v\| = 1\}$;

4) if $\mathbf{v} = (v_1, \ldots, v_{k_i(x)})$ is a basis of $E_i(x)$, then

$$\lim_{m \to \pm\infty} \frac{1}{m} \log \Gamma^{\mathbf{v}}_{k_i(x)}(m) = \chi_i(x) k_i(x);$$

where $\Gamma^{\mathbf{v}}_{k_i(x)}(m)$ is the volume of the parallelepiped generated by the vectors

$$(d_x f^m v_1, \ldots, d_x f^m v_{k_i(x)});$$

5) for any v, $w \in T_x M \setminus \{0\}$,

$$\lim_{m \to \pm\infty} \frac{1}{m} \log \angle(d_x f^m v, d_x f^m w) = 0,$$

where $\angle(v, w)$ is the angle between the vectors v and w.

Multiplicative Ergodic Theorem.

If f is a diffeomorphism of a compact smooth Riemannian manifold M, then the set of Lyapunov-Perron regular points has full measure with respect to any f-invariant Borel probability measure on M.

Non-uniform Hyperbolicity

A diffeomorphism f of a compact Riemannian manifold M is *non-uniformly hyperbolic* if there are a measurable df-invariant decomposition of the tangent space $T_xM = E^s(x) \oplus E^u(x)$ and measurable functions $c(x) > 0$, $k(x) > 0$ and $\lambda(x) \in (0,1)$ s.t. for any sufficiently small $\varepsilon > 0$ and almost any $x \in M$:

1. $\|df^n v\| \leq c(x)\lambda(x)^n\|v\|$ for $v \in E^s(x)$, $n \geq 0$;

2. $\|df^{-n}\| \leq c(x)\lambda(x)^n\|v\|$ for $v \in E^u(x)$, $n \geq 0$;

3. $\angle(E^s(x), E^u(x)) \geq k(x)$;

4. $c(f^m(x)) \leq e^{\varepsilon|m|}c(x)$, $k(f^m(x)) \geq e^{-\varepsilon|m|}k(x)$, $\lambda(f^m(x)) = \lambda(x)$, $m \in Z$.

The last property means that the rates of contraction and expansion (given by $\lambda(x)$) are constant along the trajectory and the estimates in 1. and 2. can deteriorate but with subexponential rate.

An Example of a Map With Nonzero Exponents

Starting with the hyperbolic toral automorphism A given by the matrix

$$A = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix},$$

consider the disk D_r centered at 0 of radius r. Let (s_1, s_2) be the coordinates in D_r obtained from the eigendirections of A. The map A is the time-1 map of the flow generated by the system of ODE:

$$\dot{s}_1 = s_1 \log\lambda, \quad \dot{s}_2 = -s_2 \log\lambda.$$

We obtain the Katok map by slowing down A near the origin.

Fix small $r_1 < r_0$ and consider the time-1 map g generated by the system of ODE in D_{r_1}:

$$\dot{s}_1 = s_1 \psi({s_1}^2 + {s_2}^2) \log \lambda,$$

$$\dot{s}_2 = -s_2 \psi({s_1}^2 + {s_2}^2) \log \lambda,$$

where ψ is a real-valued function on $[0, 1]$ satisfying:

1) ψ is a C^∞ function except for the origin;

2) $\psi(0) = 0$ and $\psi(u) = 1$ for $u \geq r_0$ where $0 < r_0 < 1$;

3) $\psi'(u) > 0$ for every $0 < u < r_0$;

4) $\int_0^1 \frac{du}{\psi(u)} < \infty$.

We have that $g(D_{r_2}) \subset D_{r_1}$ for some $r_2 < r_1$ and that g is of class C^∞ in $D_{r_1} \setminus \{0\}$ and coincides with A in some neighborhood of the boundary ∂D_{r_1}.

The map $G(x) = g(x)$ if $x \in D_{r_1}$ and $G(x) = A(x)$ otherwise, defines a homeomorphism of the torus which is a C^∞ diffeomorphism everywhere except for the origin. The map $G(x)$ is a slowdown of the automorphism A at 0.

1. G has nonzero Lyapunov exponents a.e.

2. G preserves a probability measure $d\nu = \kappa_0^{-1}\kappa\, dm$ where m is area and the density κ is a positive C^∞ function and is infinite at 0.

We change the coordinate system in the torus by a map ϕ s.t. the map $f = \phi \circ G \circ \phi^{-1}$ preserves area. Set

$$\phi(s_1, s_2) = \frac{1}{\sqrt{\kappa_0 \tau}} \left(\int_0^\tau \frac{du}{\psi(u)} \right)^{1/2} (s_1, s_2)$$

$(\tau = {s_1}^2 + {s_2}^2)$ in D_{r_1} and ϕ is identity otherwise. One can show that f is an area-preserving C^∞ diffeomorphism with nonzero Lyapunov exponents a.e.

Stable Manifolds

Stable Manifold Theorem.

For almost every $x \in M$ there exists a local stable manifold $V^s(x)$ such that $x \in V^s(x)$, $T_x V^s(x) = E^s(x)$, and if $y \in V^s(x)$ and $n \geq 0$ then

$$d(f^n(x), f^n(y)) \leq T(x)\lambda^n e^{\epsilon n} d(x, y),$$

where d is the distance in M induced by the Riemannian metric and $T(x) > 0$ is a Borel function satisfying

$$T(f^m(x)) \leq T(x)e^{\epsilon|m|}.$$

Stochastic Properties

f is a C^2 diffeomorphism of a compact smooth manifold M and μ an f-invariant smooth measure on M.

1. $M = \bigcup_{i \geq 0} \Lambda_i, \quad \Lambda_i \cap \Lambda_j = \emptyset;$

2. $\mu(\Lambda_0) = 0$ and $\mu(\Lambda_i) > 0$ for $i > 0$;

3. $f(\Lambda_i) = \Lambda_i$ and $f|\Lambda_i$ is ergodic for $i > 0$;

4. The Kolmogorov-Sinai entropy of f is given by the formula:

$$h_\mu(f) = \int_M \sum_{\chi_i(x) \geq 0} \chi_i(x)\, d\mu(x).$$

Nonuniformly hyperbolic dynamical systems are chaotic.

Non-uniform Hyperbolicity: Existence

Theorem (Dolgopyat, P.)

Any compact smooth Riemannian manifold M of dimension ≥ 2 admits a C^∞ volume-preserving diffeomorphism f which has nonzero Lyapunov exponents a.e. and is Bernoulli.

A. Katok: the case $\dim M = 2$;

M. Brin: the case $\dim M \geq 5$ and f has all but one nonzero Lyapunov exponent.

A Diffeomorphism With Nonzero Exponents on the 2-sphere.

We begin with a toral automorphism with 4 fixed points given by the matrix

$$A = \begin{pmatrix} 5 & 8 \\ 8 & 13 \end{pmatrix},$$

The fixed points are $x_1 = (0, 0)$, $x_2 = (1/2, 0)$, $x_3 = (0, 1/2)$, $x_4 = (1/2, 1/2)$.

For $i = 1, 2, 3, 4$ consider the disk D^i_r centered at x_i of radius r. Repeating the above arguments we construct a diffeomorphism g_i which coincides with A outside $D^i_{r_1}$. The map $G_1(x) = g_i(x)$ if $x \in D^i_{r_1}$ and $G_1(x) = A(x)$ otherwise, defines a homeomorphism of the torus which is a C^∞ diffeomorphism except for the point x_i. In each disk $D^i_{r_2}$ consider the coordinate change ϕ_i as above. We obtain a map G_2 of the torus which preserves area and has nonzero exponents a.e.

Using this map we construct a diffeomorphism of the sphere S^2 with the desired properties.

Consider the map

$$\zeta(s_1, s_2) = \left(\frac{{s_1}^2 - {s_2}^2}{\sqrt{{s_1}^2 + {s_2}^2}}, \frac{2 s_1 s_2}{\sqrt{{s_1}^2 + {s_2}^2}} \right).$$

This map is a double branched covering and is regular and C^∞ everywhere except for the points x_i, $i = 1, 2, 3, 4$ where it branches. It preserves area. The map $f = \zeta \circ G_2 \circ \zeta^{-1}$ has all the desired properties.