Delay equations with engineering applications

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Contents

Delay equations arise in mechanical systems…

… by the information system (of control), and by the contact of bodies.

- Linear stability & subcritical Hopf bifurcations
- Force control and **balancing – human and robotic**
- Contact problems
  - Shimmying wheels (of trucks and motorcycles)
  - Machine tool vibrations

Main references

Insperger T, Wahi P, Colombo A, Stepan G, Di Bernardo M, Hogan SJ, Full characterization of act-and-wait control for first-order unstable lag processes, *Journal of Vibration and Control* **16** (2010) 1209-1233.

Insperger T, Stepan G, Turi J, Delayed feedback of sampled higher derivatives, *Phil Trans of The Royal Society A – Mathematical Physical and Engineering Sciences* **368** (2010) 469-482.

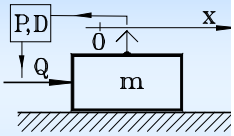
Stepan G, Introduction to Delay effects in brain dynamics, *Philosophical Transactions of The Royal Society A – Mathematical Physical and Engineering Sciences* **367** (2009) 1059-1062.

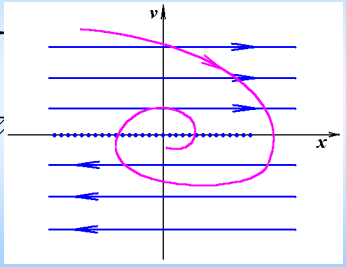
Stepan G, Delay effects in the human sensory system during balancing, *Phil Trans of The Royal Society A – Mathematical Physical and Engineering Sciences* **367** (2009) 1195-1212.

Haller, G., Stepan, G., Micro-chaos in digital control, *Journal of Nonlinear Science* **6** (1996) 415-448.

Position control

1 DoF models $\Rightarrow$ x

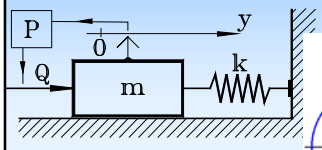


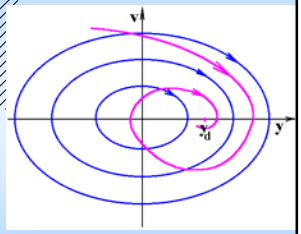


Blue trajectories:
$Q = 0$

Pink trajectories:
$Q = -Px - D\dot{x}$

Force control



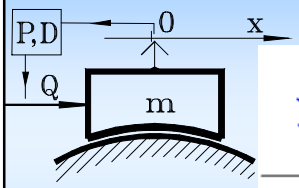


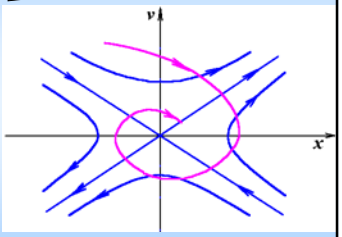
Desired contact force:
$F_d = ky_d$;

Sensed force:
$F_s = ky$

Control force: $Q = -P(F_d - F_s) - D\dot{F}_s + F_{s \text{ or } d}$

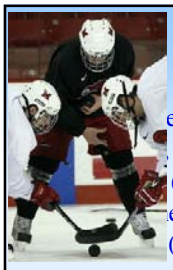
Stabilization (balancing)





Control force:
$Q = -Px - D\dot{x}$

Special case of force control: with $k < 0$



Chaos is amusing

e games – strong nonlinearities:
play cards/chess, computer games
(football, soccer, basketball… **impact**)
ear rules (tennis 6/4,0/6,6/4, snooker)
(skiing, skating, kayak, surfing,…)

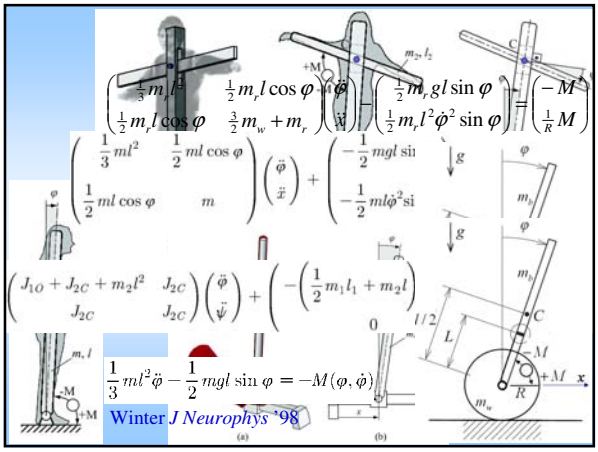
Ice-hockey (one of the most unpredictable games)
- impacts between club/puck/wall
- impacts between players/wall
- self-**balancing** of players on ice (non-holonomic)
- continuous and fast exchanging of players

Modeling balancing

Special cases of force control:
- position control with zero stiffness ($k = 0$)
- stabilization with negative stiffness ($k < 0$)

Analogue delay / human balancing

Digital effects / robotic balancing
- quantization in time: *sampling* – **linear**
- quantization in space: *round-off* errors at ADA converters – **non-linear**



$$\begin{pmatrix} \frac{1}{3}ml^2 & \frac{1}{2}ml\cos\varphi \\ \frac{1}{2}ml\cos\varphi & m \end{pmatrix}\begin{pmatrix}\ddot{\varphi}\\ \ddot{x}\end{pmatrix}+\begin{pmatrix} -\frac{1}{2}mgl\sin\varphi \\ -\frac{1}{2}ml\dot{\varphi}^2\sin\varphi \end{pmatrix}$$

$$\begin{pmatrix} J_{1O}+J_{2C}+m_2l^2 & J_{2C} \\ J_{2C} & J_{2C} \end{pmatrix}\begin{pmatrix}\ddot{\varphi}\\ \ddot{\psi}\end{pmatrix}+\begin{pmatrix} -\left(\frac{1}{2}m_1l_1+m_2l\right) \\ 0 \end{pmatrix}$$

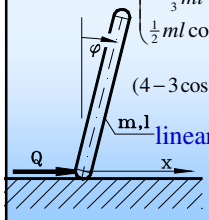
$$\frac{1}{3}ml^2\ddot{\varphi}-\frac{1}{2}mgl\sin\varphi=-M(\varphi,\dot{\varphi})$$

Winter *J Neurophys* '98

Balancing inverted pendula

All the above models lead to the same math problem

$n = 2$ DoF $\Rightarrow$ φ, x ; x – cyclic coordinate



$$\begin{pmatrix} \frac{1}{3}ml^2 & \frac{1}{2}ml\cos\varphi \\ \frac{1}{2}ml\cos\varphi & m \end{pmatrix}\begin{pmatrix}\ddot{\varphi}\\ \ddot{x}\end{pmatrix}-\begin{pmatrix} \frac{1}{2}mgl\sin\varphi \\ \frac{1}{2}ml^2\dot{\varphi}^2\sin\varphi \end{pmatrix}=\begin{pmatrix}0\\ Q\end{pmatrix}$$

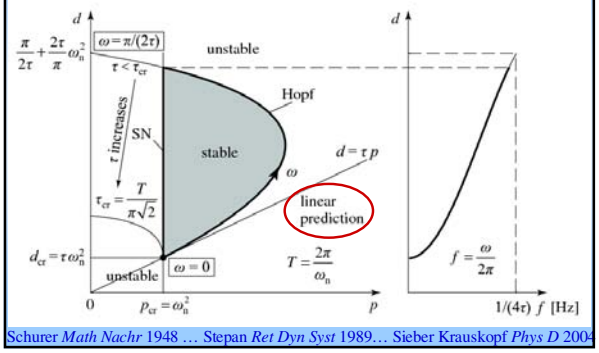
$$(4-3\cos^2\varphi)\ddot{\varphi}+\frac{3}{2}\dot{\varphi}^2\sin(2\varphi)-6\frac{g}{l}\sin\varphi=-6\frac{Q}{ml}\cos\varphi$$

linearization at $\varphi = 0$ (same for all cases)

$$\ddot{\varphi}-6\frac{g}{l}\varphi=-\frac{6}{ml}Q \qquad \ddot{\varphi}-\omega_n^2\varphi=u(\varphi,\dot{\varphi},\ldots)$$

$$\ddot{\varphi}(t)+d\dot{\varphi}(t-\tau)+p\varphi(t-\tau)-\omega_n^2\varphi(t)=0$$

$$\lambda^2+d\lambda e^{-\lambda\tau}+pe^{-\lambda\tau}-\omega_n^2=0 \;\Rightarrow\; \mathrm{Re}\,\lambda_{1,2,\ldots}<0$$



Schurer *Math Nachr* 1948 … Stepan *Ret Dyn Syst* 1989… Sieber Krauskopf *Phys D* 2004

$$\ddot{\varphi}(t)+d\dot{\varphi}(t-\tau)+p\varphi(t-\tau)-\omega_n^2\varphi(t)=0$$

$$\lambda^2+d\lambda e^{-\lambda\tau}+pe^{-\lambda\tau}-\omega_n^2=0 \;\Rightarrow\; \mathrm{Re}\,\lambda_{1,2,\ldots}<0$$

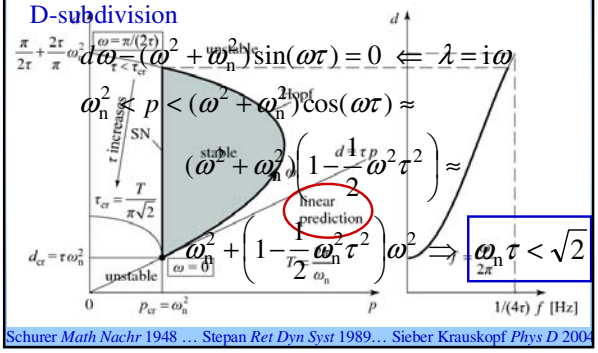
D-subdivision

$$d\omega=(\omega^2+\omega_n^2)\sin(\omega\tau)=0 \Leftarrow \lambda=i\omega$$

$$\omega_n^2<p<(\omega^2+\omega_n^2)\cos(\omega\tau)\approx$$

$$(\omega^2+\omega_n^2)\left(1-\frac{1}{2}\omega^2\tau^2\right)\approx$$

$$\omega_n^2+\left(1-\frac{1}{2}\omega_n^2\tau^2\right)\omega^2 \Rightarrow \omega_n\tau<\sqrt{2}$$



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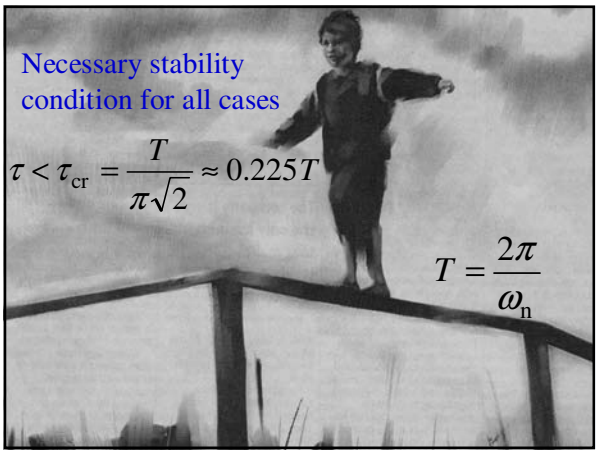


Human balancing

$$\ddot{\varphi} - 6\frac{g}{l}\varphi = -\frac{6}{ml}Q$$

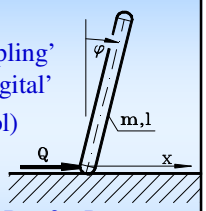
Analogous or digital?

Winking, eye-motion – 'self-sampling'

plus neurons firing… still, not 'digital'

1) $Q(t) = P\varphi(t) + D\dot{\varphi}(t)$ (PD control)

$$\ddot{\varphi} + \frac{6}{ml}D\dot{\varphi} + \frac{6}{ml}(P - mg)\varphi = 0$$

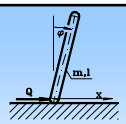


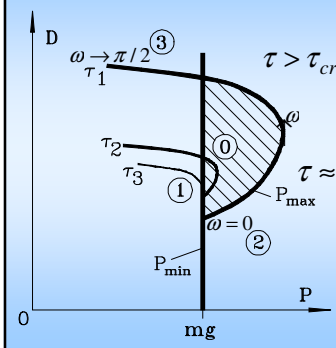
$\varphi \equiv 0$ is exponentially stable $\Leftrightarrow D > 0,\ P > mg$

2) $Q(t) = P\varphi(t-\tau) + D\dot{\varphi}(t-\tau)$ (with 'reflex' delay τ)

$$\ddot{\varphi}(t) + \frac{6}{ml}D\dot{\varphi}(t-\tau) + \frac{6}{ml}P\varphi(t-\tau) - \frac{6g}{l}\varphi(t) = 0$$

Critical reflex delay





$$\tau > \tau_{cr} = \sqrt{\frac{l}{3g}} \Rightarrow \text{instability}$$

$$l = 0.3\,[\text{m}]$$

$$\tau \approx \sqrt{0.3/(3\cdot 10)} = 0.1\,[\text{s}]$$

$$0 < \frac{\omega}{2\pi\tau} = f < \frac{1}{4\tau} \approx 2.5\,[\text{Hz}]$$

Experimental observations

Kawazoe (1992)

untrained

manual

control

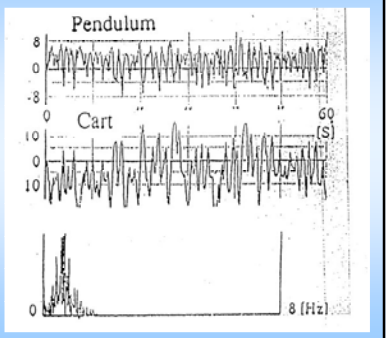
Betzke (1994)

target

shooting

0.3 – 0.7 [Hz]

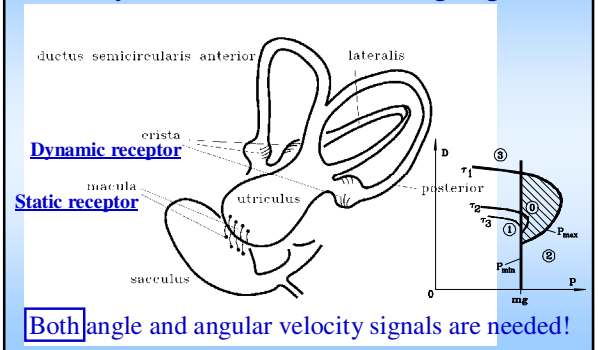
Avatar



Poise – noun, uncountable /pOIz

- A state of balance, equilibrium or stability
- Bearing or deportment of the head or body
  In the old days, when rich people lived in fine houses, young ladies were encouraged to develop a way of sitting, eating, talking, etc that would attract the right sort of man (i.e., at least as rich...). This way was limited to elegant, controlled movements. This 'way' was called poise…(in French, pose?)
- A condition of hovering, or being suspended
- Freedom from embarrassment or affectation
- A cgs unit of dynamic viscosity

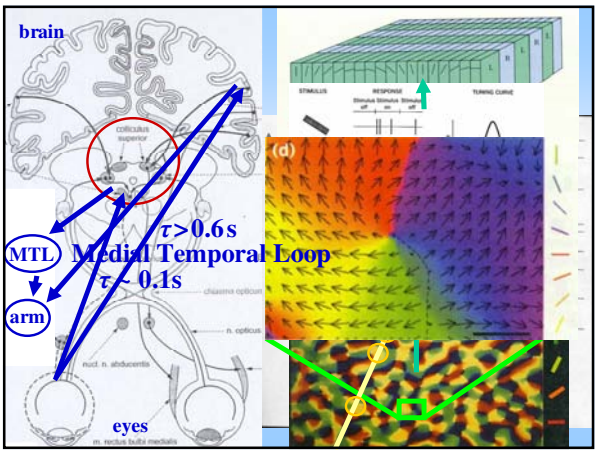
Labyrinth – human balancing organ

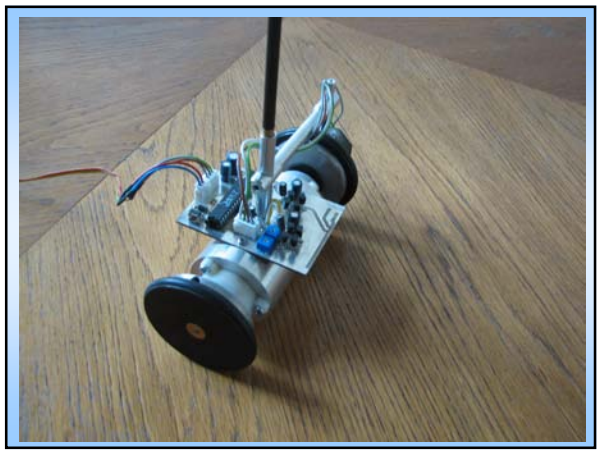


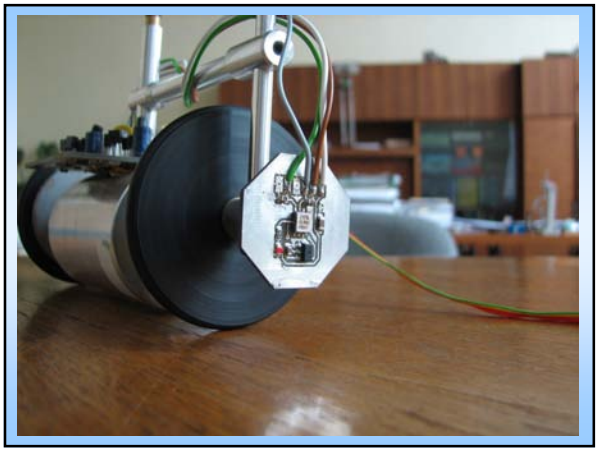
Both angle and angular velocity signals are needed!

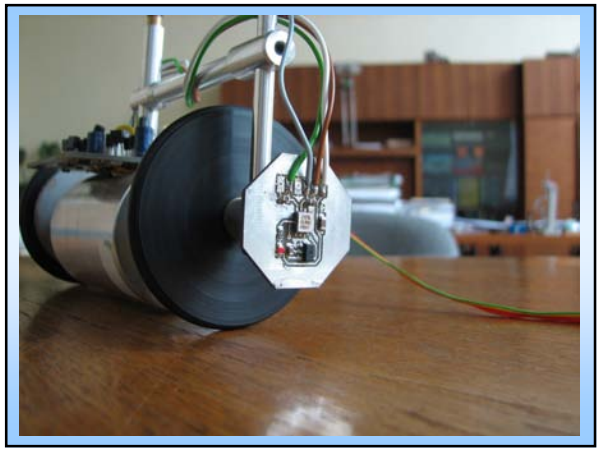
Vision and balancing

- Vision can help balancing even when labyrinth does not function properly (e.g., ‘dry ear’ effect)
- The visual system also provides the necessary **angle and angular velocity** signals!
- But: the **vertical direction is needed** (buildings, trees), otherwise it fails…
- Delay in vision – ‘thinking’, ‘recognition’,…
- Sensitivity, resolution of vision, threshold,…

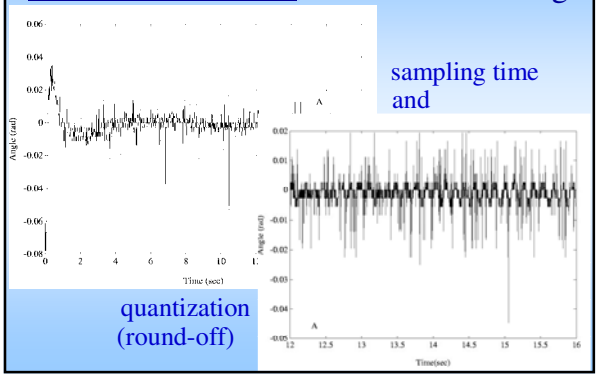








Random oscillations of robotic balancing



sampling time and quantization (round-off)

Stability of digital control – round-off

h – one digit converted to control force

$$u_j = -\frac{6}{ml} h \,\mathrm{int}\left(\frac{D\dot{\varphi}(t_j-\tau)+P\varphi(t_j-\tau)}{h}\right)$$

$$\mathbf{x}^{j+1} = \mathbf{B}\mathbf{x}^j + \mathbf{g}(\mathbf{x}^j)$$

$$\omega = \tau\sqrt{6g/l} \qquad \mathbf{g}(\mathbf{x}^j) = \begin{pmatrix} 0 \\ 0 \\ -\frac{6\tau^2}{ml} h \,\mathrm{int}\left(\frac{P}{h} x_1^j + \frac{D}{\tau h} x_2^j\right) \end{pmatrix}$$

$$\mathbf{B} = \begin{pmatrix} \mathrm{ch}\,\omega & \frac{\mathrm{sh}\,\omega}{\omega} & \frac{\mathrm{ch}\,\omega - 1}{\omega^2} \\ \omega\,\mathrm{sh}\,\omega & \mathrm{ch}\,\omega & \frac{\mathrm{sh}\,\omega}{\omega} \\ 0 & 0 & 0 \end{pmatrix}$$

$\det(\lambda\mathbf{I} - \mathbf{B}) = 0 \Rightarrow$

$\lambda_1 = \mathrm{e}^{\omega} > 1, \; \lambda_2 = \mathrm{e}^{-\omega}, \; \lambda_3 = 0$

1D cartoon – the *micro-chaos* map

Drop 2 dimensions, rescale x with h $\Rightarrow$ a ~ e^{ω}, $b \sim P$

$$x_{j+1} = a x_j - b\,\mathrm{int}(x_j)$$

A pure math approach ($p > 0$, $p < q$)

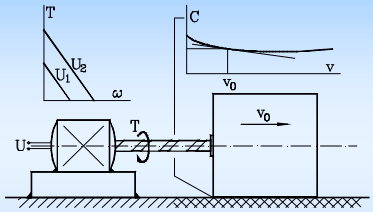
$$\dot{y}(t) = p y(t) - q\,\mathrm{int}\big(y(\mathrm{int}(t))\big)$$

solution with $x_j = y(j)$ leads to μ-chaos map,

$a = e^{p}$, $b = q(e^{p} - 1)/p$ $\Rightarrow$ $a > 1$, $(0 <) a - b < 1$

small scale: $x_{j+1} = a\,x_j$, large scale: $x_{j+1} = (a - b)\,x_j$

Digital stabilization of stick-slip



$$k = -\mu'(v_0)\,g$$

$$u(t) = v(t) - v_0$$

$$D > k$$

$$\dot{u}(t) - k u(t) = -Dh\,\mathrm{int}\frac{u(j\Delta t)}{h}, \quad t \in [j\Delta t, (j+1)\Delta t)$$

"normal form" $\dot{u}(t) = k\,u(t) - D\,\mathrm{int}\,(u(\mathrm{int}\,t))$

digital effects: round-off & sampling

Stick-slip – discrete map

Solution for one sampling interval

$$u((j+1)\Delta t) = e^{k\Delta t} u(j\Delta t) - \frac{D}{k}(e^{k\Delta t} - 1) h\,\mathrm{int}\frac{u(j\Delta t)}{h}$$

By $x_j = u(j\Delta t)$ we obtain the micro-chaos map

$$x_{j+1} = a x_j - b\,\mathrm{int}(x_j)$$

$$1 < a \qquad 0 \le a - b < 1$$

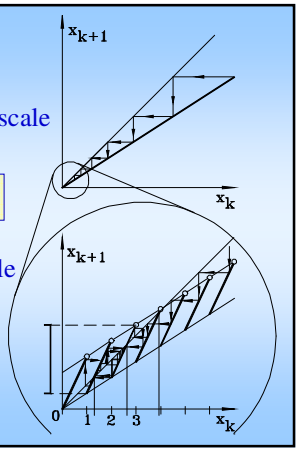
Micro-chaos map

large scale

$$x_{k+1} = a x_k - b\,\mathrm{int}(x_k)$$

small scale

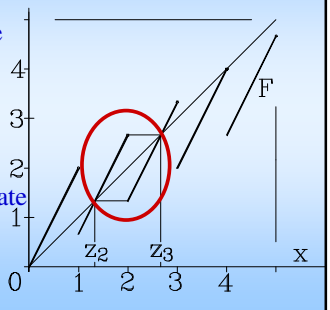
Typical in digitally controlled machines



Butterfly effect

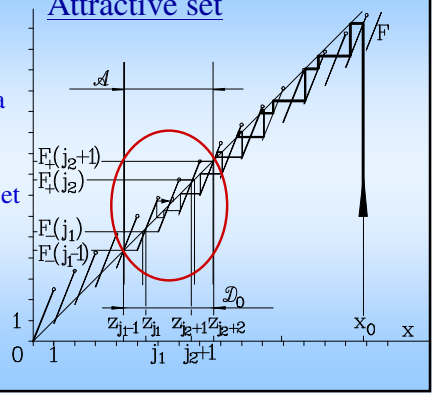
Prop. 1 The map has sensitive dependence on initial conditions

Horseshoe (Smale): invariant Cantor set on which the map is topologically conjugate to a Bernoulli shift on 2 symbols.

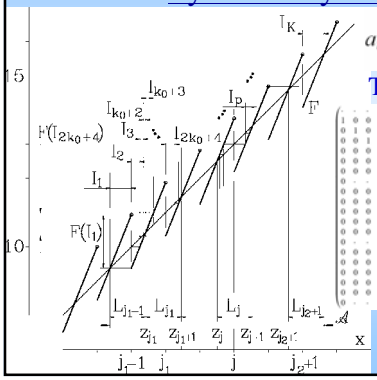


Attractive set

Prop. 2 ***A*** is a positively invariant attractive set



Symbolic dynamics

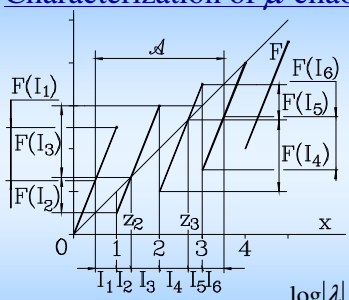


$$a_{ij} = \begin{cases} 1 & \text{if } \overline{F(I_i)} \supset I_j \\ 0 & \text{otherwise.} \end{cases}$$

Transition matrix **A**:

All elements of $\mathbf{A}^{K-1}$ are non-zero

Characterization of μ-chaos (a=5/2, b=2)



$$\mathbf{A} = \begin{pmatrix} 0&0&1&0&0&0 \\ 1&1&0&0&0&0 \\ 0&0&1&1&1&0 \\ 0&1&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&0&0&1&0&0 \end{pmatrix}$$

Fractal dim.:

$$C(\Lambda) \geq \frac{\log|\lambda|_{\max}}{\log a} = \frac{\log 2.38}{\log 2.5} = 0.94$$

2D micro-chaos map

ZOH + *delay* and round-off for 1st order process:

$$\dot{y}(t) = py(t) - q\,\mathrm{int}\big(y(\mathrm{int}(t)-1)\big) \qquad (p>0,\ p<q)$$

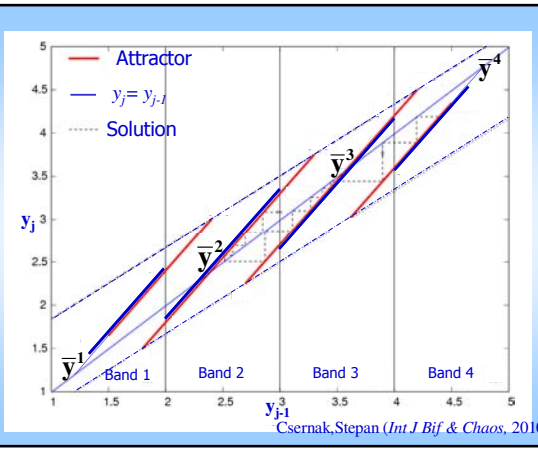
Solution and Poincare lead to

$$x_{j+1} = ax_j - b\,\mathrm{int}(x_{j-1}) \qquad (a>1,\ a-b<1)$$

Linearization at fixed points leads to eigenvalues

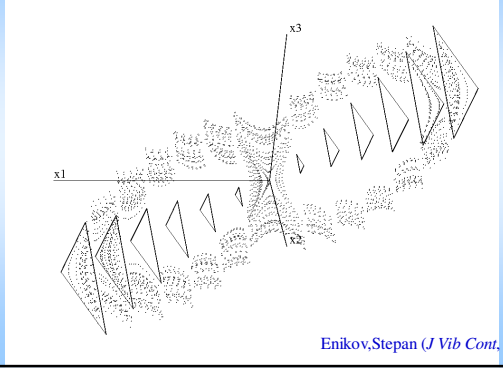
$$\begin{pmatrix} x_j \\ x_{j+1} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0 & a \end{pmatrix}\begin{pmatrix} x_{j-1} \\ x_j \end{pmatrix} + \begin{pmatrix} 0 \\ -b\,\mathrm{int}(x_{j-1}) \end{pmatrix} \Rightarrow \mu_1 = 0,\ \mu_2 = a\,(>1)$$

So in 1 step the solution settles at an attractor that has a graph similar to the 1D micro-chaos map



Csernak,Stepan (*Int J Bif & Chaos*, 2010)

3D micro-chaos



Enikov,Stepan (*J Vib Cont*, 98)

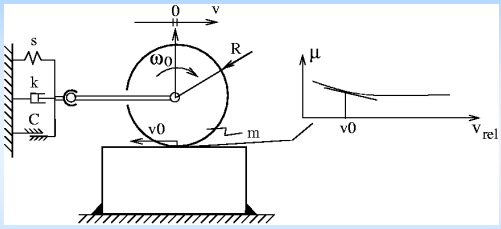
Transient micro-chaos

- PID control of machines in the presence of Coulomb friction
- Switch of robots from position control to force control, transient impacts with an elastic environment
- Stabilization of an unstable equilibrium or an unstable periodic motion of a machine (e.g.: balancing, control of chaos, …)

Transient chaos

- Unpredictable transient behavior of machines
- The transient motion disappears "suddenly"
- Exponential decay cannot be used
- Life expectancy, kick-out number, escape rate, etc. can be defined
- Examples: Lorenz repellor (Yorke, 1979), tethered satellites (Troger, 1998), shimmy, robotics, digital control, control of chaos…

Stability problems of polishing tools

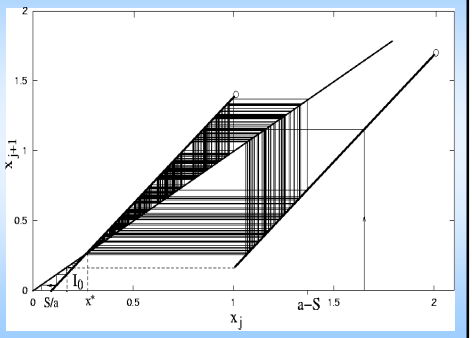


$$m\ddot{y}+(k+mg\mu'(v_0))\dot{y}+sy=mg\mu(v_0)-C\,\mathrm{sgn}(\dot{y})$$

$$y_e\in\left(\frac{mg\mu(v_0)-C}{s},\frac{mg\mu(v_0)+C}{s}\right)$$

Transient micro-chaos map

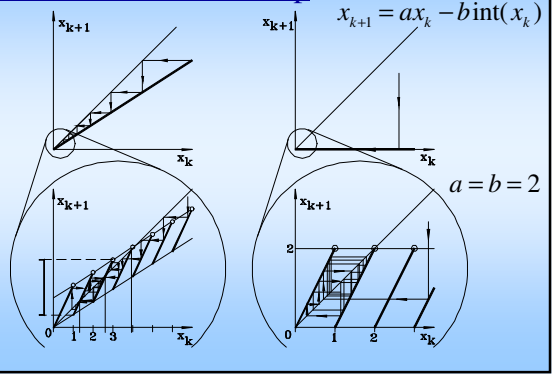
a=1.5
b=1.2
S=2/15
I_0=0.1



Trivial micro-chaos map

$$x_{k+1}=ax_k-b\,\mathrm{int}(x_k)$$

$a=b=2$



Kick-out number

Fibonacci series: $f_n=f_{n-1}+f_{n-2}$

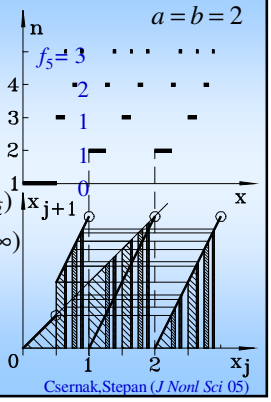
Length of intervals: $1/2^n$

$$x_j\in[1,2]\Rightarrow\sum_{n=1}^{\infty}\frac{f_n}{2^n}=1$$

$$x_{j+1}=\begin{cases}x_j, & x_j\in[0,\frac{1}{2})\\ 2x_j-2\,\mathrm{int}(x_j), & x_j\in[\frac{1}{2},\infty)\end{cases}$$

$$M=\lim_{N\to\infty}\frac{\int_0^N n\,dx}{\int_0^N dx}=\sum_{n=1}^{\infty}n\frac{f_n}{2^n}=6$$

$a=b=2$



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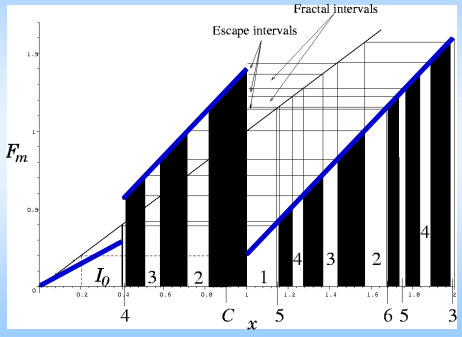
Non-trivial transient micro-chaos

a=1.4
b=1.2
I_0=0.2



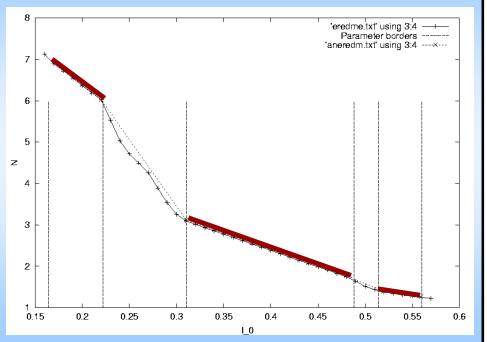
Non-trivial mean kick-out numbers

M

a=1.4

b=1.2

I_0=>



Robotic (dynamic) balancing

Even if vibration problems are all settled, there are still serious drawbacks:

- Balancing should be possible on any inclination, without knowing the exact vertical direction
- Balancing should work in space
- Balancing should incorporate gyroscopic effects

Study human balancing in more details!
elderly people, sportsmen –
delay, threshold, stochasticity

Conclusions

video1

video2

video3