Conformal Geometry and Dynamics of Quadratic Polynomials

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ABSTRACT.

Contents

Introduction

1. General terminology and notations

In this section we collect some standing (usually, standard) notations and definitions; it can be consulted as long as the corresponding objects appear in the text.

Complex plane and its affiliates

As usually, $\mathbb{N} = \{1, 2, \dots\}$ stands for the set of natural numbers;
$\mathbb{R}$ stands for the real line;
$\mathbb{C}$ stands for the complex plane,
and $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$ stands for the Riemann sphere.
For $a \in \mathbb{C}$, $r > 0$, let

$$\mathbb{D}(a, r) = \{z \in \mathbb{C} : |z - a| < r\}; \quad \bar{\mathbb{D}}(a, r) = \{z \in \mathbb{C} : |z - a| \leq r\}.$$

Let $\mathbb{D}_r \equiv \mathbb{D}(0, r)$, and let $\mathbb{D} \equiv \mathbb{D}_1$ denote the unit disk.
Let $\mathbb{T}_r = \partial \mathbb{D}_r$, and let $\mathbb{T} \equiv \mathbb{T}_1$ denote the unit circle;
$\mathbb{C}^* = \mathbb{C} \smallsetminus \{0\}$, $\mathbb{D}^* = \mathbb{D} \smallsetminus \{0\}$.
$\mathbb{A}(r, R) = \{z : r < |z| < R\}$ is an open round annulus; The notaions $\mathbb{A}[r, R]$ or $\mathbb{A}(r, R]$ for the closed or semi-closed annuli are self-explanatory.
The *equator* of $\mathbb{A}(r, R)$ is the curve $|z| = \sqrt{Rr}$.
$\mathbb{H} = \{z : \operatorname{Im} z > 0\}$ is the upper half plane, $\mathbb{H}_h = \{z : \operatorname{Im} z > h\}$;
$\mathbb{P} = \{z : 0 < \operatorname{Im} z < \pi\}$;
A *plane domain* is a domain in $\bar{\mathbb{C}}$.

General topology

In what follows, all topological spaces are assumed to satisfy the Second Countability Axiom, i.e., they have a countable base of neighborhoods. $\bar{X}$ denotes the closure of a set X; $\operatorname{int} X$ denotes its interior.
$U \Subset V$ means that U is *compactly contained in V*, i.e., $\bar{U}$ is a compact set contained in V.
A compact space is called *perfect* if it does not have isolated points.
A *Cantor set* is a totally disconnected perfect set. All compact sets are homeomorphic.

For two sets X and Y in a metric space with metric d, let

$$\text{dist}(X, Y) = \inf_{x \in X, \, y \in Y} d(x, y).$$

If one of these sets is a singleton, say $X = \{x\}$, then we use notation $\text{dist}(x, Y)$ for the distance from X to Y.

$$\text{diam}\, X = \sup_{x, y \in X} d(x, y).$$

Notation (X, Y) stands for the pair of spaces such that $X \supset Y$. A pair (X, a) of a space X and a "preferred point" $a \in X$ is called a *pointed space*.

Notation $f : (X, Y) \to (X', Y')$ means a map $f : X \to X'$ such that $f(Y) \subset Y'$. In the particular case of pointed spaces $f : (X, a) \to (X', a')$ we thus have: $f(a) = a'$.

Similar notations apply to triples, (X, Y, Z), where $X \supset Y \supset Z$, etc.

For a manifold M, $\mathrm{T}_x M$ stand for its tangent space at x, and $\mathrm{T}M$ stands for its tangent bundle.

Group actions

$\text{SL}(2, R)$ is the group of 2×2 matrices over a ring R with determinant 1 (we will deal with $R = \mathbb{C}$, $\mathbb{R}$, or $\mathbb{Z}$);

$\text{PSL}(2, R) = \text{SL}(2, R)/\{\pm I\}$, where I is the unit matrix;

$O(2) \approx \mathbb{T}$ is the group of plane rotations.

An action of a discrete group Γ on a locally compact space X is said to be *properly discontinuous* if any two points $x, y \in X$ have neighborhoods $U \ni x$, $V \ni y$ such that $\gamma(U) \cap V = \emptyset$ for all but finitely many $\gamma \in \Gamma$. The quotient of X by a properly discontinuous group action is a *Hausdorff* locally compact space.

The *stabilizer* $\text{Stab}(X)$ of a subset $Y \subset X$ is the subgroup $\{\gamma \in \Gamma : \gamma(Y) = Y\}$. A set Y called *completely invariant* under some subgroup $G \subset \Gamma$ if $G = \text{Stab}(Y)$ and $\gamma(Y) \cap Y = \emptyset$ for any $\gamma \in \Gamma \smallsetminus G$.

A group element γ is called *primitive* if it generates a maximal cyclic group.

2. Coverings

In this section we summarize for reader's convenience necessary background in the theory of covering spaces.

Let S and T be connected topological manifolds. A continuous map $f : S \to T$ is called a *covering of degree* $d \in \mathbb{N} \cup \{\infty\}$ if any point $y \in T$

has a neighborhood V such that

$$f^{-1}(V) = \bigsqcup_{i=1}^{d} U_i,$$

where each U_i is mapped homeomorphically onto V. The preimages $f^{-1}(y)$ are called *fibers* of the covering. Coverings $f : S \to T$ and $f' : S' \to T'$ are called *equivalent* if there exist homeomorphisms $\phi : S \to S'$ and $\psi : T \to T'$ such that $\psi \circ f = g \circ \phi$.

A covering is called *Galois* if there is a group Γ acting freely and properly discontinuously on S whose orbits are fibers of the covering. In this case $T \approx S/\Gamma$. The group Γ is called the *group of deck transformations* of f.

Vice versa, if a group Γ acts free and properly discontinuous on a manifold S then the quotient S/Γ is a manifold, and the natural projection $f : S \to S/\Gamma$ is a covering.

A covering $p : \hat{T} \to T$ is called *universal* if the space $\hat{T}$ is simply connected. Any manifold has a unique universal covering up to equivalence. This covering is Galois, with the fundamental group $p_1(T)$ as a group of deck transformations.

To any subsgoup G of $\pi_1(T)$ naturally corresponds a covering $f : \hat{T}/G \to T$, and this arises to one-to-one correspondence between classes of conjugate subgroups of $\pi_1(T)$ and classes of equivalent coverings over T. Moreover, the covering f is Galois if and only if the corresponding subgroup G is normal (in this case, the group of deck transformations of f is $\pi_1(T)/G$).

The crucial property of coverings is a *lifting property*. Given a curve γ on T with a marked point a ("beginninng of γ) and some point $\tilde{a} \in f^{-1}(a)$, *there exists a unique lift* of γ to a curve $\tilde{\gamma}$ on S that begins at $\tilde{a}$. If γ is a simple closed curve then $\tilde{\gamma}$ is either a simple closed curve as well or is a topological line, and the map $f : \tilde{\gamma} \to \gamma$ is a covering. If f is a Galois covering then $\mathrm{Stab}(\tilde{\gamma})$ is a cyclic group (finite or infinite), $\tilde{\gamma}$ is completely invariant under it, and $\gamma \approx \tilde{\gamma}/G$. Stabilizers of different lifts of γ are conjugate in Γ.

In case of the universal covering $p : \hat{T} \to T$, we see that to any simple closed curve γ on T corresponds the conjugacy class in $\pi_1(T)$ consisting of the generators of $\mathrm{Stab}(\tilde{\gamma})$ for various lifts $\tilde{\gamma}$. Moreover, these stabilizers do not change if replace γ with a freely homotopic curve, so the conjugacy class can be associated to the class $[\gamma]$ of freely homotopic curves.

Part 1

Conformal and quasi-conformal geometry

CHAPTER 1

Conformal geometry

1. Riemann surfaces

1.1. Topological surfaces.
1.1.1. *Definitions and examples.*

DEFINITION 1.1. A *(topological) surface* S (without boundary) is a two-dimensional topological manifold with countable base. It means that S is a topological space with a countable base and any $z \in S$ has a neighborhood $U \ni z$ homeomorphic to an open subset V of $\mathbb{R}^2$. The corresponding homeomorphism $\phi : U \to V$ is called a *(topological) local chart* on S. Such a local chart assigns to any point $z \in U$ its *local coordinates* $(x, y) = \phi(z) \in \mathbb{R}^2$.

A family of local charts whose domains cover S is called a *topological atlas* on S.

Given two local charts $\phi : U \to V$ and $\tilde{\phi} : \tilde{U} \to \tilde{V}$, the composition

$$\tilde{\phi} \circ \phi^{-1} : \phi(U \cap \tilde{U}) \to \tilde{\phi}(U \cap \tilde{U})$$

is called the *transition map* from one chart to the other.

A surface is called *orientable* if it admits an atlas with orientation preserving transition maps. Such a surface can be oriented in exactly two ways. *In what follows we will only deal with orientable (and naturally oriented) surfaces.*

Unless otherwise is explicitly said, we will assume that the surfaces under consideration are *connected*. The simplest (and most important for us) surfaces are:

• The whole plane $\mathbb{R}^2$ (homeomorphic to the open unit disk $\mathbb{D} \subset \mathbb{R}^2$).

• The unit *sphere* S^2 in $\mathbb{R}^3$ (homeomorphic via the stereographic projection to the one-point compactification of the plane); it is also called a "closed surface of genus 0" (in this context "closed" means "compact without boundary").

• A *cylinder* or *topological annulus* $C(a, b) = \mathbb{T} \times (a, b)$, where $-\infty \leq a < b \leq +\infty$. It can also be represented as the quotient of the strip $P(a, b) = \mathbb{R} \times (a, b)$ modulo the cyclic group of translations $z \mapsto z + 2\pi n$,

$n \in \mathbb{Z}$. All the cylinders $C(a, b)$ are homeomorphic to any annulus $\mathbb{A}(r, R)$, to the punctured disk $\mathbb{D}^*$ and to the punctured plane $\mathbb{C}^*$).

• The torus $\mathbb{T}^2 = \mathbb{T} \times \mathbb{T}$, also called a "closed surface of genus 1". It can also be represented as the quotient of $\mathbb{R}^2$ modulo the action of a rank 2 abelian group $z \mapsto z + \alpha m + \beta n$, $(m, n) \in \mathbb{Z}^2$, where α and β is an arbitrary basis in $\mathbb{R}^2$.

It is intuitively obvious that (up to a homeomorphism) *there are only two simply connected surfaces*: the plane and the sphere.

If we have a certain standard surface S (say, the unit disk or the unit sphere), a "topological S" (say, a "topological disk" or a "topological sphere") refers to a surface homeomorphic to the standard one.

One can also consider surfaces with boundary. The local model of a surface near a boundary point is given by a relative neighborhood of a point $(x, 0)$ in the closed upper half-plane $\bar{\mathbb{H}}$. The orientation of a surface naturally induces an orientation of its boundary (locally corresponding to the positively oriented real line).

For instance, we can consider cylinders with boundary: $C[a, b] = \mathbb{T} \times [a, b]$ or $C(a, b) = \mathbb{T} \times [a, b)$. They will be still called "cylinders" or "topological annuli". Cylinders $C(a, b)$ without boundary will be also called "open", while cylinders $C[a, b]$ will be called "closed" (according to the type of the interval involved).

Cylinders (with or without boundary) are the only topological surfaces whose fundamental group is $\mathbb{Z}$.

A *Jordan curve* $\gamma \subset S$ on a surface is a topologically embedded unit circle. A *Jordan disk* $D \subset S$ is a topological disk bounded by a Jordan curve. Both open and closed Jordan disks are allowed.

1.1.2. *New surfaces from old ones.* There are two basic ways of building new surfaces out of old ones: making holes and gluing their boundaries. Of course, any open subset of a surface is also a surface. In particular, one can make a (closed) *hole* in a surface, that is, remove a closed Jordan disk. A topologically equivalent operation is to make a *puncture* in a surface. By removing an open Jordan disk (open hole) we obtain a surface with boundary.

If we have two open holes (on a single surface or two different surfaces S_i) bounded by Jordan curves γ_i, we can *glue* these boundaries together by means of an orientation reversing homeomorphism $h : \gamma_1 \to \gamma_2$. (It can be also thought as attaching a cylinder to these curves.) We denote this operation by $S_1 \sqcup_h S_2$. For instance, by gluing together two closed disks we obtain a topological sphere: $\mathbb{D} \sqcup_h \mathbb{D} \approx \S^2$.

Combining the above operations, we obtain operations of taking *connected sums* and attaching a *handle*. To take a connected sum of

two surfaces S_1 and S_2, make an open hole in each of them and glue together the boundaris of these holes. To attach a handle to a surface S, make two open holes in it and glue together their boundaries.

If we attach a handle to a sphere, we obtain a topological torus. If we attach g handles to a sphere, we obtain a "closed surface of genus g". It turns out that any closed orientable surface is homeomorphic to one of those. Thus closed orientable surfaces are topologically classified by a single number $g \in \mathbb{Z}_+$, its genus.

One says that a surface S (with or without boundary) has a *finite topological type* if its fundamental group $\pi(S)$ is finietly generated (e.g., any compact surface is of finite type). It turns out that it is equivalent to saying that S is homeomorpic to a closed surface with finitely many open or closed holes. Clearly such a surface admits a decomposition

$$S = K \bigcup_i \sqcup_{h_i} C_i \, ,$$

where K is a compact surface and $C_i \approx \mathbb{T} \times [0,1)$ are half-open cylinders. The set $K = K_S$ is called the *compact core* of S. Note that it is obviously a deformation retract for S. We say that the cylinders C_i represent the *ends* of S.

Each end can be compactified in two ways, by adding a missing boundary curve to the cylinder, or by adding one point. In the former case, the added boundary curve is called the *ideal boundary* of the end. Let $\hat{S}$ denote the compactification of S by adding ideal boundaries to *all* ends.

1.1.3. *Euler characteristic.* Let S be a compact surface (with or without boundary) Its *Euler characteristic* is defined as

$$\chi(S) = f - e + v,$$

where f, e and v are respectively the numbers of faces, edges and vertices in any triangulation of S.

The Euler characteristic is obviously additive:

$$\chi(S_1 \sqcup_h S_2) = \chi(S_1) + \chi(S_2).$$

Since the cylinder $\mathbb{T} \times [0,1]$ has zero Euler characteristic, $\chi(\hat{S}) = \chi(K_S)$ for a surface S of finite type. We can use this as a definition of $\chi(S)$ in this case.

Making a hole in a surface drops its Euler characteristic by one; attaching a handle does not change it. Hence $\chi(S) = 2 - 2g - n$ for a surface of genus g with n holes.

Note that the above list of simplest surfaces is the full list of sufaces of finite type without boundary with *non-negative* Euler characteristic:

$$\chi(\mathbb{R}^2) = 1, \quad \chi(S^2) = 2, \quad \chi(\mathbb{T} \times (0,1)) = \chi(\mathbb{T}^2) = 0.$$

1.1.4. *Marking.* A surface S can be *marked* with an extra topological data. It can be either several marked points $a_i \in S$, or several closed curves $\gamma_i \subset S$ up to homotopy (usually but not always they form a basis of $\pi_1(S)$), or a parametrization of several boundary components $\Gamma_i \subset \partial S$, $\phi_i : \mathbb{T} \to \Gamma_i$.

The marked objects may or may not be distinguished (for instance, two marked points or the generators of π_1 may be differently colored). Accordingly, the marking is called *colored* or *uncolored*.

A homeomorphism $h : S \to \tilde{S}$ between marked surfaces should respect the marked data: marked points should go to the corresponding points ($h(a_i) = \tilde{a}_i$), marked curves γ_i should go to the corresponding curves $\tilde{\gamma}_i$ up to homotopy ($h(\gamma_i) \simeq \tilde{\gamma}_i$), and the boundary parametrizations should be naturally related ($h \circ \phi_i = \tilde{\phi}_i$).

1.2. Analytic and geometric structures on surfaces. Rough topological structure can be refined by requiring that the transition maps belong to a certain "structural pseudo-group", which often means: "have certain regularity". For example, a *smooth* structure on S is given by a family of local charts $\phi_i : U_i \to V_i$ such that all the transition maps are smooth (with a prescribed order of smoothness). A surface endowed with a smooth structure is naturally called a *smooth surface*. A local chart $\phi : U \to V$ smoothly related to the charts ϕ_i (i.e., with smooth transition maps) is referred to as a "smooth local chart". A family of smooth local charts covering S is called a "smooth atlas" on S. A smooth structure comes together with affiliated notions of smooth functions, maps and diffeomorphisms.

There is a smooth version of the connected sum operation in which the boundary curves are assumed to be smooth and the boundary gluing map h is assumed to be an orientation reversing diffeomorphism. To get a feel for it, we suggest the reader to do the following exercise:

EXERCISE 1.1. *Consider two copies D_1 and D_2 of the closed unit disk $\mathbb{D} \subset \mathbb{R}^2$. Glue them together by means of a diffeomorphism $h : \partial D_1 \to \partial D_2$ of the boundary circles. You obtain a topological sphere S^2. Show that it can be endowed with a unique smooth structure compatible with the smooth structures on D_1 and D_2 (that is, such that the tautological embeddings $D_i \to S^2$ are smooth). The boundary circles ∂D_i become smooth Jordan curves on this smooth sphere. Show that this sphere is diffeomorphic to the standard "round sphere" in $\mathbb{R}^3$.*

Real analytic structures would be the next natural refinement of smooth structures.

If $\mathbb{R}^2$ is considered as the complex plane $\mathbb{C}$ with $z = x + iy$, then we can talk about *complex analytic $\equiv$ holomorphic* transition maps and corresponding complex analytic structures and surfaces. Such surfaces are known under a special name of *Riemann surfaces*. A holomorphic diffeomorphism between two Riemann surfaces is often called an *isomorphism*. Accordingly a holomorphic diffeomorphism of a Riemann surface onto itself is called its *automorphism*.

Connected sum operation still works in the category of Riemann surfaces. In its simplest version the boundary curves and the gluing diffeomorphism should be taken real analytic. Here is a representative statement:

EXERCISE 1.2. *Assume that in Exercise 1.1 $\mathbb{R}^2 \equiv \mathbb{C}$ and the gluing diffeomorphism h is real analytic. Then S^2 can be supplied with a unique complex analytic structure compatible with the complex analytic structure on the disks $D_i \subset \mathbb{C}$. The boundary circles ∂D_i become real analytic Jordan curves on this "Riemann sphere".*

More generally, we can attach handles to the sphere by means of real analytic boundary map, and obtain an example of a Riemann surface of genus g. It is remarkable that, in fact, it can be done with only smooth gluing map, or even with a singular map of a certain class. This operation (with a singular gluing map) has very important applications in Teichmïler theory, theory of Kleinian groups and dynamics (see **??**).

If $\mathbb{R}^2$ is supplied with the standard Euclidean metric, then we can consider *conformal* transition maps, i.e., diffeomorphisms preserving angles between curves. The first thing students usually learn in complex analysis is that the class of orientation preserving conformal maps coincides (in dimension 2!) with the class of invertible complex analytic maps. Thus the notion of a *conformal structure* on an oriented surface is equivalent to the notion of a complex analytic structure (though it is worthwhile to keep in mind their conceptual difference: one comes from geometry, the other comes from analysis).

One can go further to *projective, affine, Euclidean/flat* or *hyperbolic* structures. We will specify this discussion in a due course.

One can also go in the opposite direction and consider *rough structures* on a topological surface whose structural pseudo-group is bigger then the pseudo-group of diffeomorphisms, e.g., "bi-Lipschitz structurs". Even rougher, *quasi-conformal*, structures will play an important role in our discussion.

To comfort a rigorously-minded reader, let us finish this brief excursion with a definition of a *pseudo-group* on $\mathbb{R}^2$ (in the generality adequate to the above discussion). It is a family of local homeomorphisms $f : U \to V$ between open subsets of $\mathbb{R}^2$ (where the subsets depend on f) which is closed under taking inverse maps and taking compositions (on the appropriately restricted domains). The above structures are related to the pseudo-groups of all local (orientation preserving) homeomorphisms, local diffeomorphisms, locally biholomorphic maps, local isometries (Euclidean or hyperbolic) etc.

1.3. Three geometries.

1.3.1. *Affine geometry.* Consider the complex plane $\mathbb{C}$. Holomorphic automorphisms of $\mathbb{C}$ are complex affine maps $A : z \mapsto az + b$, $a \in \mathbb{C}^*$, $b \in \mathbb{C}$. They form a group $\mathrm{Aff}(\mathbb{C})$ acting freely bi-transitively on the plane: any pair of points can be moved in a unique way to any other pair of points. Moreover, it acts freely transitively on the tangent bundle of $\mathbb{C}$.

Thus the complex plane $\mathbb{C}$ is endowed with the *affine structure* canonically affiliated with its complex analytic structure. Of course, the plane can be also endowed with a Euclidean metric $|z|^2$. However, this metric can be multiplied by any scalar $t > 0$, and there is no way to normalize it in terms of the complex structure only. All these Euclidean structures have the same group $\mathrm{Euc}(\mathbb{C})$ of *Euclidean motions* $A : z \mapsto az + b$ with $|a| = 1$. This group acts transitively on the plane with the group of rotations $z \mapsto e^{2\pi i\theta}z$, $0 \le \theta < 1$, stabilizing the origin. Moreover, it acts freely transitively on the unit tangent bundle of $\mathbb{C}$ (corresponding to any Euclidean structure).

The group Aff has very few discrete subgoups acting freely on $\mathbb{C}$: rank 1 cyclic group actions $z \mapsto z + an$, $n \in \mathbb{Z}$, and rank 2 cyclic group actions $z \mapsto an + bm$, $(m, n) \in \mathbb{Z}^2$, where (a, b) is an arbitrary basis in $\mathbb{C}$ over $\mathbb{R}$. All rank 1 actions are conjugate by an affine transformation, so that the quotients modulo these actions are all isomorphic. Taking $a = 1$ we realize these quotients as the bi-infinite cylinder $\mathbb{C}/\mathbb{Z}$. It is isomorphic to the puncured plane $\mathbb{C}^*$ by means of the exponential map $\mathbb{C}/\mathbb{Z} \to \mathbb{C}^*$, $z \mapsto e^{2\pi z}$. The quotients of rank 2 are all homeomorphic to the torus. However, they may represent different Riemann surfaces (see below 1.4.2).

Note that the above discrete groups preserve the Euclidean structures on $\mathbb{C}$. Hence these structures can be pushed down to the quotient Riemann surface. Moreover, now they can be canonically normalized: in the case of the cylinder we can normalize the lengths of the closed geodesics to be 1. In the case of the torus we can normalize its total

area. Thus, complex tori and the bi-infinite cylinder are endowed with a canonical Euclidean structure. For this reason, they are called *flat*.

1.3.2. *Spherical (projective) geometry.* Consider now the Riemann sphere $\bar{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Its bi-holomorphic automorphisms are *Möbius transformations*

$$\phi : z \mapsto \frac{az+b}{cz+d}; \quad \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} \neq 0.$$

We will denote this Möbius group by $\text{Möb}(\hat{\mathbb{C}})$. It acts freely triply transitive on the sphere: any (ordered) triple of points (a, b, c) on the sphere can be moved by a unique Möbius transformation to any other triple (a', b', c').

EXERCISE 1.3. *Show that topology of* $\text{PSL}(2, \mathbb{R})$ *and topology of uniform convergence on the sphere coincide. Given an* $\epsilon > 0$, *let us consider the set of Möbius transformations* ϕ *such that the triple* $(\phi^{-1}(0, 1, \infty)$ *is* ϵ*-separated in the spherical metric (i.e., the three points stay at least distance* ϵ *apart one from another). Show that this set is compact in* $\text{Möb}(\hat{\mathbb{C}})$.

Note that the Riemann sphere is isomorphic to the complex projective line $\mathbb{CP}^1$. For this reason Möbius transformations are also called *projective*. Algebraicly the Möbius group is isomorphic to the linear projective group $\text{PSL}(2, \mathbb{C}) = \text{SL}(2, \mathbb{C})/\{\pm I\}$ of 2×2 matrices M with $\det M = 1$ modulo reflection $M \mapsto -M$.

Any Möbius transformation has a fixed point on the sphere. Hence *there are no Riemann surfaces whose universal covering is* $\bar{\mathbb{C}}$. In fact, any non-identical Möbius transformations has either one or two fixed points, and can be classified depending on their nature.

We would like to bring a Möbius transformation to a simplest normal form by means of a conjugacy $\phi^{-1} \circ f \circ \phi$ by some $\phi \in \text{Möb}(\hat{\mathbb{C}})$. Since $\text{Möb}(\hat{\mathbb{C}})$ acts double transitively, we can find some ϕ which sends one fixed point of f to ∞ and the other (if exists) to 0. This leads to the following classification:

(i) A *hyperbolic* Möbius transformation has an attracting and repelling fixed points with multipliers[1]. λ amd λ^{-1}, where $0 < |\lambda| < 1$. Its normal form is a global linear contraction $z \to \lambda z$ (with possible spiralling if λ is unreal[2].)

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[1]The multiplier of a fixed point α is the derivative $f'(\alpha)$ calculated in *any* local chart around α, compare §14

[2]Hyperbolic Möbius transformations with unreal λ are also called *loxodromic*

(ii) An *elliptic* Möbius transformation has two fixed points with multipliers λ and λ^{-1} where $\lambda = e^{2\pi i\theta}$, $\theta \in [0, 1)$. Its normal form is the rotation $z \to e^{2\pi i\theta}z$.

(iii) (ii) A *parabolic* Möbius transformation has a single fixed point with multiplier 1. Its normal form is a translation $z \mapsto z + 1$.

EXERCISE 1.4. *Verify those of the above statements which look new to you.*

1.3.3. *Hyperbolic geometry.* Let us now consider a Riemann surface S conformally equivalent to the unit disk $\mathbb{D}$, or equivalently, to the upper half plane $\mathbb{H}$, or equivalently, to the strip $\mathbb{P}$ (we refer to such a Riemann surface as a "conformal disk"). Using the isomorphism $S \approx \mathbb{D}$, S can be naturally compactified by adding to it the *ideal boundary* $\partial S \approx \mathbb{T}$, also called the *circle at infinity* or the *absolute*.

The group $\text{Aut}(S)$ of conformal automorphisms of S looks particularly nice in the the upper half-plane model as it consist of Möbius transformations with real coefficients:

$$f : z \mapsto \frac{az + b}{cz + d}; \quad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}(2, \mathbb{R}).$$

Hence $\text{Aut}(S) \approx \text{SL}(2, \mathbb{R})/\{\pm I\} = \text{PSL}(2, \mathbb{R})$.

The above classification of Möbius transformations $M \in \text{PSL}(2, \mathbb{R})$ has a clear meaning in terms of their action on S:

(i) A hyperbolic transformation $M \in \text{PSL}(2, \mathbb{R})$ has two fixed points on the ideal boundary ∂S (and does not have fixed points in S). Its normal form in the $\mathbb{H}$-model is a dialtion $z \mapsto \lambda z$ ($0 < \lambda < 1$), and is a translation $z \mapsto z + a$ in the $\mathbb{P}$-model, where $a = \log \lambda$.

(ii) A parabolic transformation has a single fixed point on ∂S (and does not have fixed points in S). Its normal form in the $\mathbb{H}$-model is a translation $z \mapsto z + 1$.

(iii) An elliptic transformation $M \neq \text{id}$ has a single fixed point $a \in S$ (and does not have fixed points on ∂S). Its normal form in the $\mathbb{D}$-model is a rotation $z \mapsto e^{2\pi i\theta}z$.

A remarkable discovery by Poincaré is that a conformal disk S is endowed with the intrinsic hyperbolic structure, that is, there exists a Riemannian metric ρ_S on S of constant curvature -1 invariant with respect $\text{PSL}(2, \mathbb{R})$-action. In the $\mathbb{H}$-, $\mathbb{D}$- and $\mathbb{P}$-models, the length element of ρ_S is given respectively by the following expressions:

$$d\rho_{\mathbb{D}} = \frac{2|dz|}{1 - |z|^2}, \quad d\rho_{\mathbb{H}} = \frac{|dz|}{y}, \quad d\rho_{\mathbb{P}} = \frac{|dz|}{\sin y},$$

where $z = x + iy$. This metric is called *hyperbolic* .

EXERCISE 1.5. *Verify that the above three expressions correspond to the same metric on S, which has curvature -1 and is invariant under* $\mathrm{PSL}(2, \mathbb{R})$. *Moreover, any orientation preserving isometry of S belongs to* $\mathrm{PSL}(2, \mathbb{R})$.

A conformal disk S endowed with the hyperbolic metric is called the *hyperbolic plane.* In this way, $\mathrm{PSL}(2, \mathbb{R})$ assumes the meaning of the group of (orientation preserving) *hyperbolic motions* of the hyperbolic plane. It acts freely transitively on the unit tangent bundle of $\mathbb{H}$. Thus, the unit tangent bundle over $\mathbb{H}$ can be identified with $\mathrm{PSL}(2, \mathbb{R})$, while the hyperbolic plane itself can be identified with the quotient $\mathrm{PSL}(2, \mathbb{R})/O(2)$.

A *Fuchsian group* Γ is a discrete subgroup of $\mathrm{PSL}(2, \mathbb{R})$ acting on S.

EXERCISE 1.6. *Show that any Fuchsian group acts properly dicontinuously on S.*

Hence the quotient $X = S/\Gamma$ is a Hausdorff space. Moreover, if Γ acts freely on S, then the complex structure and the hyperbolic metric naturally descend from S to X, and we obtain a *hyperbolic Riemann surface.*

1.3.4. *Hyperbolic geodesics and horocycles.* Hyperbolic geodesics in the $\mathbb{D}$-model of the hyperbolic plane are Euclidean half-circles orthogonal to the absolute $\mathbb{T}$. For any hyperbolic unit tangent vector $v \in \mathrm{T}\mathbb{D}$, there exists a unique oriented hyperbolic geodesic tangent to v. For any two points x and y on the absolute, there exists a unique hyperbolic geodesic with endpoints x and y. The group $\mathrm{PSL}(2, \mathbb{R})$ acts freely and transitively on the space of oriented hyperbolic geodesics.

EXERCISE 1.7. *Verify the above assertions if they are not familiar to you.*

A *horocycle in* $\mathbb{D}$ *centered at* $x \in \mathbb{T}$ is a Euclidean circle $\gamma \subset \mathbb{D}$ tangent to $\mathbb{T}$ at x. A *horodisk* $D \subset \mathbb{D}$ is the disk bounded by the horocycle. In purely hyperbolic terms, horocycles centered at x form a foliations orthogonal to the foliation of geodesics ending at x. The stabilizer of any horocycle (and the corresponding horoball) is the parabolic group fixing its center.

In fact, the $\mathbb{H}$-model fits better for describing horocycles: in this model the horocylces centered at $x = \infty$ are horizontal lines $L_h = \{\mathrm{Im}\, z = h\}$, the corresponding horoballs are the upper half-planes $\mathbb{H}_h$, and their stabilizer is the group of translations $z \mapsto z + t$, $t \in \mathbb{R}$.

The quotient of a horoball $\mathbb{H}_h$ by a discrete cyclic group of parabolic transformations $\mathbb{Z} = < z \mapsto z + n >$ is called a *cusp*. Conformally it is the punctured disk $\mathbb{D}^*$, hyperbolically it is the pseudosphere (see Figure **??**). Simple closed curves $L_t/\mathbb{Z} \subset \mathbb{H}_h/\mathbb{Z}$, $t > h$, are also called horocycles (in the cusp).

EXERCISE 1.8. *Any cusp $\mathbb{H}_h/\mathbb{Z}$ has infinite hyperbolic diameter but a finite hyperbolic area. The hyperbolic length of the horocylcle $L_t/\mathbb{Z}$ goes to zero as $t \to \infty$.*

Let us now consider a Fuchsian group Γ and the corresponding hyperbolic Riemann surface $S = \mathbb{D}/\Gamma$. Hyperbolic geodesics on S are (obviously) projections of the hyperbolic geodesics on $\mathbb{D}$; horocycles on S are (by definition) projections of the horocycles on $\mathbb{D}$.

Let γ be a non-trivial simple closed curve on S, and let $[\gamma]$ be the class of simple closed curves freely homotopic to γ. To this class corresponds a conjugacy class $A(\gamma)$ of deck transformations (see §2). Since deck transformations cannot be elliptic, the elements of $A(\gamma)$ are either all hyperbolic or all parabolic. Accordingly, we say that the class $[\gamma]$ itself is either hyperbolic or parabolic.

PROPOSITION 1.1. *a) If the class $[\gamma]$ is hyperbolic then it is represented by a unique closed hyperbolic geodesic $\delta \in [\gamma]$. This geodesic minimizes the hyperbolic length of the closed curves in $[\gamma]$.*
b) If the class $[\gamma]$ is parabolic then S contains a neighborhood U isometric to a cusp, and $[\gamma]$ is represented by any horocycle in it. In this case, the class contains arbitrary short curves.

PROOF. Let us consider a lift $\tilde{\gamma}$ of γ, and let $G = < \phi^n >_{n \in \mathbb{Z}}$ be its stabilizer.

a) If ϕ is hyperbolic then it has two fixed points, x_- and x_+, on the absolute, and then the closure of $\tilde{\gamma}$ in $\bar{\mathbb{D}}$ is a topological interval with endpoints x_1 and x_+. Let us consider the hyperbolic geodesic $\tilde{\delta}$ in $\mathbb{D}$ with endpoints $x_\pm$. It is invariant under the action of the cyclic group G. In fact, it is completely invariant. Indeed, if $\psi(\tilde{\delta}) \cap \tilde{\delta} \neq \emptyset$ for some $\psi \in \Gamma \smallsetminus G$, then $\psi(\tilde{\gamma}) \cap \tilde{\gamma} \neq \emptyset$ as well, which is impossible since γ does not have self-intersections. Hence the projection of $\tilde{\delta}$ to S is equal to $\tilde{\delta}/G$, which is the desired simple closed geodesic representing $[\gamma]$.

b) If ϕ is parabolic then it has a single fixed point x on the absolute, and the closure of $\tilde{\gamma}$ in $\bar{\mathbb{D}}$ is a topological circle touching $\mathbb{T}$ at x (a "topological horocycle centered at x").

Let $\tilde{U}$ be the corresponding topological horoball bounded by $\tilde{\gamma}$. Let us show that it is completely invariant under G. Indeed, for $\psi \in \Gamma \smallsetminus G$,

$\psi(\tilde{U})$ is a topological horoball centered at $\beta(x) \neq x$. But since γ is a simple curve, $\psi(\tilde{\gamma}) \cap \tilde{\gamma} = \emptyset$ for any $\beta \in \Gamma \smallsetminus G$. Since two topological horoballs with disjoint boundaries are disjoint, $\psi(\tilde{U}) \cap \tilde{U} = \emptyset$.

It follows that $\tilde{U}/G$ is is isometrically embedded into $\mathbb{D}/\Gamma = S$. But $\tilde{U}/G$ is a conformal punctured disk containing some standard cusp $\mathbb{H}_h/\mathbb{Z}$. Thus, this cusp isometrically embeds into S as well, and its horocycles give us desired representatives of $[\gamma]$. $\square$

We express part b) of the above statement by saying that the class $[\gamma]$ (or, the curve γ itself) is *represented by a horocycle*, or by a *puncture*, or by a *cusp*.

A simple closed curve on S is called *peripheral* if it is either trivial or is represented by a cusp. For instance, if $S = \bar{\mathbb{C}} \smallsetminus \{x_i\}$ is a sphere with finitely many punctures then γ is non-peripheral iff each component of $\bar{\mathbb{C}} \smallsetminus \gamma$ contains at least two punctures.

EXERCISE 1.9. *Show that there is one-to-one correspondence between conjugacy classes of primitive parabolic transformations in a Fuchsian group Γ and cusps of the Riemann surface $S = \mathbb{D}/\Gamma$.*

1.4. Annulus and torus.
1.4.1. *Modulus of an annulus.* Consider an open topological annulus A. Let us endow it with a complex structure. Then A can be represented as the quotient of either $\mathbb{C}$ or $\mathbb{H}$ modulo an action of a cyclic group $< \gamma >$. As we have seen above, in the former case A is isomorphic to the flat cylinder $\mathbb{C}/\mathbb{Z} \approx \mathbb{C}^*$. In the latter case, we obtain either the punctured disk $\mathbb{D}^*$ (if γ is parabolic) or an annulus $\mathbb{A}(r, R)$ (if γ is hyperbolic). In the hyperbolic case we call A a *conformal annulus*.

EXERCISE 1.10. *Write down explicitly the covering maps $\mathbb{H} \to \mathbb{D}^*$ and $\mathbb{H} \to A(r, R)$.*

EXERCISE 1.11. *Prove that two round annuli $\mathbb{A}(r, R)$ and $\mathbb{A}(r', R')$ are conformally equivalent if and only if $R/r = R'/r'$.*

Let
$$\mathrm{mod}(A) = \frac{1}{2\pi} \log \frac{R}{r}$$
for a round annulus $A = \mathbb{A}(r, R)$. For an arbitrary conformal annulus A, define its *modulus*, $\mathrm{mod}(A)$, as the modulus of a round annulus $\mathbb{A}(r, R)$ isomorphic to A. According to the above exercise, this definition is correct and, moreover, $\mathrm{mod}(A)$ is the only conformal invariant of a conformal annulus.

If A is isomorphic to $\mathbb{C}^*$ or $\mathbb{D}^*$ then we let $\mathrm{mod}(A) = \infty$.

If A is a topological annulus with boundary whose interior is endowed with a complex structure, then $\mathrm{mod}(A)$ is defined as the modulus of the $\mathrm{int}(A)$.

The *equator* of a conformal annulus A is the image of the equator of the round annulus (see §1) under the uniformization $A(r, R) \to A$.

EXERCISE 1.12. (i) *Write down the hyperbolic metric on a conformal annulus represented as the quotient of the strip $S_h = \{0 < \mathrm{Im}\, z < h\}$ modulo the action of the cyclic group generated by $z \mapsto z + 2\pi$. (What is the relation between h and $\mathrm{mod}\, A$?)*

(ii) *Prove that the equator is the unique closed hypebolic geodesic of a conformal annulus A in the homotopy class of the generator of $\pi_1(A)$.*

(iii) *Show that the hyperbolic length of the equator is equal to $1/\mathrm{mod}(A)$. Relate it to the multiplier of the deck transformation of $\mathbb{H}$ covering A.*

Even if A is a hyperbolic annulus, it is possible to endow it with a flat, rather than hyperbolic, metric. To this end realize A as the quotient of a strip S_h modulo the cyclic group of translations (see the above exercise). Since the flat metric on S_h is translation invariant, it descends to A. In this case we call A a *flat cylinder*.

1.4.2. *Modulus of the torus.* Let us take a closer look at the actions of the group $\Gamma \approx \mathbb{Z}^2$ on the (oriented) affine plane $P \approx \mathbb{C}$ by translations (see §1.3.1). We would like to classify these actions up to affine conjugacy, i.e., two actions T and S are considered to be equivalent if there is an (orientation preserving) affine automorphism $A : P \to P$ and an algebraic automorphism $i : \Gamma \to \Gamma$ such that for any $\gamma \in \Gamma$ the following diagram is commutative:

$$
\begin{array}{ccc}
P & \xrightarrow{\;T^\gamma\;} & P \\
A \downarrow & & \downarrow A \\
P & \xrightarrow{\;S^{i(\gamma)}\;} & P
\end{array}
\qquad (1.1)
$$

This is equivalent to classifying the quotient tori P/T^Γ up to conformal equivalence (since a conformal isomorphism between the quotient tori lifts to an affine isomorphism between the universal covering spaces conjugating the actions of the covering groups).

The conjugacy A in the above definition will also be called *equivariant* with respect to the corresponding group actions.

The problem becomes easier if to require first that $i = \mathrm{id}$ in (1.1). Fix an *uncolored pair* of generators α and β of Γ. Since T acts by

translations and since P is affine, the ratio

$$\tau = \tau(T) = \frac{T^{\beta}(z) - z}{T^{\alpha}(z) - z}$$

makes sense and is independent of $z \in P$. Moreover, by switching the generators α and β we replace τ with $1/\tau$. Thus, we can color the generators in such a way that $\operatorname{Im} \tau > 0$. (With this choice, the basis of P corresponding to the generators $\{\alpha, \beta\}$ is positively oriented.)

EXERCISE 1.13. *Show that two actions T and S of $\Gamma =< \alpha, \beta >$ are affinely equivalent with $i = \operatorname{id}$ if and only if $\tau(T) = \tau(\tilde{T})$.*

According to the discussion in §1.1.4, the choice of generators of Γ means *(uncolored) marking* of the corresponding torus. Thus, the marked tori are classified by a single complex modulus $\tau \in \mathbb{H}$.

Forgetting the marking amounts to replacement one basis $\{\alpha, \beta\}$ in Γ by another, $\{\tilde{\alpha}, \tilde{\beta}\}$. If both bases are positively oriented then there exists a matrix

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \operatorname{SL}(2, \mathbb{Z})$$

such that $\tilde{\alpha} = a\,\alpha + b\,\beta$, $\tilde{\beta} = c\,\alpha + d\,\beta$. Hence

$$\tilde{\tau} = \frac{a\tau + b}{c\tau + d}.$$

Thus, the unmarked tori are parametrized by a point $\tau \in \mathbb{H}$ modulo the action of $\operatorname{SL}(2, \mathbb{Z})$ on $\mathbb{H}$ by Möbius transformations. The kernel of this action consists of two matrices, $\pm I$, so that the quotient group of Möbius transformations is isomorphic to $\operatorname{PSL}(2, \mathbb{Z}) = \operatorname{SL}(2, \mathbb{Z})/\operatorname{mod}\{\pm I\}$. This group is called *modular*. (In what follows, the modular group is identified with $\operatorname{PSL}(2, \mathbb{Z})$.)

Remark. Passing from $\operatorname{SL}(2, \mathbb{Z})$ to $\operatorname{PSL}(2, \mathbb{Z})$ has an underlying geometric reason. All tori $\mathbb{C}/\Gamma$ have a conformal symmetry $z \mapsto -z$. It change marking $\{\alpha, \beta\}$ by $-I\{\alpha, \beta\}$. Thus, remarking by $-I$ acts trivially on the space of marked tori.

The modular group has two generators, the translation $z \mapsto z + 1$ and the second order rotation $z \mapsto -1/z$. The intersection of their fundamental domains gives the standard fundamental domain Δ for this action.

EXERCISE 1.14. *a) Verify the last statement.*
b) Find all points in Δ that are fixed under some transformation of $\operatorname{PSL}(2, \mathbb{Z})$. What are the orders of their stabilizers?

c) What is the special property of the tori corresponding to the fixed points?

d) Show that by identifying the sides of Δ according to the action of the generators we obtain a topological plane $Q \approx \mathbb{R}^2$.

e) Endow the above plane with the complex structure so that the natural projection $\mathbb{H} \to Q$ is holomorphic. Show that $Q \approx \mathbb{C}$. (The corresponding holomorphic function $\mathbb{H} \to \mathbb{C}$ is called modular).

Thus, the unmarked tori are parametrized by a single modulus $\mu \in \mathbb{H}/\mathrm{PSL}(2, \mathbb{Z}) \approx \mathbb{C}$.

In the dynamical context we will be dealing with the intermadiate case of *partially marked* tori, i.e., tori with one marked generator α of the fundamental group. This space can be viewed as the quotient of the space of fully marked tori by means of forgetting the second generator, β. If we have two bases $\{\alpha, \beta\}$ and $\{\alpha, \tilde{\beta}\}$ in Γ with the same α, then $\tilde{\beta} = \beta + n\alpha$ for some $n \in \mathbb{Z}$. Hence $\tilde{\tau} = \tau + n$.

Thus, the space of partially marked tori is parametrized by $\mathbb{H}$ modulo action of the cyclic group by translations $\tau \mapsto \tau + n$. The quotient space is identified with the punctured disk $\mathbb{D}^*$ by means of the exponential map $\mathbb{H} \to \mathbb{D}^*$, $\tau \mapsto \lambda = e^{2\pi i \tau}$. So, the partially marked tori are parametrized by a single modulus $\lambda \in \mathbb{D}^*$. We will denote such a torus by $\mathbb{T}^2_\lambda$.

This modulus λ makes a good dynamical sense. Consider the covering $p : S \to \mathbb{T}^2_\lambda$ of the partially marked torus corresponding to the marked cyclic group. Its covering space S is obtained by taking the quotient of $\mathbb{C}$ by the action of the marked cyclic group $z \mapsto z + n$, $n \in \mathbb{Z}$. By means of the exponential map $z \mapsto e^{2\pi i z}$, this quotient is identified with $\mathbb{C}^*$. Moreover, the action of the complementary cyclic group $z \mapsto z + n\tau$, $n \in \mathbb{Z}$, descends to the action $\zeta \mapsto \lambda^n \zeta$ on $\mathbb{C}^*$, where the *multiplier* $\lambda = e^{2\pi i \tau}$ *is exactly the modulus of the torus*!

Thus, the partially marked torus $\mathbb{T}^2_\lambda$ with modulus $\lambda \in \mathbb{D}^*$ can be realized as the quotient of $\mathbb{C}^*$ modulo the cyclic action generated by the hyperbolic Möbius transformation $\zeta \mapsto \lambda \zeta$ with multiplier λ.

1.5. Geometry of quadratic differentials.

1.5.1. *Flat structures with cone singularities and boundary corners.* Recall that a Euclidean, or flat, structure on a surface S is an atlas of local charts related by Euclidean motions. However, for topological reasons, many surfaces do not admit any flat structure: the Gauss-Bonnet Theorem bans such a structure on any compact surface except the torus (see below). On the other hand, if we allow some simple singularities, then these obstruction disappears.

Everybody is familiar with a Euclidean cone of angle $\alpha \in (0, 2\pi)$. To give a formal definition, just take a standard Euclidean wedge of angle α and glue its sides by the isometry. It is harder to define (and even harder to visualize) a cone of angle $\alpha > 2\pi$. One possible way is to partition α into several angles $\alpha_i \in (0, 2\pi)$, $i = 0, 1, \ldots n - 1$, to take wedges W_i of angles α_i, and paste W_i to W_{i+1} by gluing the sides isometrically (where i is taken mod n) (and then to check, by taking a "common subdivision", that the result is independent of the particular choice of the angles α_i).

But there is a more natural way. Consider a smooth universal covering $\exp : \mathbb{H} \to \mathbb{D}^*$, $z \mapsto e^{iz}$, over the punctured disk, and endow $\mathbb{H}$ with the pullback of the Euclidean metric, $e^{-y}|dz|$. Let us define the wedge $W = W(\alpha)$ of angle α as the strip $\{z : 0 \leq \Re z \leq \alpha\}$ completed with one point at $\operatorname{Im} z = +\infty$. If we isometrically glue the sides of this wedge, we obtain the cone $C = C(\alpha)$ of angle α. (We can also define $C(\alpha)$ as the one-point completion at $+\infty$ of the quotient $\mathbb{H}/\alpha\mathbb{Z}$.)

EXERCISE 1.15. *Let γ be a little circle around a cone singularity of angle α. Check that the tangent vector γ' rotates by angle α as we go once around γ.*

According to the discussion in Appendix 3, *a cone singularity x with angle $\alpha = \alpha(x)$ carries curvature $2\pi - \alpha$.*

Let us now consider a compact flat surface S with boundary. Assume that the boundary is piecewise linear with corners. It means that near any boundary point, S is isometric to a wedge $W(\alpha)$ with some $\alpha > 0$. Points where $\alpha \neq \pi$ are called *corners of angle α* (as the corners are isolated, there are only finitely many of them). The *rotation $\rho(x)$* at a corner $x \in \partial S$ of angle $\alpha = \alpha(x)$ is defined as $\pi - \alpha$ (see Appendix 3).

1.5.2. *Gauss-Bonnet Formula.*

THEOREM 1.2. *If S is a compact flat surface with cone singularities and piecewise linear bounady with corners then*

$$\sum K(x) + \sum \rho(y) = 2\pi\chi(S),$$

where the first sum is taken over the cone singularities while the second sum is taken over the boundary corners.

This is certainly a particular case of the general Gauss-Bonnet formula (3.4) from Appendix 3, but in our special case we will give a direct combinatorial proof of it.

PROOF. Let us triangulate S by Euclidean triangles in such a way that all cone singularities and all boundary corners are contained in the

set of vertices. Let α_i be the list of the angles of all triangles. Summing these angles over the triangles, we obtain:

$$\sum \alpha_i = \pi(\# \text{ triangles}).$$

On the other hand, summation over the vertices gives:

$$\sum \alpha_i = 2\pi(\# \text{ regular vertices}) + \sum_{\text{cones}} \alpha(x) + \sum_{\text{corners}} \alpha(y)$$

$$= 2\pi \,(\# \text{ vertices}) - \sum_{\text{cones}} K(x) - \sum_{\text{corners}} \rho(y) + \pi(\# \text{ corners}).$$

Hence

$$\sum K(x) + \sum \rho(y) = \pi\,(2(\# \text{ vertices}) + (\# \text{ corners}) - (\# \text{ triangles})) = 2\pi\chi(S),$$

where the last equality follows from

$$3(\# \text{ triangles}) = 2(\# \text{ edges}) + (\# \text{ corners}).$$

$\square$

1.5.3. *Geodesics.* Let S be a flat surface with cone singularities. A piecewise smooth curve $\gamma(t)$ in S is called a *geodesic* if it is locally shortest, i.e., for any $x = \gamma(t)$ there exists an $\epsilon > 0$ such that for any $t_1, t_2 \in [t - \epsilon, t + \epsilon]$, $\gamma : [t_1, t_2] \to S$ is the shortest path connecting $\gamma(t_1)$ to $\gamma(t_2)$.

Obviously, any geodesic is piecewise linear: a concatenation of straight Euclidean intervals meeting at cone points. Moreover, both angles between two consecutive intervals in a geodesic must be at least π (in particular, the intervals cannot meet at a cone point with angle $< 2\pi$).

EXERCISE 1.16. *Verify these assertions by exploring geodesics on a cone $C(\alpha)$.*

THEOREM 1.3. *Let S be a closed flat surface with only negatively curved cone singularities. Then for any path $\gamma : [0, 1] \to S$, there is a unique geodesic $\delta : [0, 1] \to S$ homotopic to γ rel the endpoints.*

PROOF. *Existence.* Let L be the infimum of the lengths of smooth paths homotopic to γ rel the endpoints. We can select a minimizing sequence of picewise linear paths with the intervals of definite length. Such paths form a precompact sequece in S, so we can select a subsequence converging to a path δ in S of length L. Obviously, δ is a local minimizer, and hence is a geodesic.

Uniqueness. Let γ and δ be two geodesics on S homotopic rel the endpoints. They can be lifted to the universal covering $\hat{S}$ to geodesics

$\hat{\gamma}$ and $\hat{\delta}$ with common endpoints. We can assume without loss of generality that the endpoints a and b are the only intersection points of these geodesics (replacing them if needed by the arcs $\hat{\gamma}'$ and $\hat{\delta}'$ bounded by two consecutive intersection points). Then $\hat{\gamma}$ and $\hat{\delta}$ bound a polygon Π with vertices at a and b and some corner points x_i. Let y_j be the cone poins in int Π. By the Gauss-Bonnet formula,

$$(\pi - \rho(a)) + (\pi - \rho(b)) + \sum(\pi - \rho(x_i)) + \sum K(y_j) = 2\pi.$$

But the first two terms in the left-hand side are less than π wlile the others are negative – contradiction. $\qquad\square$

1.5.4. *Quadratic differentials and* Euc(2)-*structures.* Let S^* stand for a flat surface S with its cone singularities punctured out.

A *parallel line field* on S is a family of tangent lines $l(z) \in \mathrm{T}_z S$, $z \in S^*$, that are parallel in any local chart of S.

Let $j : \mathrm{Euc}(\mathbb{C}) \to U(2)$ be the natural projection that associates to a Euclidean motion its rotational part. Let $\mathrm{Euc}(n)$ stand for the j-preimage of the cyclic group of order n in $U(2)$. In other words, motions $A \in \mathrm{Euc}(n)$ are compositions of rotations by $2\pi k/n$ and translations. (So, the complex coordinate, they assume the form $A : z \mapsto e^{2\pi k/n}z + c$.)

LEMMA 1.4. *A flat surface S admits a parallel line field if and only if its Euclidean structure can be refined to a* Euc(2)-*structure.*

PROOF. Let S be Euc(2)-surface and let $\theta \in \mathbb{R}/\mod \pi \mathbb{Z}$. Given a local chart, we can consider the parallel line field in the θ-direction. Since the θ-direction is preserved $(\mod \pi)$ by the group Euc(2), we obtain a well defined parallel line field on S^*.

Vice versa, assume we have a parallel line field on S^*. Then we can rotate the local charts so that this line field becomes horizontal. The transit maps for this atlas are Euclidean motions preserving the horizontal direction, i.e., elements of Euc(2). $\qquad\square$

LEMMA 1.5. *S admits a parallel line field if and only if all cone angles are multiples of π.*

PROOF. Any tangent line can be parallely trnsported along any path in S^*. Since S is flat, the result is independent of the choice of a path within a certain homotopy class. S admits a parallel line field if and only if the holonomy of this parallel transport around any cone singularity is trivial, i.e., it rotates the line by a multiple of π. But the holonomy around a cone singularity of angle α rotates the line by angle α. $\qquad\square$

Next, we will relate flat geometry to complex geometry. Namely, any flat surface S is naturally a Riemann surface. Indeed, since Euclidean motions are conformal, the flat structure induces complex structure on S^*. To extend it through cone singularitites, consider a conformal isomorphism $\phi : \mathbb{H}/\alpha\mathbb{Z} \to \mathbb{D}^*$, $z \mapsto e^{2\pi i z/\alpha}$. It extend to a homeomorphism $C(\alpha) \to \mathbb{D}$ that serves as a local chart on the cone $C(\alpha)$.

EXERCISE 1.17. *Show that the extension of the conformal structures from S^* to S is unique.*

Formulate as a general statement about removing punctures.
1.5.5. *Abelian differentials and translation surfaces.*

2. Appendix 1 : Tensor calculus in complex dimesnion one

For $(n, m) \in \mathbb{Z}^2$, an (n, m)-tensor on a Riemann surface S is an object τ that can be locally written as a differential form

$$\tau(z) \, dz^n d\bar{z}^m. \tag{2.1}$$

Formally speaking, to any local chart $z = \gamma(x)$ on S corresponds a function $\tau_\gamma(z)$, and this family of functions satisfy the transforamtion rule: if $\zeta = \delta(x)$ is another local chart and $z = \phi(\zeta)$ is the transit map, then

$$\tau_\delta(\zeta) = \tau_\gamma(\phi(\zeta))\phi'(\zeta)^n\overline{\phi'(\zeta)}^m.$$

The regularity of the tensor (e.g., τ can be measurable, smooth or holomorphic) is determined by the regularity of all its local representative τ_γ.

Even when dealing with globally defined tensors, we will often use local notaion (2.1), and we will usually use the same notation for a tensor and the representing local function.

For instance, a holomorphic $(1, 0)$-tensor $\omega(z)dz$ is called an *Abelian differential*; a holomorphic $(2, 0)$-tensor $q(z)dz^2$ is called a *quadratic differential*; a $(-1, 1)$-tensor $\mu(z)d\bar{z}/dz$ is called a *Beltrami differential*. Notice that the absolute value of a Beltrami differential, $|\mu|$, is a global function on S.

(*In this book all Beltrami differentials under consideration are assumed measurable and bounded.*)

A $(1, 1)$-tensor $\rho(z)\, dzd\bar{z}$ can be interpreted either as a *conformal Riemannian metric* $\rho(z)|dz|^2$ on S, or as its area form $\rho(z)\, dz \wedge d\bar{z}$ (as all these objects are transformed under according to the same rule,

by the factor $|\phi(\zeta)|^2$).[3] In what follows we will not make distinctions between tensors of these types.

For instance, if q is a quadratic differential then $|q|$ is a $(1,1)$-form, so that we can evaluate $\int |q|$ (at least locally). If q is a quadratic differential and μ is a Beltrami differential, then $q\nu$ is again a $(1,1)$-tensor, so that the local integral $\int q\mu$ makes sense.

A $(-1,0)$-tensor $\dfrac{v(z)}{dz}$ has the same type as the vector field $v(z)\dfrac{d}{dz}$. In fact, any vector field can be viewed as such $(-1,0)$-form, up to the following convention. Let us consider a vector field

$$v = v_1\partial_x + v_2\partial_y = v\partial_z + \bar{v}\partial_{\bar{z}},$$

where

$$\partial_z = \frac{1}{2}(\partial_x - i\partial_y), \quad \partial_{\bar{z}} = \frac{1}{2}(\partial_x + i\partial_y). \tag{2.2}$$

Here $v\partial_z$ and $\bar{v}\partial_{\bar{z}}$ are called the *holomorphic* and *anti-holomorphic* parts of v, and they can be viewed as $(-1,0)$ and $(0,-1)$ forms repsectively. The projection of a vector field on its holomorphic part, $v \mapsto v/dz$, is an isomorphism between the space of vector fields and the space of $(-1,0)$-tensors, which allows us to identify these objects.

REMARK 1.1. The holomorphic part of a tangent vector can be invarinatly defined as follows. Let E be a one-dimensional complex vector space. Let is us first decompexify it to obtain a two-dimensional real vector space $E_{\mathbb{R}}$, and then compexify it to obtain a complex two dimensional space $F = (E_{\mathbb{R}})^{\mathbb{C}}$. The multiplication by i in E becomes a real linear operator $J : E_{\mathbb{R}} \to E_{\mathbb{R}}$ such that $J^2 = -I$. Its compexification $J^{\mathbb{C}} : F \to F$ has one-dimensinal eigenspaces, $F_{\pm}$ corresponding to eigenvalues $\pm i$ respectively. The first one is called the *holomorphic* part of F, while the second is called the *anti-holomorphic* part. The projection of E to F_+ parallel to F_- is a complex isomorphism that allows to identify these two spaces.

In coordinates this discussion assumes the following form. Decomplexification of E means that we introduce real coordinates $x + iy = z$. Compexification of $E_{\mathbb{R}}$ means that x and y are now considered as complex coordinates X and Y. The operator J acts in F as follows: $(X, Y) \mapsto (Y, -X)$. The coordinates that diagonalize the operator are $Z = X + iY$ and $\bar{Z} = X - iY$. In these coordinates, the projection $E \to F_+$ assumes the form $z \mapsto (z, 0)$.

---

[3]We call forms $q(z)dz \wedge d\bar{z}$ "area forms" even when they are not positive, as in the case of $dz \wedge d\bar{z} = -2idx \wedge dy$.

Let us now introduce differential operators ∂ and $\bar{\partial}$ acting on the tensors as follows:

$$\partial(\tau\, dz^n d\bar{z}^m) = \partial_z\tau dz^{n+1}d\bar{z}^m), \bar{\partial}(\tau\, dz^n d\bar{z}^m) = \partial_z\tau dz^n d\bar{z}^{m+1}).$$

EXERCISE 1.18. *Check that these operators are correctly defined.*

For instance, if v is a vector field viewed as a $(-1, 0)$ tensor, then ∂v is a Beltrami differential.

For simplicity, we will often use notation ∂ and $\bar{\partial}$ for the partial derivatives (2.2), unless it can lead to a confusion.

3. Appendix 2: Gauss-Bonnet formula for variable metrics

Formally speaking, we can skip a discussion of this general version of the Gauss-Bonnet formula as we have verified it directly in all special cases that we need. However, it does give a deeper insight into the matter. The reader can consult, e.g., [] for a proof.

Let S be a compact smooth Riemannian surface, maybe with boundary. Let $K(x)$ be the Gaussian curvature at $x \in S$, and let $\kappa(x)$ be the geodesic curvature at $x \in \partial S$. The Gauss-Bonnet formula related these gemeotric quantities to topology of S:

$$\int_S K d\sigma + \int_{\partial S} \kappa ds = 2\pi\chi(S), \tag{3.1}$$

where $d\sigma$ and ds are the area and length elements respectively.

In particular, if S is closed then

$$\int_S K d\sigma = 2\pi\chi(S), \tag{3.2}$$

which, in particular, implies that there are no flat structures on a closed surface of genus $g \neq 0$.

The boundary term in (3.1) admits a nice interpretation. Let us parametrize a closed boundary curve γ with the length parameter, so that $\gamma'(t)$ is the unit tangent vector to γ. Then for nearby points $\gamma(t)$ and $\gamma(\tau)$, where $\tau = t + \Delta t > t$, let $v(t, \tau)$ be the tangent vector $\gamma'(\tau)$ parallelly transported from $\gamma(\tau)$ back to $\gamma(t)$. Then let $\theta(t, \tau)$ be the angle between $\gamma'(t)$ and $v(t, \tau)$ (taking with positive sign if v points "into S". Summing these angles up over a partition of γ into small intervals, we obtain the *rotation number* of the tangent vector. It coincides with $\int_\gamma \kappa ds$.

Note that if ∂S consists of geodesics, the boundary term in (3.1) disappears, and it assumes the same form (3.2) as in the closed case.

If we allow the Riemannian metric to have an isolated singularity at some point $x \in S$ then using the Gauss-Bonnet formula for a small disk arround x, we can assign the Gaussian curvature to x:

$$K(x) = 2\pi - \lim_{\gamma \to x} \int_\gamma \kappa ds, \tag{3.3}$$

provided the limit exists. (Here γ is a small circle around x, and $K(x)$ is assumed to be integrable.)

If we allow a corner of angle $\alpha \in (0, \infty)$ at a boundary point $y \in \partial S$ (see §1.5.1), we can assign the rotation number $\rho(y) = \pi - \alpha \in (\pi, -\infty)$ to it as the angle between the incoming and outgoing tangent vectors.

Then the Gauss-Bonnet formula is still valid for surfaces with singularities and boundary corners, assuming the following form:

$$\int_S K d\sigma + \sum_{\text{sing}} K(x) + \int_{\partial S} \kappa ds + \sum_{\text{corners}} \rho(y) = 2\pi \chi(S). \tag{3.4}$$

4. Uniformization Theorem

4.1. The following theorem of Riemann and Koebe is the most fundamental result of complex analysis:

THEOREM 1.6. *Any simply connected Riemann surface is conformally equivalent to either the Riemann sphere $\bar{\mathbb{C}}$, or to the complex plane $\mathbb{C}$, or the unit disk $\mathbb{D}$.*

4.2. Classification of Riemann surfaces. Consider now any Riemann surface S. Let $\pi : \hat{S} \to S$ be its universal covering. Then the complex structure on S naturally lifts to $\hat{S}$ turning S into a simply connected Riemann surface which holomorphically covers S. Thus, we come up with the following classification of Riemann surfaces:

THEOREM 1.7. *Any Riemann surface S is conformally equivalent to one of the following surfaces:*
- *The Riemann sphere $\bar{\mathbb{C}}$ (spherical case);*
- *The complex plane $\mathbb{C}$, or the punctured plane $\mathbb{C}^*$, or a torus $\mathbb{T}^2_\tau$, $\tau \in \mathbb{H}$ (parabolic case);*
- *The quotient of the hyperbolic plane $\mathbb{H}^2$ modulo a discrete group of isometries (hyperbolic case).*

Thus, *any* Riemann surface comes endowed with one of the three geometries described in §1.3: projective, affine, or hyperbolic.

4.3. Simply connected plane domains.

4.4. Thrice punctured sphere and modular function λ. Let us now consider the case of the biggest hyperbolic plane domain, the thrice punctured sphere $\mathbb{U} = \mathbb{C} \setminus \{0, 1\}$[4]. In this case, there is a simple explicit construction of the universal covering. Namely, let us consider an ideal triangle Δ in the hyperbolic plane, that is, the geodesic triangle with vertices on the absolute[5] (see Figure **??**). By the Riemann Mapping Theorem, it can be conformally mapped onto the upper half plane $\mathbb{H}$ so that its vertices go to the points $0, 1$ and ∞. By the Schwarz Reflection Principle, this conformal map can be extended to the three symmetric ideal triangles obtained by reflection of Δ in its edges. Each of these symmetric rectangles will be mapped onto the lower half-plane $\mathbb{H}^-$. Then we can extend this map further to the six symmetic rectangles each of which will be mapped onto $\mathbb{H}$ again, etc. Proceeding in this way, we obtain the desired universal covering $\lambda : \mathbb{D} \to \mathbb{U}$ called a *modular function*.

EXERCISE 1.19. *Verify the follwing properties:*
a) The union of these triangles tile the whole disk $\mathbb{D}$;
b) The modular function λ is the desired universal covering;
c) Its group of deck transforamations is the congruent group Γ_2, that is, the subgroup of $\mathrm{PSL}(2, \mathbb{Z})$ consisting of matrices congruent to I mod 2.

4.5. Do we need the full strength of the Uniformization Theorem?

4.6. Appendix: harmonic functions.
4.6.1. *Definitions.* There are three equivalent definitions of a harmonic function $h : U \to \mathbb{R}$ on a domain $U \subset \mathbb{C}$:

• $h \in C^2(U)$ and $\Delta h = 0$ where $\Delta = \partial_x^2 + \partial_y^2$ is the usual Euclidean Laplacian ;

• h is continuous and for any disk $|\mathbb{D}(a, r) \subset U$, $h(a) = \int_0^{2\pi} \frac{1}{2\pi} h(a + re^{i\theta}) d\theta$.

• Locally $h = \Re \phi$ for some holomortphic function ϕ.
    distributional stuff? comment on proof?
    The last definition makes obvious that harmonicity is well-defined on an arbitrary Riemann surface S. It can be also seen from the first definition as follows. In terms of the differential operators ∂ and $\bar{\partial}$ (see

---

[4]Note that all thrice punctured spheres are equivalent under the action of the Möbius group $\mathrm{M\ddot{o}b}(\hat{\mathbb{C}})$.
    [5]Note that all these triangles are equivalent under the action of $\mathrm{PSL}(2, \mathbb{R})$.

§2) we have: $\partial_z \partial_{\bar{z}} = \frac{1}{4}\Delta$, so that

$$\partial\bar{\partial}h = \partial_z \partial_{\bar{z}}h\, dz \wedge d\bar{z} = \frac{1}{2i}\Delta h\, dx \wedge dy.$$

Since $\partial\bar{\partial}$ is a well-defined operator (from functions to 2-forms) on a Riemann surface S, we see that $\Delta h\, dx \wedge dy$ is a well defined 2-form on S. In particular, zeros of Δh, and hence harmonicity of h, are well defined.

4.6.2. *Maximum Principle.* Given a domain U on a Riemann surface S, let $H(U)$ stand for the space of harmonic functions in U , and let $H(\bar{U})$ stand for the subspace of $H(U)$ consisting of functions that admit continuous extension to $\bar{U}$.

Maximum Principle. *If a harmonic function h on U has a local maximum/minimum in U then it is constant.*

COROLLARY 1.8. *Let $U \Subset S$ be a compactly embedded domain in a Riemann surface S, and let $h : H(\bar{U})$. Then h attains its maximum and minimum on ∂U.*

COROLLARY 1.9. *Under the above circumstances, h is uniquely determined by its boundary values, $h|\partial U$.*

4.6.3. *Poisson Formula.* The Poisson Formula allows us to recover a harmonic function $h \in H(\bar{\mathbb{D}})$ from its boundary values.

Poisson Formula. *Any continuous function $g \in C(\mathbb{T})$ on the unit circle admits a unique harmonic extension $h \in H(\bar{U})$ to the unit disk (so that $g = h|\mathbb{T}$). This extension is given by the following formula:*

$$h(z) = \frac{1}{2\pi}\int_0^{2\pi} g(\zeta)\frac{1 - |z|^2}{|z - \zeta|^2}d\theta, \quad z \in \mathbb{D}, \ \zeta = e^{i\theta} \in \mathbb{T}.$$

PROOF. For $z = 0$, this formula just says that

$$h(0) = \int_0^{2\pi} h(e^{i\theta})d\theta.$$

It implies the formula at any point $z \in \mathbb{D}$ by making a conformal change of variable $\phi_z : \bar{\mathbb{D}} \to \bar{\mathbb{D}}$

$$\zeta \mapsto \frac{\zeta + z}{1 - \bar{z}\zeta}$$

that moves 0 to z. Since the space $H(\bar{\mathbb{D}})$ is invariant under such changes of variable, we have:

$$h(z) = (h \circ \phi_z)(0) = \int_0^{2\pi} h \circ \phi_z(e^{i\theta})d\theta = \int_0^{2\pi} h(e^{i\theta})d\theta_z,$$

where
$$d\theta_z = (\phi_z)_*(d\theta) = |(\phi_z^{-1})'(\theta)|d\theta,$$

and the latter derivative is equal to the Poisson kernel $\dfrac{1 - |z|^2}{|z - \zeta|^2}$ (check it!).

Uniqueness of the extension follows from the Maximum Principle.
$\square$

5. Principles of the hyperbolic metric

5.1. Schwarz Lemma. In terms of the hyperbolic metric, the elementary Schwarz Lemma can be brought to a conformally invariant form that plays an outstanding role in holomorphic dynamics:

Schwarz Lemma. *Let $\phi : S \to S'$ be a holomorphic map between two hyperbolic Riemann surfaces. Then*
• either ϕ is a strict contraction, i.e., $\|D\phi(z)\| < 1$ for any $z \in S$, where the norm of the differential is evaluated with respect to the hyperbolic metrics of S and S';
• or else, ϕ is a covering map.

PROOF. Given a point $z \in S$, let $\pi : (\mathbb{D}, 0) \to (S, z)$ and $\pi' : (\mathbb{D}, 0) \to (S', \phi(z))$ be the universal coverings of the Riemann surfaces S and S' respectively. Then ϕ can be lifted to a holomorphic map $\tilde{\phi} : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$. By the elementary Schwarz Lemma, $|\tilde{\phi}'(0)| < 1$ or else $\tilde{\phi}$ is a conformal automorphism of $\mathbb{D}$ (in fact, rotation). This yields the desired dichotomy for ϕ. $\square$

In particular, if $S \subset S'$ then $\rho_S \geq \rho_{S'}$ ("a smaller Riemann surface is more hyperbolic").

5.2. Hyperbolic metric blows up near the boundary. For a domain $U \subset \bar{\mathbb{C}}$, let $d(z)$ stand for the *spherical distance* from $z \in U$ to ∂U.

EXERCISE 1.20. *Show that $d\rho_{\mathbb{D}^*}(z) = -\dfrac{|dz|}{|z| \log |z|}$;*

LEMMA 1.10. *Let $\mathbf{S}$ be a Riemann surface, $x \in \mathbf{S}$, and assume that the punctured surface $S = \mathbf{S} \smallsetminus \{x\}$ is hyperbolic with the hyperbolic metric ρ. Then*
$$d\rho(z) \asymp -\frac{|dz|}{|z| \log |z|},$$
where z is a local coordinate on $\mathbf{S}$ with $z(x) = 0$.

PROOF. By Proposition 1.1, a standard cusp $\mathbb{H}_h/\mathbb{Z}$ is isometrically embedded into S so that its puncture corresponds to x. On the other hand, by means of the exponential maps $\mathbb{H} \to \mathbb{D}^*$, $z \mapsto e^{2\pi i z}$, the cusp $\mathbb{H}_h/\mathbb{Z}$ is isometric to the punctured disk $\mathbb{D}_r^*$, $r = e^{-2\pi h}$, in the hyperbolic metric of $\mathbb{D}^*$. By the previous Excercise, the latter has the desired form in the plane coordinate of $\mathbb{D}_r^*$ (which extends to a local coordinate on $\mathbf{S}$ near x). Hence it has the desired form in any other local coordinate on $\mathbf{S}$ near x. $\square$

PROPOSITION 1.11. *For any hyperbolic plane domain $U \subset \bar{\mathbb{C}}$, there exists $\kappa = \kappa(U) > 0$ such that:*

$$\frac{d\rho_U}{d\sigma}(z) \geq -\frac{\kappa}{d(z)\log d(z)},$$

where σ is the spherical metric.

PROOF. Take some point $z \in U$, and find the closest to it point $a \in \partial U$. Since ∂U consists of at least three points, we can find two more points, $b, c \in \partial U$, such that the points a, b, c are ϵ-spearated on $\bar{\mathbb{C}}$, where $\epsilon > 0$ depends only on U. Let us consider the Möbius transformation ϕ that moves (a, b, c) to $(0, 1, \infty)$. By Exercise 1.3, thes transformations are uniformly bi-Lipschitz in the spherical metric, which reduces the problem to the case when $(a, b, c) = (0, 1, \infty)$. But in this case, $\rho_U(z)$ dominates the hyperbolic metric on $\mathbb{U} = \mathbb{C} \smallsetminus \{0, 1\}$, and the desired estimate follows from Lemma 1.10. $\square$

EXERCISE 1.21. *More generally, let $\mathbf{S}$ be a Reimann surface endowed with a conformal Riemannian metric σ, and let K be a compact subset of $\mathbf{S}$ such that $\mathbf{S} \smallsetminus K$ is a hyperbolic Riemann surface with hyperbolic metric ρ. Then*

$$\frac{d\rho}{d\sigma}(z) \geq -\frac{\kappa}{d(z)\log d(z)}, \quad \text{as } z \to K,\ z \in S,$$

where $d(z) = \text{dist}(z, K)$.

5.3. Normal families and Montel's Theorem. Let U be a Riemann surface, and let $\mathcal{M}(U)$ be the space of meromorphic functions $\phi : U \to \bar{\mathbb{C}}$. Supply the *target* Riemann sphere $\bar{\mathbb{C}}$ with the spherical metric d_s and the space $\mathcal{M}(U)$ with the topology of uniform convergence on compact subsets of U. Thus $\phi_n \to \phi$ if for any compact subset $K \subset U$, $d_s(\phi_n(z), \phi(z)) \to 0$ uniformly on U. Since locally uniform limits of holomorphic functions are holomorphic, $\mathcal{M}(U)$ is *closed* in the space $C(U)$ of continuous functions $\phi : U \to \bar{\mathbb{C}}$ (endowed with the topology of uniform convergence on compact subsets of U).

EXERCISE 1.22. *Endow $\mathcal{M}(U)$ with a metric compatible with the above convergence that makes $\mathcal{M}(U)$ a complete metric space.*

It is important to remember that the target should be supplied with the *spherical* rather than *Euclidean* metric even if the original family consists of *holomorphic* functions. In the limit we can still obtain a meromorphic function, though of a very special kind:

EXERCISE 1.23. *Let $\phi_n : U \to \mathbb{C}$ be a sequence of holomorphic functions converging to a meromorphic function $\phi : U \to \bar{\mathbb{C}}$ such that $\phi(z) = \infty$ for some $z \in U$. Then $\phi(z) \equiv \infty$, and thus $\phi_n(z) \to \infty$ uniformly on compact subsets of U.*

A family of meromorphic functions on U is called *normal* if it is precompact in $\mathcal{M}(U)$.

EXERCISE 1.24. *Show that normality is the local property: If a family is normal near each point $z \in U$, then it is normal on U.*

EXERCISE 1.25. *If a domain $U \subset \mathbb{C}$ is supplied with the Euclidean metric $|dz|$ while the target $\bar{\mathbb{C}}$ is supplied with the spherical metric $|dz|/(1 + |z|^2)$, then the corresponding "ES norm" of the differential $D\phi(z)$ is equal to $|\phi'(z)|/(1 + |\phi(z)|^2)$, $z \in U$. Show that a family of meromorphic functions $\phi_n : U \to \bar{\mathbb{C}}$ is normal if and only if the ES norms $\|D\phi_n(z)\|$ are uniformly bounded on compact subsets of U.*

THEOREM 1.12 (Little Montel). *Any bounded family of holomorphic functions is normal.*

PROOF. It is because the derivative of a holomorphic function can be estimated via the function itself. Indeed by the Cauchy formula

$$|\phi'(z)| \leq \frac{\max_{\zeta \in U} |\phi(\zeta)|}{\mathrm{dist}(z, \partial U)}.$$

Thus, if a family of holomorphic functions ϕ_n is uniformly bounded, their derivatives are uniformly bounded on compact subsets of U. By the Arzela-Ascoli Criterion, this family is precompact in the space $C(U)$ of continuous functions. Making use of Exercise 1.22), we see that the original family is precompact in the space $\mathcal{M}(U)$. $\square$

EXERCISE 1.26. *A sequence of holomorphic functions is normal if and only if from any subsequence one can extract a further subsequence which is either bounded or divergent to ∞.*

THEOREM 1.13 (Montel). *If a family of meromorphic functions $\phi_n : U \to \bar{\mathbb{C}}$ does not assume three values then it is normal.*

PROOF. Since normality is a local property, we can assume that U is a disk. Let us endow it with the hyperbolic metric ρ. Let a, b, c be omitted values on $\bar{\mathbb{C}}$, and let ρ' be the hyperbolic metric on the thrice punctured sphere $\bar{\mathbb{C}} \smallsetminus \{a, b, c\}$.

By the Schwarz Lemma, all the functions ϕ_n are contractions with respect to these hyperbolic metrics. By Exercise 1.11 (iii), the spherical metric is dominated by ρ', so the ϕ_n are uniformly Lipschitz from metric ρ to the spherical metric. Normality follows.   □

THEOREM 1.14 (Refined Montel). *Let $\{\phi_n : U \to \bar{\mathbb{C}}\}$ be a family of meromorphic functions. Assume that there exists three meromorphic functions $\psi_i : U \to \bar{\mathbb{C}}$ such that for any $z \in U$ and $i \neq j$ we have: $\psi_i(z) \neq \psi_j(z)$ and $\phi_n(z) \neq \psi_i(z)$. Then the family $\{\phi_n\}$ is normal.*

PROOF. Let us consider the holomorphic family of Möbius transformations $h_z : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ depending on $z \in U$ as a parameter such that $h_z : (\psi_1(z), \psi_2(z), \psi_3(z)) \mapsto (0, 1, \infty)$. Then the family of functions $\Phi_n(z) = h_z(\phi_n(z))$ omits value $0, 1, \infty$, and hence is normal by Theorem 1.13. It follows that the original family is normal as well.   □

EXERCISE 1.27. *Show that the theorem is still valid if we allow $\psi_i(z) = \psi_j(z)$ for some $z \in U$.*

Given a family $\{\phi_n\}$ of meromorphic functions on U, we can define its *set of normality* as the maximal open $F \subset U$ set on which this family is normal.

5.4. Koebe Distortion Theorem. We are now going to discuss one of the most beautiful and important theorems of the classical geometric functions theory.

The *inner radius* $r_{D,a} \equiv \text{dist}(a, \partial D)$ of a pointed disk (D, a) is as the biggest round disk $\mathbb{D}(a, \rho)$ contained in D. The *outer radius* $R_{D,a} \equiv$ H-dist$(a, \partial D)$ is the radius of the smallest disk $\bar{\mathbb{D}}(a, \rho)$ containing D. (If $a = 0$, we will simply write r_D and R_D.) The *eccentricity* of a pointed disk (D, a) is the ratio $R_{D,a}/r_{D,a}$.

THEOREM 1.15 (Koebe Distortion). *Let $\phi : (\mathbb{D}, 0) \to (D, a)$ be a conformal map, and let $k \in (0, 1)$, $D_k = \phi(\mathbb{D}_k)$. Then there exist constants $C = C(k)$ and $L = L(k)$ (independent of a particular ϕ!) such that*

$$\frac{|\phi'(z)|}{|\phi'(\zeta)|} \leq C(k) \text{ for all } z, \zeta \in \mathbb{D}_k \qquad (5.1)$$

and

$$L(k)^{-1}|\phi'(0)| \leq r_{D_k,a} \leq R_{D_k,a} \leq L(k)\,|\phi'(0)|. \qquad (5.2)$$

In particular, the inner radius of the image $\phi(\mathbb{D})$ around a is bounded from below by an absolute constant times the derivative at the origin:

$$r_{\phi(D),a} \geq \rho|\phi'(0)| > 0. \tag{5.3}$$

The expression in (5.1) is called the *distortion* of ϕ, its logarithm is called the *non-linearity* of ϕ. Thus estimate (5.1) tells us that the function ϕ restricted to $\mathbb{D}_k$ has a *uniformly bounded distortion*. Estimate (5.2) tells that the eccentricity of the domain D_k around a is uniformly bounded. Note that since *any* topological disk in $\mathbb{C}$, except $\mathbb{C}$ itself, can be uniformized by $\mathbb{D}$, there could be no possible bounds on the distortion and eccentricity in the whole unit disk $\mathbb{D}$. However, once the disk is shrunk a little bit, the bounds appear!

The Koebe Distortion Theorem is equivalent to the normality of the space of normalized univalent functions:

THEOREM 1.16. *The space $\mathcal{U}$ of univalent functions $\phi : (\mathbb{D}, 0) \rightarrow (\mathbb{C}, 0)$ with $|\phi'(0)| = 1$ is compact (in the topology of uniform convergence on compact subsets of $\mathbb{D}$).*

PROOF. Note first that the image $\phi(\mathbb{D})$ cannot contain the whole unit circle $\mathbb{T}$. Otherwise the inverse map ϕ^{-1} would be well-defined on some disk $\mathbb{D}_r$ with $r > 1$, and by the Schwarcz Lemma, $|D\phi^{-1}(0)| \leq 1/r < 1$ contrary to the normalization assumption.

Hence for any $\phi \in \mathcal{U}$ there is a $\theta \in \mathbb{R}$ such that the rotated function $e^{i\theta}\phi$ does not assume value 1. Since the group of rotation is compact, it is enough to prove that the space $\mathcal{U}_0 \subset \mathcal{U}$ of univalent functions $\phi \in \mathcal{U}$ which do not assume value 1 is compact.

Let us puncture $\mathbb{D}$ at the origin, and restrict all the functions $\phi \in \mathcal{U}_0$ to the punctured disk $\mathbb{D}^*$. Since all the ϕ are univalent, they do not assume value 0 in $\mathbb{D}^*$. By the Montel Theorem, the family $\mathcal{U}_0$ is normal on $\mathbb{D}^*$.

Let us show that it is normal at the origin as well. Take a Jordan curve $\gamma \subset \mathbb{D}^*$ around 0, and let Δ be the disk bounded by γ. Restrict all the functions $\phi \in \mathcal{U}_0$ to γ. By normality in $\mathbb{D}^*$, the family $\mathcal{U}_0$ is either uniformly bounded on γ, or admits a sequence which is uniformly going to ∞. But the latter is impossible since all the curves $\phi_n(\gamma)$ intersect the interval $[0,1]$ (as they go once around 0 and do not go around 1). Thus, the family $\mathcal{U}_0$ is uniformly bounded on γ. By the Maximum Principle, it is is uniformly bounded, and hence normal, on Δ as well.

Thus, the family $\mathcal{U}_0$ is precompact. What is left, is to check that it contains all limiting functions. By the Argument Principle, limits of univalent functions can be either univalent or constant. But the latter is not possible in our situation because of normalization $|\phi'(0)| = 1$. $\square$

EXERCISE 1.28. *(a) Show that a family $\mathcal{F}$ of univalent functions $\phi : \mathbb{D} \to \mathbb{C}$ is precompact in the space of all univalent functions if and only if there exists a constant $C > 0$ such that*

$$|\phi(0)| \leq C \quad and \quad C^{-1} \leq |\phi'(0)| \leq C \text{ for all } \phi \in \mathcal{F}.$$

b) Let (Ω, a) be a pointed domain in $\mathbb{C}$ and let $C > 0$. Consider a family $\mathcal{F}$ of univalent functions $\phi : \Omega \to \mathbb{C}$ such that $|\phi(a)| \leq C$. Show that this family is normal if and only if there exists $\rho > 0$ such that each function $\phi \in \mathcal{F}$ omits some value ζ with $|\zeta| < \rho$.

Proof of the Koebe Distortion Theorem. Compactness of the family $\mathcal{U}$ immediately yields that functions $\phi \in \mathcal{U}$ and their derivatives are uniformly bounded on any smaller disk $\mathbb{D}_k$, $k \in (0,1)$. Combined with the fact that all functions of $\mathcal{U}$ are univalent, compactness also implies a lower bound on the inner radius $r_{\phi(D_k)}$ and on the derivative $\phi'(z)$ in $\mathbb{D}_k$. These imply estimates (5.1) and (5.2) on the dsitortion and eccentricity by normalizing a univalent function $\phi : \mathbb{D} \to \mathbb{C}$, i.e., considering

$$\tilde{\phi}(z) = \frac{\phi(z) - a}{f'(0)} \in \mathcal{U}.$$

(Note that this normalization does not change either distortion of the function, or the eccentricity of the image.)

Estimate (5.3) is an obvious consequence of the left-hand side of (5.2). $\square$

We have given a qualitative version of the Koebe Distortion Theorem, which will be sufficient for all our purposes. The quantitative version provides sharp constants $C(k)$, $L(k)$, and ρ, all attained for a remarkable extremal *Koebe funcion* $f(z) = z/(1-z)^2 \in \mathcal{U}$. The sharp value of the constant ρ is particularly famous:

Koebe 1/4-Theorem. Let $\phi : (\mathbb{D}, 0) \to (\mathbb{C}, 0)$ be a univalent function with $\phi'(0) = 1$. Then $\phi(\mathbb{D}) \supset \mathbb{D}_{1/4}$, and this estimate is attained for the Koebe function.

We will sometimes refer to the Koebe 1/4-Theorem rather than its qualitatve version (5.3), though as we have mentioned, the sharp constants never matter for us.

EXERCISE 1.29. *Find the image of the unit disk under the Koebe function.*

Let us finish with an invariant form of the Koebe Distortion Theorem:

THEOREM 1.17. *Consider a pair of conformal disks $\Delta \Subset D$. Let $\mathrm{mod}(D \smallsetminus \Delta) \geq \mu > 0$. Then any univalent function $\phi : D \to \mathbb{C}$ has a bounded (in terms of μ) distortion on Δ:*

$$\frac{|\phi'(z)|}{|\phi'(\zeta)|} \leq C(\mu) \ \text{for all } z, \zeta \in \Delta.$$

The proof will make use of one important property of the modulus of an annulus: if an annulus is getting pinched, then its modulus is vanishing:

LEMMA 1.18. *Let $0 \in K \subset \mathbb{D}$, where K is compact. If*

$$\mathrm{mod}(D \smallsetminus K) \geq \mu > 0$$

then $K \subset \mathbb{D}_k$ where the radius $k = k(\mu) < 1$ depends only on μ.

PROOF. Assume there exists a sequence of compact sets K_i satisfying the assumptions but such that $R_i \to 1$, where R_i is the outer radius of K_i around 0. Let us uniformize $D \smallsetminus K_i$ by a round annulus, $h_i : \mathbb{A}(\rho_i, 1) \to \mathbb{D} \smallsetminus K_i$. Then $\rho_i \leq \rho \equiv e^{-\mu} < 1$. Thus, the maps h_i are well-defined on a common annulus $A = \mathbb{A}(\rho, 1)$. By the Little Montel Theorem, they form a normal family on A, so that we can select a converging subsequence $h_{i_n} \to h$.

Let $\gamma \subset A$ be the equator of A. Then $h(\gamma)$ is a Jordan curve in $\mathbb{D}$ which separates the sets K_{i_n} (with sufficiently big n) from the unit circle - contradiction. $\qquad \square$

Remark. The extremal compact sets in the above lemma (minimizing k for a given μ) are the straight intervals $[0, ke^{i\theta}]$.

Proof of Theorem 1.17 Let us uniformize D by the unit disk, $h : \mathbb{D} \to D$, in such a way that $h(0) \in \Delta$. Let $\tilde{\Delta} = h^{-1}\Delta$ and $\tilde{\phi} = \phi \circ h$. By Lemma **??**, $\tilde{\Delta} \subset \mathbb{D}_k$, where $k = k(\mu) < 1$. By the Koebe Theorem, the distortion of the functions h and $\tilde{\phi}$ on $\tilde{\Delta}$ is bounded by some constant $C = C(k)$. Hence the distortion of ϕ is bounded by C^2. $\square$

We will often use the following informal formulation of Theorem 1.17: "If $\phi : D \to \mathbb{C}$ is a univalent function and $\Delta \subset D$ is *well inside D*, then ϕ has *a bounded distortion* on Δ".

Or else: "If a univalent function $\phi : \Delta \to \mathbb{C}$ has a *definite space* around Δ, then it has a bounded distortion on Δ".

5.5. Hyperbolic metric on simply connected domains. For simply connected plane domains, the hyperbolic metric can be very well controlled:

LEMMA 1.19. *Let $D \subset \mathbb{C}$ be a conformal disk endowed with the hyperbolic metric ρ_D. Then*

$$\frac{1}{4} \frac{|dz|}{\mathrm{dist}(z, \partial D)} \leq d\rho_D(z) \leq \frac{|dz|}{\mathrm{dist}(z, \partial D)} \,.$$

Remark. Of course, particular constants in the above estimates will not matter for us.

PROOF. Let $r = \mathrm{dist}(z, \partial D)$; then $\mathbb{D}(z, r) \subset D$. Consider a linear map $h : \mathbb{D} \to \mathbb{D}(z, r)$ as a map from $\mathbb{D}$ into D. By the Schwarz Lemma, it contracts the hyperbolic metric. Hence

$$d\rho_D(z) \leq h_*(d\rho_\mathbb{D}(0)) = h_*(|d\zeta|) = |dz|/r.$$

To obtain the opposite inequality, consider the Riemann mapping $\psi : (\mathbb{D}, 0) \to (D, z)$. By definition of the hyperbolic metric,

$$d\rho_D(z) = \psi_*(d\rho_\mathbb{D}(0)) = \psi_*(|d\zeta|) = \frac{|dz|}{|\psi'(0)|} \,.$$

But by the Koebe 1/4-Theorem, $r \leq |\psi'(0)|/4$, so that $d\rho_D(z) \geq |dz|/4r$. $\qquad\square$

The *1/d-metric* on a plane domain U is a continuous Riemannian metric with the length element $|dz|/d(z)$. The previous lemma tells us that the hyperbolic metric on a simply connected domain is equivalent to the 1/d-metric.

5.6. Hyperbolic metric on the thick part.

6. Proper maps and branched coverings

A continuous map $f : S \to T$ between two topological spaces is called *proper* if for any compact set $K \subset T$, its full preimage $f^{-1}K$ is compact. In other words, $fz \to \infty$ in T as $z \to \infty$ in S (where the neighborhoods of "∞" are defined as complements of compact subsets). Full preimages of points under a proper map will also be called its *fibers*. Note that discrete fibers are finite. If a proper map $f : S \to T$ is injective then we say that S is *properly embedded into* T.

EXERCISE 1.30. *Assume that S & T are precompact domains in some ambient surfaces and that f admits a continuos extension to the closure $\bar{S}$. Then f is proper if and only if $f(\partial S) \subset \partial T$.*

EXERCISE 1.31. *Let $V \subset T$ be a domain and $U \subset S$ be a component of $f^{-1}V$. If $f : S \to T$ is proper, then the restriction $f : U \to V$ is proper as well.*

Let now S and T be topological surfaces, and f be a *topologically holomorphic map*. The latter means that for any point $a \in S$, there exist local charts $\phi : (U, a) \to (\mathbb{C}, 0)$ and $\psi : (V, fa) \to (\mathbb{C}, 0)$ such that $\psi \circ f \circ \phi^{-1}(z) = z^d$, where $d \in \mathbb{N}$. This number $d \equiv \deg_a f$ is called the *(local) degree* of f at a. If $\deg_a f > 1$, then a is called a *branched* or *critical* point of f, and $f(a)$ is called a branched or critical value of f. We also say that d is the multiplicity of a as a preimage of $b = f(a)$.

EXERCISE 1.32. *Show than any non-constant holomorphic map between two Riemann surfaces is topologically holomorphic.*

A basic property of topologically holomorphic proper maps is that they have a global degree:

PROPOSITION 1.20. *Let $f : S \to T$ be a topologically holomorphic proper map between two surfaces. Assume that T is connected. Then all points $b \in T$ have the same (finite) number of preimages counted with multiplicities. This number is called the degree of f, $\deg f$.*

PROOF. Since the fibers of a topologically holomorphic map are discrete, they are finite. Take some point $b \in T$, and consider the fiber over it, $f^{-1}b = \{a_i\}_{i=1}^l$. Let $d_i = \deg_{a_i} f$. Then there exists a neighborhood V of b and neighborgood U_i of a_i such that any point $z \in V$, $z \neq b$, has exactly d_i preimages in U_i, and all of them are regular.

Let us show that if V is sufficiently small then all preimages of $z \in V$ belong to $\cup U_i$. Otherwise there would exist sequences $z_n \to b$ and $\zeta_n \in S \smallsetminus \cup U_i$ such that $f(\zeta_n) = z_n$. Since f is proper, the sequence $\{\zeta_n\}$ would have a limit point $\zeta \in S \smallsetminus \cup U_i$. Then $f(\zeta) = b$ while ζ would be different from the a_i - contradiction.

Thus all points close to b have the same number of preimages counted with multiplicities as b, so that this number is locally constant. Since T is connected, this number is globally constant. $\square$

COROLLARY 1.21. *Topologically holomorphic proper maps are surjective.*

The above picture for proper maps suggests the following generalization. A topologically holomorphic map $f : S \to T$ between two surfaces is called a *branched covering* of degree $d \in \mathbb{N} \cup \{\infty\}$ if any point $b \in T$ has a neighborhood V with the following property. Let $f^{-1}b = \{a_i\}$ and let U_i be the components of $f^{-1}V$ containing a_i. Then these components are pairwise disjoint, and there exist maps $\phi_i : (U_i, a_i) \to (\mathbb{C}, 0)$ and $\psi : (V, b) \to (\mathbb{C}, 0)$ such that

$\psi \circ f \circ \phi_i^{-1}(z) = z_i^d$. Moreover, $\sum d_i = d$. (A branched covering of degree 2 will be also called a *double branched covering*.)

We see that *a topologically holomorphic map is proper if and only if it is a branched covering of finite degree.* All terminilogy developed above for proper maps immediately extends to arbitrary branched coverings.

Note that if $V \subset T$ is a domain which does not contain any critical values, then the "map f is unbranched over V", i.e., its restriction $f^{-1}V \to V$ is a covering map. In particular, if V is simply connected, then $f^{-1}V$ is the union of d disjoint domains U_i each of which homeomorphically projects onto V. In this case we have d well-defined branches $f_i^{-1} : V \to U_i$ of the inverse map. (We will often use the same notation f^{-1} for the inverse branches.)

Let us finish with a beautiful relation between topology of the surfaces S and T, and branching properties of f.

Riemann - Hurwitz formula. *Let $f : S \to T$ be a branched covering of degree d between two topological surfaces of finite type. Let C be the set of branched points of f. Then*

$$\chi(S) = \deg f \cdot \chi(T) - \sum_{a \in C} (\deg_a f - 1).$$

Let us define the *multiplicity* of $a \in C$ as a critical point to be equal to $\deg_a f - 1$ (in the holomorphic case it is the multiplicity of a as the root of the equation $f'(a) = 0$). Then the sum in the right-hand side of the Riemann-Hurwitz formula is equal to the *number of critical points of f counted with multiplicities.*

EXERCISE 1.33. *A double branched covering between two topological disks has a single branched point of degree 2.*

6.1. Topological Argument Principle. Consider the punctured plane $\mathbb{R}^2 \smallsetminus \{b\}$. If $\gamma : S^1 \to \mathbb{R}^2 \smallsetminus \{b\}$ is a smooth oriented Jordan curve then one can define the winding number of γ around b as

$$w_b(\gamma) = \int_\gamma d(\arg(x - b)).$$

Since the 1-form $d(\arg(x-b))$ is closed, the winding number is the same for homotopic curves. Hence we can define the winding number $w_b(\gamma)$ for any continuous Jordan curve $\gamma : S^1 \to \mathbb{R}^2 \smallsetminus \{b\}$ by approximating it with a smooth Jordan curves.

Furthermore, the winding number can be linearly extended to any simplicial 1-cycle in $\mathbb{R}^2 \smallsetminus \{b\}$ with integer coefficients (i.e., a formal

combination of oriented Jordan curves in $\mathbb{R}^2 \smallsetminus \{b\}$) and then factored
to the first homology group. It gives an isomorphism

$$w : H_1(\mathbb{R}^2 \smallsetminus \{b\}) \to \mathbb{Z}, \quad [\gamma] \mapsto w_b(\gamma). \tag{6.1}$$

EXERCISE 1.34. *Prove the last statement.*

Let $x \in D$ be an isolated preimage of $b = fx$. Then one can define
the $\mathrm{ind}_x(f)$ as follows. Take a disk $V \subset D$ around x that does not
contain other preimages of $b = fx$. Take a positively oriented Jordan
loop $\gamma \subset V \smallsetminus \{x\}$ around x whose image does not pass through b, and
calculate the winding number of the curve $f : \gamma \to \mathbb{R}^2 \smallsetminus \{b\}$ around b:

$$\mathrm{ind}_x(f) = w_{fx}(f \circ \gamma).$$

Clearly it does not depend on the loop γ, since the curves corresponding
to different loops are homotopic without crossing b.

PROPOSITION 1.22. *Let $D \subset \mathbb{R}^2$ be a domain bounded by a Jordan
curve Γ, and let $f : \bar{D} \to \mathbb{R}^2$ be a continuous map such that the curve $f \circ
\Gamma$ does not pass through some point $b \in \mathbb{R}^2$. Assume that the preimage
of this point $f^{-1}b$ is discrete in D. Then*

$$\sum_{x \in f^{-1}b} \mathrm{ind}_x(f) = w_b(f \circ \Gamma),$$

provided Γ is positively oriented.

PROOF. Note first that since $f^{-1}b$ is a discrete subset of a compact
set $\bar{D}$, $f^{-1}x$ is actually finite, so that the above sum makes sense.

Select now small Jordan loops γ_i around points $x_i \in f^{-1}b$, and
orient them anti-clockwise. Since Γ and these loops bound a 2-cell,
$[\Gamma] = \sum[\gamma_i]$ in $H_1(\bar{D} \smallsetminus f^{-1}b)$. Hence $f_*[\Gamma] = \sum f_*[\gamma_i]$ in $H_1(\mathbb{R}^2 \smallsetminus \{b\})$.
Applying the isomorphism (6.1), we obtain the desired formula. $\square$

EXERCISE 1.35. *Let $f : D \to \mathbb{R}^2$ be a continuous map, and let
$a \in D$ be an isolated point in the fiber $f^{-1}b$, where $b = f(a)$. Assume
that $\mathrm{ind}_a(f) \neq 0$. Then f is locally surjective near a, i.e., for any $\epsilon > 0$
there exists a $\delta > 0$ such that $f(\mathbb{D}_\epsilon(a)) \supset \mathbb{D}_\delta(b)$.*

Hint: For a small ϵ-circle γ around a, the curve $f \circ \gamma$ stays some
positive distance δ from b. Then for any $b' \in \mathbb{D}_\delta(b)$ we have: $\mathrm{ind}_b(f \circ
\gamma) = \mathrm{ind}_b(f \circ \gamma) \neq 0$. But if $b' \notin f(\mathbb{D}_\epsilon(a))$ then the curve $f \circ \gamma$ could
be shrunk to b without crossing b'.

6.1.1. *Degree of proper maps.*

6.2. Lifts.

LEMMA 1.23. *Let $f : (S, a) \to (T, b)$ and $\tilde{f} : (\tilde{S}, \tilde{a}) \to \tilde{T}, \tilde{b})$ be two double branched between topological disks (with or without boundary) coverings branched at points a and $\tilde{a}$ respectively. Then any homeomorphism $h : (T, b) \to (\tilde{T}, \tilde{b})$ lifts to a homeomorphism $H : (S, a) \to (\tilde{S}, \tilde{a})$ which makes the diagram*

$$
\begin{array}{ccc}
(S, a) & \xrightarrow{\ \ H\ \ } & (\tilde{S}, \tilde{a}) \\
f \downarrow & & \downarrow \tilde{f} \\
(T, b) & \xrightarrow{\ \ h\ \ } & (\tilde{T}, \tilde{b})
\end{array}
$$

commutative. Moreover, the lift H is uniquely determined by its value at any unbranched point $z \neq a$. Hence there exists exactly two lifts.

If the above surfaces are Riemann and the map h is holomorphic then then the lifts H are holomorphic as well.

PROOF. Puncturing all the surfaces at their preferred points, we obtain four topological annuli. The maps f and $\tilde{f}$ restrict to the unbranched double coverings between respective annuli, while h restricts to a homeomorphism. The image of the fundamental group $\pi_1(S \smallsetminus \{a\})$ under f consist of homotopy classes of curves with winding number 2 around b, and similar statement holds for $\tilde{f}$. Since the winding number is preserved under homeomorphisms,

$$
h_*(f_*(\pi_1(S \smallsetminus \{a\}))) = \tilde{f}_*(\pi_1(\tilde{S} \smallsetminus \{\tilde{a}\})). \tag{6.2}
$$

By the general theory of covering maps, h admits a lift

$$
H : S \smallsetminus \{a\} \to \tilde{S} \smallsetminus \{\tilde{a}\}
$$

which makes the "punctured" diagram (6.2) commutative. Moreover, this lift is uniquely determined by the value of H at any point $z \in S \smallsetminus \{a\}$.

Extend now H at the branched point by letting $H(a) = \tilde{a}$. It is clear from the local structure of branched coverings that this extension is continuous (as well as the inverse one), so that it provides us with the desired lift.

If all the given maps are holomorphic then the lift H is also holomorphic on the punctured disk $S \smallsetminus \{a\}$. Since isolated singularities are removable for bounded holomorphic maps, the extension of H to the whole disk is also holomorphic. $\qquad\square$

EXERCISE 1.36. *Similar statement holds for branched coverings f and $\tilde{f}$ with a single branched point (of any degree). Analyse the situation with two branched points.*

EXERCISE 1.37. *Assume that all the topological disks in the above lemma are $\mathbb{R}$-symmetric and that all the maps commute with the reflection σ with respect to $\mathbb{R}$. Assume also that $h(f(T \cap \mathbb{R})) = \tilde{f}(\tilde{T} \cap \mathbb{R})$. Then both lifts H also commute with σ (in particualar, they preserve the real line).*

7. Extremal length and width

7.1. Definitions. Let us now introduce one of the most powerful tools of conformal geometry. Given a family Γ of curves in a Riemann surface U, we will define a conformal invariant $\mathcal{L}(\Gamma)$ called the extremal length of Γ. Consider a measurable conformal metric $\rho|dz|$ on $\mathbb{C}$ with finite total mass

$$m_\rho(U) = \int \int \rho^2 dx \wedge dy$$

(such metrics will be called *admissible*). Let

$$\rho(\gamma) = \int_\gamma \rho|dz|,$$

stand for the length of $\gamma \in \Gamma$ in this metric (with the convention $\rho(\gamma) = \infty$ if γ is non-rectifiable, or $\rho|\gamma$ is not measurable, or else it is not integrable[6]. Define the *ρ-length of* Γ as

$$\rho(\Gamma) = \inf_{\gamma \in \Gamma} \rho(\gamma).$$

Normalize it in the scaling invariant way:

$$\mathcal{L}_\rho(\Gamma) = \frac{\rho(\Gamma)^2}{m_\rho(U)},$$

and define the *extremal length* of Γ as follows:

$$\mathcal{L}(\Gamma) = \sup_\rho \mathcal{L}_\rho(\Gamma),$$

where the supremum is taken over all admissible metrics.

A metric ρ on which this supremum is attained (if exists) is called *extremal*.

EXERCISE 1.38. *Show that the value of $\mathcal{L}(\Gamma)$ does not change if one uses only continuous admissible metrics ρ.*

Let us summarize immediate consequences of the definition:

---

[6]For this to make sense, we should think of ρ as an actual *function* rather than a class of functions up to modification on null-sets. It is also convenient to assume that ρ is defined *everywhere*.

EXERCISE 1.39. • Extension of the family: *If a family of curves* Γ' *contains a family* Γ, *then* $\mathcal{L}(\Gamma') \leq \mathcal{L}(\Gamma)$.

• Overflowing: *If* Γ *overflows* Γ' *(i.e., each curve of* Γ *contains some curve of* Γ'*), then* $\mathcal{L}(\Gamma) \geq \mathcal{L}(\Gamma')$.

• Independence of the embient surface: *If* $U \subset U'$ *and* Γ *is a family of curves in* U *then* $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$. *(This justifies skipping of "U" in the notation.)*

The *extremal width* of the family Γ is defined as the inverse to its length: $\mathcal{W}(\Gamma) = \mathcal{L}(\Gamma)^{-1}$. One can also conveniently define it as follows:

EXERCISE 1.40.
$$\mathcal{W}(\Gamma) = \inf m_\rho(U),$$
where the infimum is taken over all admissible metrics with $\rho(\gamma) \geq 1$ *for all curves* $\gamma \in \Gamma$.

REMARK 1.2. One should think that a family is "big" if it has big extremal width. So, big families are short.

The extremal lenght and width are *conformal invariants*:
If $\phi : U \to U'$ is a conformal isomorphism between two Riemann surfaces such that $\phi(\Gamma) = \Gamma'$, then $\mathcal{L}(\Gamma) = \mathcal{L}(\Gamma')$. This immediately follows from the observartion that ϕ tranfers the family of admissible metrics on U onto the family of admissible metrics on U'.

7.2. Electric circuits laws. We will now formulate two crucial properties of the extremal length and width which show that the former behaves like the resistance in electric circuits, while the latter behaves like conductance.

Let Γ_1, Γ_2 and Γ be three families of curves on U. We say that Γ *disjointly overflows* Γ_1 and Γ_2 if any curve $\gamma \in \Gamma$ contains a pair of disjoint curves $\gamma_1 \in \Gamma_1$ and $\gamma_2 \in \Gamma_2$.

Series Law. *Assume that a family* Γ *disjointly overflows families* Γ_1 *and* Γ_2. *Then*
$$\mathcal{L}(\Gamma) \geq \mathcal{L}(\Gamma_1) + \mathcal{L}(\Gamma_2),$$
or equivalently,
$$\mathcal{W}(\Gamma) \leq \mathcal{W}(\Gamma_1) \oplus \mathcal{W}(\Gamma_2).$$

PROOF. Let ρ_1 and ρ_2 be arbitrary admissible metrics. By appropriate scalings, we can normalize them so that
$$\rho_i(\Gamma_i) = m_{\rho_i}(U) = \mathcal{L}_{\rho_i}(\Gamma_i), \quad i = 1, 2.$$

Let $\rho = \max(\rho_1, \rho_2)$. Since any $\gamma \in \Gamma$ contains two disjoint curves $\gamma_i \in \Gamma_i$, we have:

$$\rho(\gamma) \geq \rho_1(\gamma_1) + \rho_2(\gamma_2) \geq \rho_1(\Gamma_1) + \rho_2(\Gamma_2) = \mathcal{L}_{\rho_1}(\Gamma_1) + \mathcal{L}_{\rho_2}(\Gamma_2).$$

Taking the infimum over all $\gamma \in \Gamma$, we obtain:

$$\rho(\Gamma) \geq \mathcal{L}_{\rho_1}(\Gamma_1) + \mathcal{L}_{\rho_2}(\Gamma_2).$$

On the other hand,

$$m_\rho(U) \leq m_{\rho_1}(U) + m_{\rho_2}(U) = \mathcal{L}_{\rho_1}(\Gamma_1) + \mathcal{L}_{\rho_2}(\Gamma_2).$$

Hence

$$\mathcal{L}_\rho(\Gamma) \geq \mathcal{L}_{\rho_1}(\Gamma_1) + \mathcal{L}_{\rho_2}(\Gamma_2).$$

Taking the supremum over all normalized metrics ρ_1 and ρ_2, we obtain the desired inequality. $\qquad\square$

We say that two families of curves, Γ_1 and Γ_2, are *disjoint* if they are contained in disjoint measurable sets.

Parallel Law. *Let $\Gamma = \Gamma_1 \cup \Gamma_2$. Then*

$$\mathcal{W}(\Gamma) \leq \mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2).$$

Moreover, if Γ_1 and Γ_2 are disjoint then

$$\mathcal{W}(\Gamma) = \mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2).$$

PROOF. This time, let us normalize admissible metrics ρ_1 and ρ_2 so that $\rho_i(\Gamma_i) \geq 1$, and let again $\rho = \max(\rho_1, \rho_2)$. Then $\rho(\Gamma) \geq 1$ as well, and hence

$$\mathcal{W}(\Gamma) \leq m_\rho(U) \leq m_{\rho_1}(U) + m_{\rho_2}(U).$$

Taking the infimum over the metrics ρ_i, we obtain the desired inequality.

Assume now that Γ_1 and Γ_2 are disjoint. Let X_1 and X_2 be two disjoint measurable sets supporting the respective families. Take any admissible metric ρ with $\rho(\Gamma) \geq 1$, and let $\rho_i = \rho| X_i$. Then $\rho_i(\Gamma_i) \geq 1$ as well, and hence

$$m_\rho(U) = m_{\rho_1}(U) + m_{\rho_2}(U) \geq \mathcal{W}(\Gamma_1) + \mathcal{W}(\Gamma_2).$$

Taking the infimum over ρ, we obtain the opposite inequality. $\qquad\square$

REMARK 1.3. Both laws obviously extend to the case of n families $\Gamma_1, \ldots, Gamma_n$.

7.3. Modulus of an annulus revisited.

7.3.1. *Modulus as the extremal length.* We will now calculate the modulus of an annulus (see §1.4.1) in terms of the extremal length. Consider a flat cylinder $C = C[l, h] = (\mathbb{R}/l\mathbb{Z}) \times [0, h]$ with circumferance l and height h. Curves joining the top to the bottom of C will be called *vertical.* Among these curves there are *genuinly vertical*, that is, straight intervals perpendicular to the top and the bottom. *Horizontal curves* in C are closed curves homotopic to the top and the bottom of C. Among them there are *genuinly horizontal*, that is, the circles parallel to the top and the bottom. Genuinly verical and horizontal curves form the vertical and horizontal *foliations* respectively.

If A is an open conformal annulus, then it is isomorphic to a flat cylinder, $A \approx C(0, h)$, and we will freely identify them. In particular, curves in A corresponding to vertical/horizontal curves in the cylinder will be also referred to as vertical/horizontal.[7]

PROPOSITION 1.24. *Let Γ be a family of vertical curves in the annulus A containing almost all genuinly vertical ones. Then $\mathcal{L}(\Gamma) = \mathrm{mod}(A)$.*

PROOF. We will identify A with the cylinder $C(l, h)$. Take first the flat metric e on the cylinder.[8] Then $e(\gamma) \geq h$ for any $\gamma \in \Gamma$, so that, $e(\Gamma) = h$. On the other hand, $m_e(\Gamma) = lh$. Hence

$$\mathcal{L}(\Gamma) \geq \mathcal{L}_e(\Gamma) = h^2/lh = \mathrm{mod}(A).$$

Take now any admissible metric ρ on A. Let γ_θ be the genuinly vertical curve through $\theta \in \mathbb{R}/l\mathbb{Z}$. Then $\rho(\Gamma) \leq \rho(\gamma_\theta)$ for any $\theta \in \mathbb{R}/l\mathbb{Z}$. Integrating this over $\mathbb{R}/l\mathbb{Z}$ (using that $\gamma_\theta \in \Gamma$ for a.e. $\theta \in \mathbb{R}/l\mathbb{Z}$) and applying the Cauchy-Schwarz inequality, we obtain:

$$(l \cdot \rho(\Gamma))^2 \leq \left(\int_{\mathbb{R}/l\mathbb{Z}} \rho(\gamma_\theta) \, d\theta \right)^2 = \left(\int_A \rho \, dm_e \right)^2 \leq lh \, m_\rho(A).$$

Hence $\mathcal{L}_\rho(A) \leq \mathrm{mod}(A)$, and the conclusion follows.   □

There is also the "dual" way to evaluate the same modulus:

EXERCISE 1.41. *Let Γ be a family of horizontal curves in A containing almost all genuinly horizontal curves. Then*

$$\mathrm{mod}(A) = \mathcal{W}(\Gamma).$$

---

[7]Notice that if $A \subset \mathbb{C}$ but ∂A is not locally connected, then vertical curves do not have to land at some points of ∂A.

[8]As we will see, e will happen to be the extremal metric.

7.3.2. *Gröztsch Inequality.* The following inequality plays an out-standing role in holomorphic dynamics (the name we use for it is widely adopted in the dynamical literature):

PROPOSITION 1.25. *Consider a conformal annulus A containing n disjoint conformal annuli $A_1, \ldots A_n$ homotopically equivalent to A. Then*

$$\operatorname{mod}(A) \geq \sum \operatorname{mod} A_k.$$

PROOF. Let Γ_k be the horizontal family of A_k and Γ be the horizontal family in A. By the Parallel Law, $\mathcal{W}(\Gamma) \geq \sum \mathcal{W}(\Gamma_k)$, and the concusion follows from Exercise 1.41. (Dually, one can apply the Series Law to the extremal length of the vertical families.)

□

7.3.3. *Euclidean geometry of an annulus.* The length-area method allows one to relate $\operatorname{mod}(A)$ to the Euclidean geometry of A. As a simple illustration, let us show that $\operatorname{mod}(A)$ is bounded by the "distance between the inner and the outer complements of A rel the size of the inner complement":

define

LEMMA 1.26. *Consider a topological annulus $A \subset \mathbb{C}$. Let K and Q stand for its inner and outer complements respectively. Then*

$$\operatorname{mod}(A) \leq C \operatorname{dist}(K, Q) / \operatorname{diam} K.$$

PROOF. Let Γ be the family of horizontal curves in A. According to the last Exercise, we need to bound $\lambda(\Gamma)$ from below.

Take points $a \in K$ and $c \in Q$ on minimal distance $\operatorname{dist}(K, Q)$, and then select a point $b \in K$ such that $\operatorname{dist}(a, b) > \operatorname{diam} K/2$. Consider a family Δ of closed Jordan curves $\gamma \subset \mathbb{C} \smallsetminus \{a, b, c\}$ with winding number 1 around a and b and winding number 0 around c. Since $\Gamma \subset \Delta$, $\lambda(\Gamma) \geq \lambda(\Delta)$.

Let us estimate $\lambda(\Delta)$ from below. Rescale the configuration $\{a, b, c\}$ (without changing notations) so that $|a - b| = 1$ and $|a - c| = d$, where

$$\frac{1}{2} \operatorname{dist}(K, Q) / \operatorname{diam} K \leq d \leq \operatorname{dist}(K, Q) / \operatorname{diam} K.$$

Consider a unit neighborhood B of the union $[a, b] \cup [a, c]$ of two intervals, and endow it with the Euclidean metric E (extended by 0 outside B). Then $l_E(\Delta) \geq 1$ while $m_E(B) \leq Ad$. Hence $\lambda_E(\Delta) \geq 1/Ad$, and we are done.   □

EXERCISE 1.42. *For an annulus A as above, prove a lower bound:*

$$\operatorname{mod}(A) \geq \mu(\operatorname{dist}(K, Q) / \operatorname{diam}(K)) > 0.$$

7.3.4. *Shrinking nests of annuli.* Let $X \subset \mathbb{C}$ be a compact connected set. Let us say that a sequence of disjoint annuli $A_n \subset \mathbb{C}$ is *nested around X* if for any any n, A_n separates both A_{n+1} and X from ∞. (We will also call it a "nest of annuli around X".)

COROLLARY 1.27. *Consider a nest of annuli A_n around X. If $\sum \mod A_n = \infty$ then X is a single point.*

PROOF. Only the first annulus, A_1, can be unbounded in $\mathbb{C}$. Take some disk $D = \mathbb{D}_R$ containing A_2, and consider the annulus $D \smallsetminus X$. By the Gröztsch Inequality,

$$\mod(D \smallsetminus X) \geq \sum_{n \geq 2} \mod A_n = \infty.$$

Hence X is a single point. □

7.3.5. *Quadrilaterals.* Given a standard flat recatangle $\Pi[l, h] = [0, l] \times [0, h]$, we can naturally define (genuinly) vertical/horizontal curves in it. We let $\mod \Pi = h/l$. Two rectangels Π and Π' are called *conformally equivalent* if there is a conformal isomorphism $\Pi \to \Pi'$ that maps the horizontal sides of Π to the horizontal sides of Π'.

EXERCISE 1.43. *Two rectangles Π and Π' are conformally equivalent if and only if $\mod \Pi = \mod \Pi'$.*

EXERCISE 1.44. *Let Γ be a family of vertical curves in $\Pi[l, h]$ that contains almost all genuinly vertical curves. Then $\mathcal{L}(\Gamma) = \mod(\Pi)$.*

A *quadrilateral Q* is a conformal disk with four marked points on its ideal boundary. It has four ideal boundary sides. *Marking* of a quadrilateral is a choice of pair of opposite sides called "horizontal" (and then the other pair is naturally called "vertical"). Any marked quadrilateral can be conformally mapped onto a rectangle $\Pi(l, h)$ so that the horizontal sides of Q go to the horizontal sides of $\Pi(l, h)$. At this point, we can naturally define (genuinly) vertical/horizontal curves in Q, and also let $\mod Q = \mod \Pi(l, h)$. With this at hands, Exercises 1.43 and 1.44 immediately extend to general marked quadrilaterals.

7.3.6. *Tori.* Let us now consider a flat torus $\mathbb{T}^2$. Given a non-zero homology class $\alpha \in H_1(\mathbb{T}^2)$, we let Γ_α be the family of closed curves on $\mathbb{T}^2$ representing α (we call them α-*curves*). Among these curves, there are closed geodesics, α-*geodesics* (they lift to straight lines in the universal covering $\mathbb{R}^2$) . They form a foliation. All these geodesics have the same length, l_α.

EXERCISE 1.45. *Let Γ be a family of α-curves containing all α-geodesics. Then*

$$\mathcal{W}(\Gamma) = \frac{\text{area}\,\mathbb{T}^2}{l_\alpha^2}.$$

An annulus A emebedded into $\mathbb{T}^2$ is called an α-*annulus* if its horizontal curves represent the class α. The following obseravation finds interesting applictions in dynamics and geometry:

PROPOSITION 1.28. *Let $A_1, \ldots, A_n$ be a family of disjoint α-annuli. Then*

$$\sum \text{mod}\, A_i \leq \frac{\text{area}\,\mathbb{T}^2}{l_\alpha^2}.$$

PROOF. Let Γ_i be the family of horizontal curves of the annulus A_i. Then by the Parallel Law, $\sum \mathcal{W}(\Gamma_i) \leq \mathcal{W}(\Gamma_\alpha)$, and the result follows from Exercises 1.41 and 1.45. $\square$

7.4. Dirichlet integral.

7.4.1. *Definition.* Consider a Riemann surface S endowed with a smooth conformal metric ρ. The *Dirichlet integral (D.I.)* of a function $\chi : S \to \mathbb{C}$ is defined as

$$D(\chi) = \int \|\nabla \chi\|_\rho \, dm_\rho,$$

where the norm of the gradient and the area form are evaluated with respect to ρ. However:

EXERCISE 1.46. *The Dirichlet integral is independent of the choice of the conformal metric ρ. In particular, it is invariant under conformal changes of variable.*

In the local coordinates, the Dirichlet integral is expressed as follows:

$$D(h) = \int (|h_x|^2 + |h_y|^2) dm = \int (|\partial h|^2 + |\bar\partial h|^2) dm.$$

In particular, for a conformal map $h : U \hookrightarrow \mathbb{C}$ we have the area formula:

$$D(h) = \int |h'(z)|^2 dm = \text{area}\, h(U).$$

7.4.2. *D.I. of a harmonic function.*

EXERCISE 1.47. *Consider a flat cylinder $A = S^1 \times (0, h)$ with the unit circumference. Let $\chi : A \to (0, 1)$ be the projection to the second coordinate (the "height" function) divided by h. Then $D(\chi) = 1/h$.*

Note that the function χ in the exercise is a harmonic function with boundary values 0 and 1 on the boundary components of the cylinder (i.e., the solution of the Dirichlet problem with such boundary values).

EXERCISE 1.48. *Such a harmonic function is unique up to switching the boundary components of A, which leads to replacement of χ by $1-\chi$.*

Due to the conformal invariance of the Dirichlet integral (as well as the modulus of an annulus and harmonicity of a function), these trivial remarks immediately yield a non-trivial formula:

PROPOSITION 1.29. *Let us consider a conformal annulus A. Then there exist exactly two proper harmonic function $\chi_i : A \to (0,1)$ (such that $\chi_1 + \chi_2 = 1$) and $D(\chi_i) = 1/\operatorname{mod}(A)$.*

7.4.3. *Multi-connected case.* Let S be a compact Riemann surface with boundary. Let $\partial S = (\partial S)_0 \sqcup (\partial S)_1$, where each $(\partial S)_i \neq \emptyset$ is the union of several boundary components of ∂S. Let us consider two families of curves: the "vertical family" Γ^v consisting of arcs joining $(\partial S)_0$ to $(\partial S)_1$, and the "horizontal family" Γ^h consisting of Jordan multi-curves separating $(\partial S)_0$ from $(\partial S)_1$. (A *multicurve* is a finite union of Jordan curves.)

Let $\chi : S \to [0,1]$ be the solution of the Dirichlet problem equal to 0 on $(\partial S)_0$ and equal to 1 on $(\partial S)_1$.

THEOREM 1.30.

$$\mathcal{L}(\Gamma^v) = \mathcal{W}(\Gamma^h) = \frac{1}{D(h)}.$$

The *modulus of S rel the boundaries* $(\partial S)_0$ and $(\partial S)_1$ is defined as the above extremal length:

$$\operatorname{mod}((\partial S)_0, (\partial S)_1) = \mathcal{L}(\Gamma^v).$$

Remark. Physically, we can think of the pair $(\partial S)_0$ and $(\partial S)_1$ in S as an electric condensator. The harmonic function χ represents the potential of the electric field created by the uniformly distributed charge on $(\partial S)_1$. The Dirichlet integral $D(\chi)$ is the *energy* of this field. Thus, $\operatorname{mod}((\partial S)_0, (\partial S)_1) = 1/D(\chi)$ is equal to the ratio of the charge to the energy, that is, to the *capacity* of the condensator.

8. Boundary values

8.1. Prime ends.

8.2. Local connectivity and continuous extensions.

8.3. Landing rays. Let $K \subset \mathbb{C}$ be a continuum.

If $\mathcal{R}_1$ and $\mathcal{R}_2$ are two rays landing at some point $a \in K$, then $\mathcal{R}_1 \cup \mathcal{R}_2 \cup \{a\}$ is a simple curve whose both ends go to ∞. By the Jordan Theorem, it divides $\mathbb{C}$ into two domains called the *sectors bounded by* $\mathcal{R}_1$ and $\mathcal{R}_2$.

LEMMA 1.31. *If two rays $\mathcal{R}_1$ and $\mathcal{R}_2$ land at the same point $a \in K$ then each sector bounbed by these rays contains a component of $K \smallsetminus \{a\}$.*

For this reason, landing points of at least two rays are called *dividing points* of K.

9. Carathéodory topology

9.1. Hausdorff convergence. Let (X, d) be a metric space. The *Hausdorff distance* between two subsets Y and Z in X is defined as follows:

$$\text{H-dist}(Y, Z) = \max(\sup_{y \in Y} d(y, Z), \ \sup_{z \in Z} d(Y, z))$$

Note that $\text{H-dist}(Y, Z) < \epsilon$ means that Z is contained in an ϵ-neighborhood of Y and vice versa.

Let $\mathcal{X}$ be the space of closed subsets in X.

EXERCISE 1.49. *(i) Show that that* H-dist *defines a metric on $\mathcal{X}$;*
(ii) If X is complete then $\mathcal{X}$ is complete as well;
(iii) If X is compact then $\mathcal{X}$ is compact as well.

9.2. Carathéodory convergence. Let us consider the space $\mathcal{D}$ of all pointed conformal disks (D, a) in the complex plane. This space can be endowed with a natural topology called *Carathéodory*. We will describe it it terms of convergence:

DEFINITION 1.2. A sequence of pointed disks $(D_n, a_n) \in \mathcal{D}$ converges to a disk $(D, a) \in \mathcal{D}$ if:

(i) $a_n \to a$;

(ii) Any compact subset $K \subset D$ is eventually contained in all disks D_n:

$$\exists N : \ K \subset D_n \ \forall n \geq N;$$

(iii) If K is a topological disk contained in infinitely many domains D_n then K is contained in D.

Note that this definition allows one to pinch out big bubbles from the domains D_n (see Figure ...).

EXERCISE 1.50. *a) Describe a topology on $\mathcal{D}$ which generates the Carathéodory convergence.*

b) Show that if ∂D_n converges to ∂D in the Hausdorff metric then the disks D_n converge to D in the Carathéodory sense.

The above purely geometric definition can be reformulated in terms of the uniformizations of the disks under consideration. Let us uniformize any pointed disk $(D, a) \in \mathcal{D}$ by a conformal map $\phi : \mathbb{D} \to D$ *normalized* so that $\phi(0) = a$ and $\phi'(0) > 0$.

PROPOSITION 1.32. *A sequence of pointed disks $(D_n, a) \in \mathcal{D}$ converges to a pointed disk $(D, a) \in \mathcal{D}$ if the corresponding sequence of normalized uniformizations $\phi : D_n \to D$ converges to D uniformly on compact subsets of $\mathbb{D}$.*

PROOF. □

Recall that $r_{D,a}$ stands for the inner radius of the domain D with respect to $a \in D$ (see §5.4). For $r \in (0, 1)$, let $\mathcal{D}_r$ stand for the family of pointed disks $(D, a) \in \mathcal{D}$ with $r \le r_{D,a} \le 1/r$.

COROLLARY 1.33. *The space $\mathcal{D}_r$ is compact.*

PROOF. Let $\phi_D : (\mathbb{D}, 0) \to (D, a)$ be the normalized uniformization of D. Then

$$r \le \phi_D'(0) \le \frac{1}{4r}$$

(The left-hand estimate follows from the Schwarz Lemma applied to $\phi^{-1} : \mathbb{D}(a, r) \to \mathbb{D}$. The right-hand estimate follows from the Koebe 1/4-Theorem applied to ϕ_D itself.)   By the Koebe Distortion Theorem, the family of univalent functions ϕ_D, $D \in \mathcal{D}_r$, is compact. By Proposition 1.32, the space $\mathcal{D}_r$ is compact as well.

□

CHAPTER 2

Quasi-conformal geometry

7. Analytic definition and regularity properties

7.1. Linear discussion.

7.1.1. *Hyperbolic metric on the space of conformal structures.* Let $V \approx \mathbb{R}^2$ be a real two-dimensional vector space. A *conformal structure* on V is a Euclidean structure (v, w) up to scaling. Equivalently, it is an ellipse centered at the origin up to scaling. Let $\mathrm{Conf}(V)$ stand for the space of conformal structures on V.

Let us fix some reference Euclidean structure on V, and call the corresponding conformal structure σ (given by "round circles") "standard". Let $A \in \mathrm{GL}_+(2, \mathbb{R})$ be an orientation preserving invertible linear operator on V. It preserves the standard conformal structure if and only if it is a *similarity*, i.e., a composition of a rotation and a scalar operator. Let $\mathrm{Sim}(2)$ stand for the group of similarities.

Any operator $A \in \mathrm{GL}_+(2, \mathbb{R})$ determines a new Euclidean structure (Av, Aw) on V, the pullback of the standard one under A. Euclidean structures corresponding to operators A and A' are conformally equivalent (i.e., proportional) if and only if $A' = SA$ where $S \in \mathrm{Sim}(2)$. Thus, the space $\mathrm{Conf}(V)$ is naturally identified with the symmetric space $\mathrm{Sim}(2) \, \mathrm{GL}(2, \mathbb{R}) = \mathrm{O}(2) \, \mathrm{SL}(2, \mathbb{R})$.

But recall from §**??** that the hyperbolic plane $\mathbb{H}$ is naturally identified with the symmetric space $\mathrm{PSL}(2, \mathbb{R})/\mathrm{P}\mathbb{R}(2) \approx \mathrm{SL}(2, \mathbb{R})/\mathrm{O}(2)$. Since the left and right symmetric spaces are isomorphic by the inversion $A \mapsto A^{-1}$, we obtain a natural identification $\mathrm{Conf}(V) \approx \mathbb{H}$. Thus, the hyperbolic metric can be transferred from $\mathbb{H}$ to $\mathrm{Conf}(V)$. Since the hyperbolic metric on $\mathbb{H}$ is invariant under the left action of $\mathrm{PSL}(2, \mathbb{R})$, the corresponding metric on $\mathrm{Conf}(V)$ is invariant under the right action of $\mathrm{PSL}(2, \mathbb{R})$, which is induced by action of $\mathrm{GL}_+(2, \mathbb{R})$ by pullbacks:

$$\mathrm{dist}_{\mathrm{hyp}}(\mu, \nu) = \mathrm{dist}_{\mathrm{hyp}}(T^*\mu, T^*\nu)$$

for any (orientation preserving invertible) linear operator $T : V \to V$.

To calculate this metric, let us select an orthonormal basis in V and consider first a diagonal matrix $A \in \mathrm{SL}(2, \mathbb{R})$ with eigenvalues $e^{\pm K/2}$, $K \geq 1$. On the one hand, it represents the point $A(i) \in \mathbb{H}$; on the other

hand, it represents the conformal structure with ellipse $E = A^{-1}(\mathbb{T})$. Hence

$$\text{dist}_{\text{hyp}}(\mu, \sigma) = \text{dist}_{\text{hyp}}(A(i), i) = \log K.$$

But K is the ratio of the axes of the ellipse E, which will be called its *shape* (and the "shape" of the corresponding conformal structure). So, *the hyperbolic distance from a conformal structure μ to the standard one is equal to the logarithm of the shape of μ.*

Now, if μ and ν are arbitrary conformal structures then we can find an operator $T : V \to V$ such that $T^*(\nu) = \sigma$, and by invariance of the hyperbolic metric we have: $\text{dist}_{\text{hyp}}(\mu, \nu) = \text{dist}_{\text{hyp}}(T^*\mu, \sigma)$. Thus, $\text{dist}_{\text{hyp}}(\mu, \nu)$ can be interpreted as the *logarithm of the shape of μ rel to ν.*

According to a well-known structural theorem for linear operators, any operator A can be decomposed into a product of a self-adjoint operator S and a rotation O, $A = O \cdot S$. This decomposition is unique up to multiplying S and O by -1. We can normalize it so that the eigenvalue $\sigma_{\max}$ of S with the maximal absolute value is positive.

Let $\sigma_{\min}$ stands for the eigenvalue of S with the minimal absolute value; it is positive or negative depending on whether A preserves or reverses the orientation. The operator A is a similarity if and only if S is scalar, i.e., $\sigma_{\max} = \sigma_{\min}$. Otherwise we have two uniquely defined eigenlines $l_{\max}$ and $l_{\min}$ corresponding to $\sigma_{\max}$ and $\sigma_{\min}$ respectively, and moreover, these eigenlines are orthogonal. In this case, $E := A^{-1}(\mathbb{T}) = S^{-1}(\mathbb{T})$ is an ellipse with the big axis of length $1/|\sigma_{\min}|$ on $l_{\min}$ and the small axis of length $1/\sigma_{\max}$ on $l_{\max}$. The *shape* of this ellipse, i.e., the ratio of the axes, is equal to $\sigma_{\max}/|\sigma_{\min}|$. This shape will be also called the *dilatation* of A, $\text{Dil}\, A$.

picture

This ellipse E is the unit circle of a new Euclidean structure

$$(Av, Av) = (Sv, Sv)$$

on V. If A is postcomposed with a conformal linear map, then the ellipse E is scaled and the Euclidean structure is replaced by a conformally equivalent (i.e., proportional). Thus, an invertible operator $A : V \to V$ up to left multiplication by a similarity determines a conformal structure on V, and vice versa. So, the space of conformal structures on V is naturally identified with the quotient $\text{Sim}\, \text{GL}(2, \mathbb{R}) = O(2)\, \text{SL}(2, \mathbb{R})$.

7.1.2. *Beltrami coefficients.* Let $V = \mathbb{C}_{\mathbb{R}}$ be the complex plane viewed as the two-dimensional oriented real Euclidean space (with $\{1, i\}$ being a positively oriented orthonormal basis), and let $A : \mathbb{C}_{\mathbb{R}} \to \mathbb{C}_{\mathbb{R}}$ be an invertible linear operator.

Let us calculate the above quantities in coordinates $z, \bar{z}$ of $\mathbb{C}_{\mathbb{R}}$. The operator A can be represented as $z \mapsto az + b\bar{z} = a(z + \mu\bar{z})$, where $\mu = b/a$ is called the *Beltrami coefficient* of A. Let $\mu = e^{2i\theta}$, where $\theta \in \mathbb{R}/(\pi\mathbb{Z})$. Then the maximum of A on the unit circle $\mathbb{T} = \{e^{i\phi}\}$ is attained at the direction $\phi = \theta \mod \pi\mathbb{Z}$, while the minimum is attained at the orthognal direction $\theta + \pi/2 \mod \pi\mathbb{Z}$. These are the eigenlines $l_{\max}$ and $l_{\min}$ of S. The corresponding eigenvalues are equal to $\sigma_{\max} = |a|(1 + |\mu|) = |a| + |b|$ and $\sigma_{\min} = |a|(1 - |\mu|) = |a| - |b|$. Thus

$$\mathrm{Dil}(A) = \frac{1 + |\mu|}{1 - |\mu|}, \quad \det(A) = |a|^2 - |b|^2 = \sigma_{\min} \, \mathrm{Dil}(A).$$
$$(7.1)$$

So the shape and orientation of the ellipse E is controlled by $|\mu|$ and $\arg \mu$ respectively. We also see that A is orientation preserving if and only if $|b| < |a|$, i.e., $|\mu| < 1$, and A is conformal if and only if $\mu = 0$.

7.1.3. *Infinitesimal notation.* Let us now interpret the above discussion in infinitesimal terms. Consider a map $h : U \to \mathbb{C}$ on a domain $U \subset \mathbb{C}$ differentiable at a point $z \in U$, and apply the above considerations to its differential $Dh(z) : \mathrm{T}_z U \to \mathrm{T}_{hz}\mathbb{C}$. In the $(dz, d\bar{z})$-coordinates of the tangent spaces, it assumes the form

$$\partial h + \bar{\partial} h = \partial_z h \, dz + \partial_{\bar{z}} h \, d\bar{z},$$

where the partial derivatives ∂_z and $\bar{\partial}_z$ and the corresponding operators ∂ and $\bar{\partial}$ are defined in §2. Moreover,

$$Dh(z) = \partial_z h(z) \, dz \left(1 + \mu_h(z)\frac{dz}{d\bar{z}}\right),$$

where $\mu_h = \partial_{\bar{z}} h / \partial_z h$ is the *Beltrami coefficient* of h at z. In fact, the above expression suggests that we should instead consider the $(-1, 1)$-tensor

$$\bar{\partial} h / \partial h = \mu_h \frac{d\bar{z}}{dz}$$

called *Beltrami differential* of h. However, in what follows we will not make a notational difference between the Beltrami differential and the coefficient.

Assume that $Dh(z)$ is non-singular, i.e., $|\mu_h| \neq 1$. The map f is orientation preserving at z if and only if $|\mu_h(z)| < 1$, and h is conformal at z if and only if $\mu_h(z) = 0$, which is equivalent to the *Cauchy-Riemann equation* $\bar{\partial} h(z) = 0$.

Let us consider an infinitesimal ellipse

$$E_h(z) \equiv Dh(z)^{-1}(\mathbb{T}_{hz}) \subset \mathrm{T}_z\mathbb{C}, \qquad (7.2)$$

where $\mathbb{T}_{hz}$ is a round circle in the tangent space $\mathrm{T}_{hz}U$. If h is not conformal at z, then $E_h(z)$ is a genuine (not round) ellipse with the small axis in the direction $\arg(\mu(z))/2 \mod \pi$ and the shape

$$\mathrm{Dil}_h(z) = \frac{1 + |\mu_h(z)|}{1 - |\mu_h(z)|}. \tag{7.3}$$

Moreover, by the second formula of (7.1), we have:

$$\mathrm{Jac}_h(z) = |\partial h(z)|^2 - |\bar{\partial}h(z)|^2 = \sigma_{\min}(z)\,\mathrm{Dil}_h(z), \tag{7.4}$$

where $\mathrm{Jac}_h(z) \equiv \det Dh(z)$ and $\sigma_{\min}(z) = \inf_{|v|=1} Dh(z)\,v$.

7.2. Conformal structures. A (measurable) *conformal structure* on a domain $U \subset \mathbb{C}$ is a measurable family of conformal structures in the tangent planes $\mathrm{T}_z U$, $z \in U$. In other words, it is a measurable family of infinitesimal ellipses $E(z) \subset T_z U$ defined up to scaling by a measurable function $\rho(z) > 0$, $z \in U$. (As always in the measurable category, all the above objects are defined almost everywhere.) According to the linear discussion, any conformal structure is determined by its *Beltrami coefficient* $\mu(z)$, $z \in U$, a measurable function in z assuming its values in $\mathbb{D}$, and vice versa. Thus conformal structures on U are described analytically as elements μ from the unit ball of $L^\infty(U)$. We say that a conformal structure has a *bounded dilatation* if the eccentricities of the ellipses $E(z)$ are bounded almost everywhere. In terms of Beltrami coefficients, it means that $\|\mu\|_\infty < 1$. The *standard* conformal structure σ is given by the family of infinitesimal circles. The corresponding Beltrami coefficient vanishes almost everywhere: $\mu = 0$.

Denote by $\mathrm{DH}^+(U, V)$ (standing for "differentiable homeomorphisms") the space of orientation preserving homeomorphisms $f : U \to V$, which are differentiable almost everywhere (with respect to the Lebesgue measure) with a non-singular differential $Df(z)$ measurably depending on z. (If we do not need to specify the domain and the range of f we write simply $f \in \mathrm{DH}^+$; if we do not assume that f is orientation preserving, we skip "+"). Consider some homeomorphism $f \in \mathrm{DH}^+(U, V)$ between two domains in $\mathbb{C}$. Then by the above discussion we obtain a measurable family $\mathcal{E}$ of infinitesimal ellipses $E_f(z) = Df(z)^{-1}(\mathbb{T}_{fz}) \subset \mathrm{T}_z U$. If f is postcomposed with a conformal map $\phi : V \to \mathbb{C}$, then the family of ellipses is scaled by a real factor (depending on z). Thus any homeomorphism $f \in DH^+(U, V)$ (defined up to a postcomposition with a conformal map) determines a (measurable) conformal structure $\mathcal{E}_f = f^*\sigma$ on U. The Beltrami coefficient of this structure is equal to $\mu_f(z) = \bar{\partial}f(z)/\partial f(z)$. It is also called the Beltrami coefficient of f.

picture

We say that f has a bounded dilatation if the corresponding conformal structure $\mathcal{E}_f$ does. In this case we let

$$\mathrm{Dil}(f) = \mathrm{Dil}(\mathcal{E}_f) = \frac{1 + \|\mu_f\|_\infty}{1 - \|\mu_f\|_\infty}.$$

What happens with conformal structures under conformal changes of variable? Let us consider a conformal map $\phi : \tilde{U} \to U$. Let $E(z)$ be an infinitesimal ellipse in $T_z U$ and $\tilde{E}(\tilde{z}) = D\phi^{-1} E(z)$ be the corresponding ellipse in $T_{\tilde{z}}\tilde{U}$. Then the dilatations of these ellipses are equal, while the small axis of $E(z)$ is obtained from the small axis of $\tilde{E}(\tilde{z})$ by rotation through the angle $\arg f'(z)$. It follows that $\tilde{\mu}(\tilde{z})/\mu(z) = f'(z)/\overline{f'}(z)$, so that the differential (-1,1)-form $\mu(z)d\bar{z}/dz$ is invariant under the above change of variable.

This allows us to generalize the above discussion to arbitrary Riemann surfaces. A (measurable) conformal structure on a Riemann surface S is a measurable family of infinitesimal ellipses defined up to scaling. Analytically it is described as a measurable *Beltrami differential* (i.e., (1,-1)-differential form) μ with $\|\mu\|_\infty < 1$. To any homeomorphism $f \in \mathrm{DH}^+(S, S')$ between two Riemann surfaces corresponds a conformal structure $\mathcal{E}_f = f^*\sigma$ on S with the Beltrami differential $\mu_f = \bar{\partial}f/\partial f$ (where $\bar{\partial}f$ and ∂f are now understood as differential 1-forms). Note that the ellipses $E_f(z)$ are well-defined only up to scaling since the round circles on S' are well-defined only up to scaling (as there is no preferred metric on S').

Remark. A key problem is whether any conformal structure $\mathcal{E}$ is associated to a certain map f. This problem has a remarkable positive solution in the category of quasi-conformal maps.

Let us consider a smaller class $\mathrm{AC}^+(U, V) \subset \mathrm{DH}^+(U, V)$ of absolutely continuous orientation preserving homeomorphisms from U onto V. (Reminder: f is absolutely continuous if for any set X of zero Lebesgue measure, the preimage $f^{-1}X$ has also zero measure.) Then we can naturally pull back any measurable conformal structure $\mathcal{E}'$ on S' to obtain a structure $\mathcal{E} = f^*(\mathcal{E}')$ on S. If f^{-1} is also absolutely continuous then we can push forward the structures: $\mathcal{E}' = f_*(\mathcal{E})$. We will use similar notations for pull-backs and push-forwards of Beltrami differentials. *In fact, in what follows we will not make notational differences between conformal structures and Beltrami differentials.*

EXERCISE 2.1. *Calculate the Beltrami differential $f^*\mu$ in terms of μ and Df. Show that $\mathrm{Dil}(f^*\mu(z)) \leq \mathrm{Dil}\, Df(z) \cdot \mathrm{Dil}\, \mu(f(z))$. Moreover, dilatation behaves submultiplicatively under compositions:*

$$\mathrm{Dil}(f \circ g) \leq \mathrm{Dil}(f) \cdot \mathrm{Dil}(g).$$

Thus, if a conformal structure ν on S' has a bounded dilatation and f has a bounded dilatation, then the pull-back structure $f^*\nu$ has a bounded dilatation as well.

More generally, let us consider a (non-invertible) map $f : U \to V$ which locally belongs to class AC^+ outside a finite set of "critical points". For such maps the push-forward operation is not well-defined, but the pull-back $\nu = f^*\mu$ is still well-defined. The fact that f has critical points does not cause any troubles since we need to know μ only almost everywhere. The property that $\mathrm{Dil}(f^*\mu) \leq \mathrm{Dil}(f) \cdot \mathrm{Dil}(\mu)$ is obviously valid in this generality.

7.3. Distributional derivatives and absolute continuity on lines. Let U be a domain in $\mathbb{C} \equiv \mathbb{C}_\mathbb{R}$. All functions below are assumed to be complex valued. A *test function* ϕ on U is an infinitely differentiable function with compact support. One says that a locally integrable function $f : U \to \mathbb{C}$ has *distributional partial derivatives of class L^1_{loc}* if there exist functions h and g of class L^1_{loc} on U such that for any test function ϕ,

$$\int_U f \cdot \partial\phi \, dm = -\int_U h\phi \, dm; \quad \int_U f \cdot \bar{\partial}\phi \, dm = -\int_U g\phi \, dm,$$

where m is the Lebesgue measure. In this case h and g are called ∂ and $\bar{\partial}$ derivatives of f in the sense of distributions. Clearly this notion is invariant under smooth changes of variable, so that it makes sense on any smooth manifold (and for all dimensions).

EXERCISE 2.2. *Prove that a function f on the interval $(0, 1)$ has a destributional derivative of class L^1_{loc} if and only if it is absolutely continuous. Moreover, its classical derivative $f'(x)$ coincides with the distributional derivative.*

There is a similar criterion in the two-dimensional setting. A continuous function $f : U \to \mathbb{C}$ is called *absolutely continuous on lines* if for any family of parallel lines in any disk $D \Subset U$, f is absolutely continuous on almost all of them. Thus, taking a typical line l of the above family, the curve $f : l \to \mathbb{C}$ is rectifiable. Clearly such functions have classical partial derivatives almost everywhere.

PROPOSITION 2.1. *Consider a homeomorphism $f : U \to V$ between two domains in the complex plane. It has distributional partial derivatives of class L^1_{loc} if and only if it is absolutely continuous on lines.*

In fact, in the proof of existence of distributional partial derivatives (the easy direction of the above Proposition), just two transversal

families of parallel lines are used. Thus one can relax the definition of absolutele continuity on lines by taking any two directions ("horizontal" and "vertical").

PROPOSITION 2.2. *Consider a homeomorphism $f : U \to V$ which is absolutely continuous on lines. Then for almost any $z \in U$, f is differentiable at z in the classical sense, i.e., $f \in$ DH.*

This result can be viewed as a measurable generalization of the elementary fact that existence of continuous partial derivatives implies differentiability.

7.4. Definition. We are now ready to give a definition of quasi-conformality. An orientation preserving homeomorphism $f : S \to S'$ between two Riemann surfaces is called quasi-conformal if

- It has locally integrable distributional partial derivatives;
- It has bounded dilatation.

Note that the second property makes sense because the first property implies that f is differentiable a.e. in the classical sense (by the results of §7.3).

We will often abbreviate "quasi-conformal" as "qc". A qc map f is called K-qc if $\mathrm{Dil}(f) \leq K$.

A map $f : S \to S'$ is called K- *quasi-regular* if for any $z \in S$ there exist K-qc local charts $\phi : (U, z) \to (\mathbb{C}, 0)$ and $\psi : (V, f(z)) \to (\mathbb{C}, 0)$ such that $\psi \circ f \circ \phi^{-1} : z \mapsto z^d$. Sometimes we will abbreviate K-quasi-regular maps as "K-qr". A map is called quasi-regular if it is K-qr for some K.

EXERCISE 2.1. *Show that any quasi-regular map $f : S \to S'$ can be decomposed as $g \circ h$, where $h : S \to T$ is a qc map to some Riemann surface T and $g : T \to S'$ is holomorphic. In particular, if $S = S' = \bar{\mathbb{C}}$ then also $T = \mathbb{C}$ and $g : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a rational map.*

7.5. Absolute continuity and Sobolev class H. We will now prove several important regularity properties of quasi-conformal maps. Let us define a Sobolev class $H = H(U)$ as the space of uniformly continuous functions $f : U \to \mathbb{C}$ whose distributional partial derivatives on U belong to $L^2(U)$. The norm on H is the maximum of the uniform norm of f and L^2-norm of its partial derivatives. Infinitely smooth functions are dense in H. This can be shown by the standard regularization procedure: convolute f with a sequence of functions $\phi_n(x) = n^2\phi(n^{-1}x)$, where ϕ is a non-negative test function on U with $\int \phi \, dm = 1$ (see [**?**, Ch V, §2.1] or [**LV**, Ch. III, Lemma 6.2]).

PROPOSITION 2.3. *Quasiconformal maps are absolutely continuous with respect to the Lebesgue measure, and thus for any Borel set $X \subset U$,*

$$m(fX) = \int_X \text{Jac}(f, z) \, dm.$$

The partial derivatives ∂f and $\bar{\partial} f$ belong to L^2_{loc}.

PROOF. Since both statements are local, we can restrict ourselves to homeomorphisms $f : U \to U'$ between domains in the complex plane. Consider the pull-back of the Lebesgue measure on U', $\mu = f^*m$. It is a Borel measure defined as follows: $\mu(X) = m(fX)$ for any Borel set $X \subset U$. Let us decompose it into absolutely continuous and singular parts: $\mu = h \cdot m + \nu$. By the Lebesgue Density Points Theorem, for almost all $z \in U$, we have:

$$\frac{1}{\pi\epsilon^2} \int_{\mathbb{D}(z,\epsilon)} h \, dm \to h(z); \quad \frac{1}{\pi\epsilon^2} \nu(\mathbb{D}(z,\epsilon)) \to 0 \quad \text{as} \quad \epsilon \to 0.$$

Summing up we obtain:

$$\frac{m(f(\mathbb{D}(z,\epsilon)))}{m(\mathbb{D}(z,\epsilon))} = \frac{\mu(\mathbb{D}(z,\epsilon))}{m(\mathbb{D}(z,\epsilon))} \to h(z) \quad \text{as} \quad \epsilon \to 0.$$

But if f is differentiable at z then the left hand-side of the last equation goes to $\text{Jac}(f, z)$. Hence $\text{Jac}(f, z) = h(z)$ a.e. It follows that for any Borel set X,

$$\int_X \text{Jac}(f, z) \, dm = \int_X h \, dm \le \mu(X) = m(fX). \quad (7.5)$$

But $\text{Jac}(f, z) = |\bar{\partial} f(z)|^2 - |\partial f(z)|^2 \ge (1 - k^2)\, |\partial f(z)|^2$, where $k = \|\mu_f\|_\infty$. Thus

$$\int_X |\partial f|^2 \, dm \le \frac{1}{1 - k^2}\, m(fX); \quad \int_X |\bar{\partial} f|^2 \, dm \le \frac{k^2}{1 - k^2}\, m(fX), \quad (7.6)$$

and we see that the partial derivatives of f are locally square integrable.

What is left is to prove the opposite to (7.5). As we have just shown, f locally belongs to the Sobolev class H. Without loss of generality we can assume that it is so on the whole domain U, i.e., $f \in H(U)$. Let us approximate f in $H(U)$ by a sequence of C^∞ functions f_n. Take a domain $D \Subset U$ with piecewise smooth boundary (e.g., a rectangle).

Let $V_n \subset f_n D$ be the set of regular values of f_n. By Sard's Theorem, it has full measure in $f_n D$. Let $R = f_n^{-1} V_n \cap D$. Note that the

$\int_{R_n} \operatorname{Jac} f_n \, dm$ is equal to the area of the image of $f_n|R_n$ counted with multiplicities:

$$\int_{R_n} \operatorname{Jac}(f_n, z) \, dm = \int_{V_n} \operatorname{card}(f_n^{-1}\zeta) \, dm \geq m(V_n) = m(f_n D).$$

Since $f_n \to f$ uniformly on D, $\liminf m(f_n D) \geq m(fD)$. Since $\operatorname{Jac}(f_n) \to \operatorname{Jac}(f)$ in $L^1(U)$,

$$\int_R \operatorname{Jac}(f_n, z) \, dm \to \int_R \operatorname{Jac}(f, z) \, dm \leq \int_D \operatorname{Jac}(f, z) \, dm.$$

Putting the last estimates together, we obtain the desired estimate for D.

For an arbitrary Borel set $X \subset U$, the result follows by a simple approximation argument using a covering of X by a union of rectangles D_i with disjoint interiors such that $m(\cup D_i \smallsetminus X) < \epsilon$. $\qquad\square$

8. Geometric definitions

Besides the analytic definition given above, we will give two geometric definitions of quasi-conformality, in terms of quasi-invariance of moduli, and in terms of bounded circular dilatation (or, "quasi-symmetricity").

8.1. Quasi-invariance of moduli. In this section we will show, by the length-area method, that the moduli of annuli are quasi-invariant under qc maps. This will follow from a more general result on quasi-invariance of extremal length:

LEMMA 2.4. *Let $h : U \to \tilde{U}$ be a K-qc map. Let Γ and $\tilde{\Gamma} = f(\Gamma)$ be two families of rectifiable curves in the respective domains such that h is absolutely continuous on all curves of Γ. Then $\mathcal{L}(\Gamma) \leq K\mathcal{L}(\tilde{\Gamma})$.*

PROOF. To any measurable metric ρ on U, we are going to associate a metric $\tilde{\rho}$ on $\tilde{U}$ such that $h^*(\tilde{\rho}) \geq \rho$ while $h^*(m_{\tilde{\rho}}) \leq Km_\rho$ (so, the map h is expanding with respect to these metrics, with area expansion bounded by K). Then $\rho(\tilde{\gamma}) \geq \rho(\gamma)$ for any $\gamma \in \Gamma$ and $\tilde{\gamma} = f(\gamma) \in \tilde{\Gamma}$, while $m_{\tilde{\rho}}(\tilde{U}) \leq Km_\rho(U)$. Hence $\mathcal{L}_{\tilde{\rho}}(\tilde{\Gamma}) \geq K^{-1}\mathcal{L}_\rho(\Gamma)$. Taking the supremum over all metrics ρ, we obtain the desired estimate.

To define correspondence $\rho \mapsto \tilde{\rho}$, recall formula (7.4) relating the Jacobian and the minimal expansion. Letting $\tilde{\rho}(hz) = \rho(z)/\sigma_{\min}(z)$, we obtain for a.e. $z \in U$ and any unit tangent $v \in \mathrm{T}_z U$:

$$|h^*(d\tilde{\rho}) \, v| = \tilde{\rho}(hz)|Dh(z)v| \geq d\rho \, (v)$$

and

$$h^*(dm_{\tilde{\rho}}) = \tilde{\rho}(hz)^2 \operatorname{Jac} h(z) \, dxdy = K(z)\rho(z)^2 dxdy \leq Kdm_\rho,$$

which are the required properties of the metrics.   □

PROPOSITION 2.5. *Consider a K-qc map $h : A \to \tilde{A}$ between two topological annuli. Then*

$$K^{-1}\operatorname{mod}(\tilde{A}) \leq \operatorname{mod}(A) \leq K\operatorname{mod}(\tilde{A}).$$

PROOF. Let $\tilde{\Gamma}$ be the family of genuinely vertical paths on $\tilde{A}$ on which h^{-1} is absolutely continuous, and let $\Gamma = h^{-1}(\tilde{\Gamma})$. By Proposition 1.24, $\operatorname{mod}\tilde{A} = \mathcal{L}(\tilde{\Gamma})$, while $\operatorname{mod}A \leq \mathcal{L}(\Gamma)$. By Lemma 2.4, $\mathcal{L}(\Gamma) \leq K\mathcal{L}(\tilde{\Gamma})$, which yields the desired right hand-side estimate. The left-hand side estimate is obtained by replacing h with h^{-1}.   □

EXERCISE 2.2. *Show that the moduli of rectangles are quasi-invariant in the same sense as for the annuli.*

EXERCISE 2.3. *Prove that $\mathbb{C}$ and $\mathbb{D}$ are not qc equivalent.*

8.2. Bounded circular dilatation. For a homeomorphism $h : U \to V$ between Riemannian surfaces with metric d, let is consider its maximal and minimal expansions of the circle of radius ϵ:

$$M_h(z, r) = \max_{d(\zeta,z)=r} d(h(\zeta), h(z)), \quad m_h(z, r) = \min_{d(\zeta,z)=r} d(h(\zeta), h(z)),$$

and let

$$L_h(z, r) = \frac{M_h(z, r)}{m_h(z, r)}, \quad L_h(z) = \limsup_{r \to 0} L_h(z, r).$$

The latter quantity is called the *dilatation* of h at z. We say that a map h has an *L-bounded circular dilatation* if $L_h(z) \leq L$ for *all* $z \in U$. (Notice that L is independent of the particular choice of the Riemannian metric.)

PROPOSITION 2.6. *Any K-qc map has an L-bounded circular dilatation where $L = L(K)$.*

PROOF. Of course, we can assume that U and V are contained in the complex plane endowed with the Euclidean metric. For notational convenience, let us normalize h so that $z = h(z) = 0$, and let $m(r) = m_h(0, r)$, $M(r) = M_h(0, r)$. Let a and b be two points on the circle $\mathbb{T}_r$ for which $|h(a)| = m(r)$ and $|h(b)| = M(r)$. Assume that r is so small that the disk $\mathbb{D}_{M(r)}$ is contained in V. Then let us consider an annulus $A' = \mathbb{A}(m(r), M(r))$ and let $A = h^{-1}(A')$. The inner component of $\mathbb{C} \smallsetminus A$ contains points 0 and $a \in \mathbb{T}_r$, while its outer component of $\mathbb{C} \smallsetminus A$ contains $b \in \mathbb{T}_r$. By Lemma 1.26, $\operatorname{mod}A$ is bounded by an absolute constant C. By Lemma 2.5,

$$\frac{1}{2\pi} \log \frac{M(r)}{m(r)} = \operatorname{mod}A' \leq K\operatorname{mod}A \leq KC,$$

and we are done. ☐

Let X and Y be two metric spaces, and let $\eta : \mathbb{R}_+ \to \mathbb{R}_+$. A homeomorphism $h : X \to Y$ is called η-quasi-symmetric ("η-qs") if for any $x, y, z \in X$ we have:

$$\frac{\text{dist}(h(z), h(x))}{\text{dist}(h(y), h(x))} \leq \eta \left(\frac{\text{dist}(z, x)}{\text{dist}(y, x)} \right).$$

EXERCISE 2.4. *Assume that X and Y are compact Riemannian manifolds. Then $h : X \to Y$ is η-qs if and only if there exists a constant $\kappa \geq 1$ such that for any $x, y, z \in X$ with $\text{dist}(z, x) = \text{dist}(y, x)$ we have:*

$$\frac{\text{dist}(h(z), h(x))}{\text{dist}(h(y), h(x))} \leq \kappa. \tag{8.1}$$

Homeomorphisms satisfying (8.1) will also be called κ-qs (we trust that this slight terminological inconsistency will not cause confusion).

PROPOSITION 2.7. *Any K-qc homeomorphism $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ fixing $0, 1$ and ∞ is κ-qs, where κ depends only on K.*

EXERCISE 2.5. *Without normalization, the above proposition would fail.*

8.3. Back to the analytic definition.

PROPOSITION 2.8. *If a homeomorphism $h : U \to V$ between domains U and V has an L-bounded circular dilatation then it is L-qc.*

PROOF. Since the L-bounded circular dilatation implies the L-bounded infinitesimal dilatation at any point of differentiability, all we need to show is that h has the required regularity, i.e., it is absolutely continuous on almost all parallel lines. Since this is a local property, we can assume that U us the unit square, and that the parallel lines in question are horizontal.

Let $U_b = \{z \in U : \text{Im } z \leq b\}$. Since the area function

$$\mu : b \mapsto \text{area}(h(U_b))$$

is monotonic, it is differentiable for a.e. b. Let us take such a point b where μ is differentiable, and prove absolute continuity of h on the corresponding line $\gamma_b = \{z : \text{Im } z = b\}$.

For $K \in \mathbb{N}$, let $X_K = \{z \in \gamma_b : L_h(x, \epsilon) \leq K/2 \text{ for } \epsilon \leq 1/\text{K}\}$. Since the dilatation of h is bounded[1], we have: $\bigcup X_K = \gamma_b$. Hence it is enough to prove that $h| X_K$ is absolutely continuous.

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[1]For the regularity purposes, it is sufficient to assume that the circular dilatation is finite everywhere.

Let $Q \subset X_K$ be a set of zero length. We want to show that $h(Q)$ has zero length as well. By approximation, it is sufficinet to show for closed sets. Then Q can be covered with finitely many disks $D_i = \mathbb{D}(z_i, \epsilon)$ ($z_i \in \gamma_b$, $i = 1, \ldots, n$) with intersection multiplicity at most 2 and an arbitrary small total length. Hence for any $\delta > 0$, we have $n\epsilon \leq \delta$ once ϵ is sufficiently small.

Let $M_i = M_h(z_i, \epsilon)$ and $m_i = m_h(z_i, \epsilon)$. Then $M_i \leq km_i$, $l(hX) \leq \sum M_i$, and by the Cauchy-Bunyakovsky inequality,

$$l(hX)^2 \leq n \sum M_i^2 \leq nK^2 \sum m_i^2 \leq \frac{K^2 \delta}{\pi} \cdot \frac{\text{area}(h \cup D_i))}{\epsilon}.$$

But the last ratio is bounded by

$$\frac{\mu(b+\epsilon) - \mu(b-\epsilon)}{\epsilon} \to \frac{1}{2} \mu'(b) \quad \text{as } \epsilon \to 0,$$

and the desired conclusion follows.   □

9. Further important properties of qc maps

9.1. Weyl's Lemma. This lemma asserts that a 1-qc map is conformal. In other words, if a qc map is infiniesimally conformal on the set of full measure (i.e., $\bar{\partial} h(z) = 0$ a.e.), then it is conformal in the classical set. Since $\bar{\partial} h(z) = 0$ is just the Cauchy-Riemann equation, this statement is classical for smooth maps.

Let us formulate a more general version of Weyl's Lemma:

LEMMA 2.9 (Weyl). *Assume that a continuous function $h : U \to \mathbb{C}$ has distributional derivatives of class L^1_{loc}. If $\bar{\partial} h(z) = 0$ a.e., then h is holomorphic.*

PROOF. By approximation, Weyl's Lemma can be reduced to the classical statement. Since the statement is local, we can assume without loss of generality that the partial derivatives of h belong to $L^1(U)$. Convoluting h with smooth bump-functions we obtain a sequence of smooth functions $h_n = h * \theta_n$ converging to h uniformly on U with derivatives converging in $L^1(U)$. Let us show that $\bar{\partial} h_n = 0$. For a test function η on U, we have:

$$\int \bar{\partial} h_n(z)\, \eta(z)\, dm(z) = - \int h_n(z)\, \bar{\partial}\eta(z)\, dm(z)$$

$$= - \int h(\zeta)\, dm(\zeta) \int \theta_n(z - \zeta) \bar{\partial}\eta(z)\, dm(z)$$

$$= \int h(\zeta)\, dm(\zeta) \int \bar{\partial}\theta_n(z - \zeta) \eta(z)\, dm(z)$$

$$= \int \eta(z)\, dm(z) \int h(\zeta)\, \bar{\partial}\theta_n(z - \zeta)\, dm(\zeta)$$

$$= \int \eta(z)\, dm(z) \int \bar{\partial}h(\zeta)\, \theta_n(z - \zeta)\, dm(\zeta) = 0$$

Here the first and the third equalities are the classical integration by parts, the next to the last one comes from the definition of the distributional derivative, and the intermediate ones come from the Fubini Theorem.

It follows that the smooth functions h_n satisfy the Cauchy-Riemann equations and hence holomorphic. Since uniform limits of holomorphic functions are holomorphic, h is holomorphic as well. $\square$

9.2. Devil Staircase. The following example shows that Weyl's Lemma is not valid for homeomorphisms of class DH (i.e., differentiable a.e.). The technical assumption that the classical derivative can be understood in the sense of distributions (which allows us to integrate by parts) is thus crucial for the statement.

Take the standard Cantor set $K \subset [0,1]$ and construct a devil staircase $h : [0,1] \to [0,1]$, i.e., a continuous monotone function which is constant on the complementary gaps to K.

EXERCISE 2.3. *Do the construction. (Topologically it amounts to showing that by collapsing the gaps to points we obtain a space homeomorphic to the interval.)*

Consider a strip $S = [0,1] \times \mathbb{R}$ and let $f : (x,y) \mapsto (x, y + h(x))$. This is a homeomorphism on S which is a rigid translation on every strip $G \times R$ over a gap $G \subset [0,1] \smallsetminus K$. Since $m(K \times \mathbb{R}) = 0$, this map is conformal a.e. However it is obviously not conformal on the whole strip P.

Clearly f in not absolutely continuous on the horizontal lines: it translates them to devil staircases.

9.3. Quasiconformal Removability and Gluing. A closed set $K \subset \mathbb{C}$ is called *qc removable* if any homeomorphism $h : U \to \mathbb{C}$ defined on an neighborhood U of K, which is quasiconformal on $U \smallsetminus K$, is quasiconformal on U.

Remark. We will see later on (§**??**) that qc removable sets have zero measure and hence $\mathrm{Dil}(f|U) = \mathrm{Dil}(f|U \smallsetminus K)$.

EXERCISE 2.4. *Show that isolated points are removable.*

PROPOSITION 2.10. *Smooth Jordan arcs are removable.*

PROOF. Let us consider a smooth Jordan arc $\Gamma \subset U$ and a homeomorphism $f : U \to \mathbb{C}$ which is quasi-conformal on $U \smallsetminus \Gamma$. We should check that f is absolutely continuous on lines near any point $z \in \Gamma$. Take a small box B centered at z whose sides are not parallel to $T_z\Gamma$. Then any interval l in B parallel to one of its sides intersects Γ at a sinle point ζ. Since for a typical l, f is absolutely continuous on the both sides of $l \smallsetminus \{\zeta\}$, it is absolutely continuous on the whole interval l as well.

Moreover, $\mathrm{Dil}(f)$ is obviously bounded since it is so on $U \smallsetminus \Gamma$ and Γ has zero measure.   □

The above statement is simple but important for holomorphic dynamics. It will allow us to construct global qc homeomorphisms by gluing together different pieces without spoiling dilatation.

Let us now state a more delicate gluing property:

LEMMA 2.11 (Bers). *Consider a closed set $K \subset \bar{\mathbb{C}}$ and two its neighborhoods U and V. Assume that we have two quasi-conformal maps $f : U \smallsetminus K \to \bar{\mathbb{C}}$ and $g : V \to \bar{\mathbb{C}}$ that match on ∂K, i.e., the map*

$$h(z) = \begin{cases} f(z), & z \in U \smallsetminus K \\ g(z), & z \in K \end{cases}$$

is continuous. Then h is quasi-conformal and $\mu_h(z) = \mu_g(z)$ for a.e. $z \in K$.

PROOF. Consider a map $\phi = f^{-1} \circ h$. It is well-defined in a neighborhood Ω of K, is identity on K and is quasi-conformal on $\Omega \smallsetminus K$. Let us show that it is quasi-conformal on Ω. Again, the main difficulty is to show that h is absoultely continuous on lines near any point $z \in K$.

Take a little box near some point $z \in K$ with sides parallel to the coordinate axes. Without loss of generality we can assume that $z \neq \infty$ and ϕB is a bounded subset of $\mathbb{C}$. Let ψ denote the extension of $\partial\phi/\partial x$ from $B \smallsetminus K$ onto the whole box B by 0. By (7.6), ψ is square integrable on B and hence it is square integrable on almost all horizontal sections of B. All the more, it is integrable on almost all horizontal sections. Take such a section I, and let us show that ϕ is absolutely continuous on it.

Let $I_j \subset I$ be a finite set of disjoint intervals; $\Delta\phi_j$ denote the increment of ϕ on I_j. We should show that

$$\sum |\Delta\phi_j| \to 0 \quad \text{as} \quad \sum |I|_j \to 0. \tag{9.1}$$

Take one interval I_j and decompose it as $L \cup J \cup R$ where $\partial J \subset K$ and int L and int R belong to $B \setminus K$. Then

$$|\Delta \phi_j| \leq |J| + \int_{L \cup R} g \, dx \leq |I_j| + \int_{I_j} g \, dx.$$

Summing up the last estimates over j and using integrability of g on I_j, we obtain (9.1).

Absolute continuity on the vertial lines is treated in exactly the same way. $\square$

9.4. Weak topology in L^2. Before going further, let us briefly recall some background in functional analysis. Consider the space $L^2 = L^2(X)$ on some measure space (X, m). A sequence of functions $h_n \in L^2$ weakly converges to some function $h \in L^2$, $h_n \underset{w}{\to} h$, if for any $\phi \in L^2$, $\int h_n \phi \, dm \to \int h \phi \, dm$. The main advantage of this topology is the property that the balls of L^2 are weakly compact (see e.g., [**?**,]). Note also that vice versa, any weakly convergent sequence belongs to some ball in L^2 (Banach-Schteinhaus [**?**,]).

However, one should handle the weak topology with caution: for instance, product is not a weakly continuous operation:

EXERCISE 2.6. *Show that* $\sin nx \underset{w}{\to} 0$ *in* $L^2[0, 2\pi]$, *while* $\sin^2 nx \underset{w}{\to} 1/2$.

At least, the weak topology respects the order:

EXERCISE 2.7. *Let* $h_n \underset{w}{\to} h$.

- *If* $h_n \geq 0$ *then* $h \geq 0$;
- *If* $h_n = 0$ *a.e. on some subset* $Y \subset X$, *then* $h = 0$ *a.e. on* Y;
- *After selecting a further subsequence,*

$$(h_n)_+ \underset{w}{\to} h_+ \ \text{and} \ (h_n)_- \underset{w}{\to} h_-, \quad \text{so that} \ |h_n| \underset{w}{\to} |h|.$$

Here $h_+(z) = \max(h(z), 0)$, $h_(z) = \min(h(z), 0)$.

9.5. Compactness. We will proceed with the following fundamental property of qc maps:

THEOREM 2.12. *The space of K-qc maps $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ fixing $0, 1$ and ∞ is compact in the topology of uniform convergence on $\bar{\mathbb{C}}$*

PROOF. It will be more convenient to consider the space $\mathcal{X}$ of K-qc maps h such that $h\{0, 1, \infty\} = \{0, 1, \infty\}$. First, we will show that the family of maps $h \in \mathcal{X}$ is equicontinuous. Otherwise we would have an $\epsilon > 0$, a sequence of maps $h_n \in \mathcal{X}$, and a sequence of points $z_n, \zeta_n \in \bar{\mathbb{C}}$

such that such that $d(z_n, \zeta_n) \to 0$ while $d(h_n z_n, h_n \zeta_n) \geq \epsilon$, where d stands for the sperical metric. By compactness of $\bar{\mathbb{C}}$, we can assume that the $z_n, \zeta_n \in \bar{\mathbb{C}}$ converge to some point a and the $h_n(a)$ converge to some b. Postcomposing or/and precomposing if necessary the maps h_n's with $z \mapsto 1/z$, we can make $|a| \leq 1$, $|b| \leq 1$.

Consider a sequence of annuli $A_n = \{z : r_n < |z - a| < 1/2\}$ where $r_n = \max(|z_n - a|, |\zeta_n - a|) \to 0$. Since the disk $\mathbb{D}(a, 1/2)$ does not contain one of the points 0 or 1, its images $h_n(\mathbb{D}(a, 1/2))$ have the same property. Hence the Euclidean distance from the point $h_n(a)$ (belonging to the inner complement of $h_n(A_n)$) to the outer complement of that annulus is eventually bounded by 3. On the other hand, the diameter of the inner complement of $h_n(A_n)$ is bounded from below by $\epsilon > 0$. By Lemma 1.26, $\mod(h_n(A_n))$ is bounded from above. But $\mod(A_n) = 1/r_n \to 0$ contradicting quasi-invariance of the modulus (Proposition 2.5).

Hence $\mathcal{X}$ is precompact in the space of continuous maps $\bar{\mathbb{C}} \to \bar{\mathbb{C}}$. Since $\mathcal{X}$ is invariant under taking the inverse $h \mapsto h^{-1}$, and the composition is a continuous operation in the uniform topology, $\mathcal{X}$ is precompact in $\mathrm{Homeo}(\bar{\mathbb{C}})$. Since $\mathrm{Homeo}^+(\bar{\mathbb{C}})$ is closed in $\mathrm{Homeo}(\bar{\mathbb{C}})$, $\mathcal{X}$ is precompact in the former space as well.

To complete the proof, we should show that the limit functions are also K-qc homeomorphisms. Let a sequence $h_n \in \mathcal{X}$ uniformly converges to some h. Given a point $a \in \bar{\mathbb{C}}$, we will show that in some neighborhood of a, f has distributional derivatives of class L^2. Without loss of generality we can assume that $a \in \mathbb{C}$. Take a neighborhood $B \ni a$ such that $h(B)$ is a bounded subset of $\mathbb{C}$. Then the neighborhoods $h_n(B)$ are eventually uniformly bounded. By (7.6), the partial derivatives ∂h_n and $\bar{\partial} h_n$ eventually belong to a fixed ball of $L^2(D)$. Hence they form weakly precompact sequences, and we can select limits along subsequences (without changing notations):

$$\partial h_n \underset{w}{\to} \phi \in L^2(D); \qquad \bar{\partial} h_n \underset{w}{\to} \psi \in L^2(D).$$

It is straightforward to show that ϕ and ψ are the distributional partial derivatives of h. Indeed, for any test functions η we have:

$$\int h \, \partial \eta \, dm = \lim \int h_n \, \partial \eta \, dm = -\lim \int \partial h_n \, \eta \, dm = -\int \phi \, \eta \, dm, \tag{9.2}$$

and the similarly for the $\bar{\partial}$-derivative.

What is left is to show that $|\phi(z)| \leq k|\psi(z)|$ for a.e. z, where $k = (K-1)/(K+1)$. To see this, select a further subsequence in such

a way that $|\partial h_n| \underset{w}{\to} |\phi|$, $\quad |\bar{\partial} h_n| \underset{w}{\to} |\psi|$ and use the fact that the weak topology respects the order (see Exercise 2.7). $\qquad\square$

EXERCISE 2.8. *Fix any three points a_1, a_2, a_3 on the sphere $\mathbb{C}$. A family $\mathcal{X}$ of K-qc maps $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is precompact in the space of all K-qc homeomorphisms of the sphere (in the uniform topology) if and only if the reference points are not moved close to each other (or, in formal words: there exists a $\delta > 0$ such that $d(ha_i, ha_j) \geq \delta$ for any $h \in \mathcal{X}$ and $i \neq j$, where d is the spherical metric). Consider first the case $K = 0$.*

We will also need a disk version of the above Compactness Theorem:

COROLLARY 2.13. *The space of K-qc homeomorphisms $f : \mathbb{D} \to \mathbb{D}$ fixing 0 is compact in the topology of uniform convergence on $\mathbb{D}$.*

PROOF. Let $\mathcal{Y}$ be the space of K-qc homeomorphisms $h : \mathbb{D} \to \mathbb{D}$ fixing 0, and $\mathcal{X}$ be the space of $\mathbb{T}$-symmetric K-qc homeomorphisms $H : \mathbb{C} \to \mathbb{C}$ fixing 0 and ∞. (To be $\mathbb{T}$-symmetric means to commute with the involution $\tau : \mathbb{C} \to \mathbb{C}$ with respect to the circle.) Clearly maps $H \in \mathcal{X}$ preserve the unit circle (the set of fixed points of τ); in particular, they do not move 1 close to 0 and ∞. By Theorem 2.12 (and the Exercise following it), $\mathcal{X}$ is compact.

Let us show that $\mathcal{X}$ and $\mathcal{Y}$ are homeomorphic. The restriction of a map $H \in \mathcal{X}$ to the unit disk gives a continuous map $i : \mathcal{X} \to \mathcal{Y}$. The inverse map $i^{-1} : \mathcal{Y} \to \mathcal{X}$ is given by the following extension procedure. First, extend $h \in \mathcal{Y}$ continuously to the closed disk $\bar{\mathbb{D}}$ (Theorem **??**), and then reflect it symmetrically to the exterior of the disk, i.e., let $H(z) = \tau \circ h \circ \tau(z)$ for $z \in \bar{\mathbb{C}} \smallsetminus \bar{\mathbb{D}}$. Since τ is an (orientation reversing) conformal map, H is K-qc on $\bar{\mathbb{C}} \smallsetminus \mathbb{T}$. By Lemma 2.10, it is K-qc everywhere, and hence belongs to $\mathcal{X}$.

Hence $\mathcal{Y}$ is compact as well. $\qquad\square$

10. Measurable Riemann Mapping Theorem

We are now ready to prove one of the most remarkable facts of analysis: any measurable conformal structure with bounded dilatation is generated by a quasi-conformal map:

THEOREM 2.14 (Measurable Riemann Mapping Theorem). *Let μ be a measurable Beltrami differential on $\bar{\mathbb{C}}$ with $\|\mu\|_\infty < 1$. Then there is a quasi-conformal map $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ which solves the Beltrami equation: $\bar{\partial} h / \partial h = \mu$. This solution is unique up to post-composition with a Möbius automorphism of $\bar{\mathbb{C}}$. In particular, there is a unique solution fixing three points on $\bar{\mathbb{C}}$ (say, $0, 1$ and ∞).*

The local version of this result sounds as follows:

THEOREM 2.15 (Local integrability). *Let μ be a measurable Beltrami differential on a domain $U \subset \mathbb{C}$ with $\|\mu\|_\infty < 1$. Then there is a quasi-conformal map $h : U \to \mathbb{C}$ which solves the Beltrami equation: $\bar\partial h / \partial h = \mu$. This solution is unique up to post-composition with a conformal map.*

The rest of this section will be occupied with a proof of these two theorems.

10.1. Uniqueness. Uniqueness part in the above theorems is a consequence of Weyl's Lemma. Indeed, if we have two solutions h and g, then the composition $\psi = g \circ h^{-1}$ is a qc map with $\bar\partial \psi = 0$ a.e. on its domain. Hence it is conformal.

10.2. Local vs global. Of course, the global Riemann Measurable Riemann Theorem immediately yields the local integrability (e.g., by zero extantion of μ from U to the whole sphere). Vice versa, the global result follows from the local one and the classical Uniformization Theorem for the sphere . Indeed, by local integrability, there is a finite covering of the sphere $S^2 \equiv \bar{\mathbb{C}}$ by domains U_i and a family of qc maps $\phi_i : U_i \to \mathbb{C}$ solving the Beltrami equation on U_i. By Weyl's Lemma, the gluing maps $\phi_i \circ \phi_j^{-1}$ are conformal. Thus the family of maps $\{\phi_i\}$ can be interpreted as a complex analytic atlas on S^2, which endows it with a new complex analytic structure m (compatible with the original qc structure). But by the Uniformization Theorem, all complex analytic structures on S^2 are equivalent, so that there exists a biholomorphic isomorphism $h : (S^2, m) \to \bar{\mathbb{C}}$. It means that the maps $h \circ \phi_i^{-1}$ are conformal on $\phi_i U_i$. Hence h is quasi-conformal on each U_i and $h_*(\mu) = (h \circ \phi_i^{-1})_* \sigma$ over there. Since the atlas is finite, h is a global quasi-conformal solution of the Beltrami equation.

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10.3. Strategy. The further strategy of the proof will be the following. First, we will solve the Beltrami equation locally assuming that the coefficient μ is real analytic. It is a classical (and elementary) piece of the PDE theory. By the Uniformization Theorem, it yields a global solution in the real analytic case. Approximating a measurable Beltrami coefficient by real analytic ones and using compactness of the space of normalized K-qc maps, we will complete the proof.

10.4. Real analytic case. Assume that μ is a real analytic Beltrami coefficient in a neighborhood of 0 in $\mathbb{R}^2 \equiv \mathbb{C}_\mathbb{R}$ with $|\mu(0)| < 1$. Then it admits a complex analytic extension to a neighborhood of 0 in the complexification $\mathbb{C}^2$. Let (x, y) be the standard coordinates in $\mathbb{C}^2$,

and let $u = x + iy, \; v = x - iy$. In these coordinates the complexified Beltrami equation assumes the form:

$$\frac{\partial h}{\partial v} - \mu(u, v)\frac{\partial h}{\partial u} = 0. \tag{10.1}$$

This is a linear equation with variable coefficients, which can be solved by the standard method of characteristics. Namely, let us consider a vector field $W(u, v) = (1, -\mu(u, v))$ near 0 in $\mathbb{C}^2$. Since the left-hand side of (10.1) is the derivative of h along X, we come to the equation $Wh = 0$. Solutions of this equation are the first integrals of the ODE $\dot{w} = W$. But since W is non-singular at 0, this ODE has a non-singular local first integral $h(u, v)$. Restricting h to $\mathbb{R}^2$, we obtain a local solution $h : (\mathbb{R}^2, 0) \to \mathbb{C}$ of the original Beltrami equation. Since h is non-singular at 0, it is a local (real analytic) diffeomorphism.

By means of the Uniformization Theorem, we can now pass from local to global solutions of the Beltrami equation with a real analytic Beltrami differential $\mu(z)d\bar{z}/dz$ on the sphere (see §10.2). Note that the global solution is real analytic as well since the complex structure generated by the local solutions is compatible with the original real analytic structure of the sphere (as local solutions are real analytic).

EXERCISE 2.9. *For a real analytic Beltrami coefficient*

$$\mu(z) = \sum a_{n,m} z^n \bar{z}^m$$

on $\mathbb{C}$, find the condition of its real analyticity at ∞.

There is also a "semi-local" version of this result:

If μ is a real analytic Beltrami differential on the disk $\mathbb{D}$ with $\|\mu\|_\infty < 1$, then there is a quasi-conformal (real analytic) diffeomorphism $h : \mathbb{D} \to \mathbb{D}$ solving the Beltrami equation $\bar{\partial}h/\partial h = \mu$.

To see it, consider the complex structure m on the disk generated by the local solutions of the Beltrami equation. We obtain a simply connected Riemann surface $S = (\mathbb{D}, m)$. By the Uniformization Theorem, it is conformally equivalent to either the standard disk $(\mathbb{D}, \sigma)$ or to the complex place $\mathbb{C}$. But S is quasi-conformally equivalent to the standard disk via the identical map $\mathrm{id} : (\mathbb{D}, m) \to (\mathbb{D}, \sigma)$. By Exercise 2.3, it is then conformally equivalent to the standard disk, and this equivalence $h : (\mathbb{D}, m) \to (\mathbb{D}, \sigma)$ provides a desired solution of the Beltrami equation.

By §10.1 Such a solution is unique up to a postcomposition with a Möbius automorphism of the disk.

10.5. Approximation. Let us consider an arbitrary measurable Beltrami coefficient μ on a disk $\mathbb{D}$ with $\|\mu\| < \infty$. Select a sequence

of real analytic Beltrami coefficients μ_n on $\mathbb{D}$ with $\|\mu_n\|_\infty \leq k < 1$, converging to μ a.e.

EXERCISE 2.10. *Construct such a sequence (first approximate μ with continuous Beltrami coefficients).*

Applying the results of the previous section, we find a sequence of quasi-conformal maps $h_n : (\mathbb{D}, 0) \to (\mathbb{D}, 0)$ solving the Beltrami equations $\bar{\partial} h_n / \partial h_n = \mu_n$. The dilatation of these maps is bounded by $K = (1 + k)/(1 - k)$. By Corollary 2.13, they form a precompact sequence in the topology of uniform convergence on the disk. Any limit map $h : \mathbb{D} \to \mathbb{D}$ of this sequence is a quasi-conformal homeomorphism of $\mathbb{D}$. Let us show that its Beltrami coefficient is equal to μ.

By (7.6), the partial derivatives of the h_n belong to some ball of the Hilbert space $L^2(\mathbb{D})$. Hence we can select weakly convergent subsequences $\partial h_n \to \phi$, $\bar{\partial} h_n \to \psi$. We have checked in (9.2) that $\phi = \partial h$ and $\psi = \bar{\partial} h$. What is left is to check that $\psi = \mu \phi$. To this end, it is enough to show that $\mu_n \, \partial h_n \to \mu \phi$ weakly (to appreciate it, recall that the product is not weakly continuous, see Exercise 2.6). For any test function $\eta \in L^2(\mathbb{D})$, we have:

$$\left| \int (\eta \mu \phi - eta \mu_n \, \partial h_n) \, dm \right| \leq$$

$$\leq \left| \int \eta \mu (\phi - \partial h_n) \, dm \right| + \int |\eta(\mu - \mu_n) \, \partial h_n| \, dm.$$

The first term in the last line goes to 0 since the ∂h_n weakly converge to ϕ. The second term is estimated by the Cauchy-Schwarz inequality by $\|\eta(\mu - \mu_n)\|_2 \|\partial h_n\|_2$, which goes to 0 since $\mu_n \to \mu$ a.e. and the ∂h_n belong to some Hilbert ball. This yields the desired.

It proves the Measurable Riemann Mapping Theorem on the disk $\mathbb{D}$, which certainly implies the local integrability. Now the global theorem on the sphere follows from the local integrability by §10.2. This completes the proof.

10.6. Conformal and complex structures. Let us discuss the general relation between the notions of complex and conformal structures. Consider an oriented surface S endowed with a *qs structure*, i.e., supplied with an atlas of local charts $\psi_i : V_i \to \mathbb{C}$ with uniformly qc transit maps $\psi_i \circ \psi_j^{-1}$ ("uniformly qc" means "with uniformly bounded dilatation"). Note that a notion of a measurable conformal structure with bounded dilatation makes perfect sense on such a surface (in what follows we call it just a "conformal structure").

Endow S with a complex structure compatible with its qs structure. By definition, it is determined by an atlas $\phi_i : U_i \to \mathbb{C}$ on S of uniformly qc maps such that the transit maps are complex analytic. Then the conformal structures $\mu_i = \phi_i^*(\sigma)$ on U_i coincide on the intersections of the local charts and have uniformly bounded dilatations. Hence they glue into a global conformal structure on S.

Vice versa, any conformal structure μ determines by the Local Integrability Theorem a new complex structure on the surface S compatible with its qc structure (see §10.2).

Thus the notions of conformal and complex structures on a qc surface are equivalent. In what follows we will not distinguish them either conceptually or notationally.

Fixing a reference complex structure on S (so that S becomes a Riemann surface), complex/conformal structures on S get parametrized by measurable Beltrami differentials μ on S with $\|\mu\|_\infty < 1$.

10.7. Moduli spaces. Consider some qc surface S (with or without boundary, possibly marked or partially marked).

The *moduli space* $\mathcal{M}(S)$, or the *deformation space of S* is the space of all conformal structures on S compatible with the underlying qc structure, up to the action of qc homeomorphisms perserving the marked data. In other words, $\mathcal{M}(S)$ is the space of all Riemann surfaces qc equivalent to S, up to conformal equivalence relation (respecting the marked data).

If we fix a reference Riemann surface S_0, then its deformations are represented by qc homeomorphisms $h : S_0 \to S$ to various Riemann surfaces S. Two such homeomorphisms h and $\tilde{h}$ represent the same point of the moduli space if there exists a conformal isomorphism $A : S \to \tilde{S}$ such that the composition $H = \tilde{h}^{-1} \circ A \circ h : S_0 \to S_0$ respects all the marked data. In particular, $H = \text{id}$ on the marked boundary. In the case when the whole fundamental group is marked, H must be homotopic to the id relative to the marked boundary.

For instance, if S has a finite conformal type, i.e., S is a Riemann surface of genus g with n punctures (without marking), then $\mathcal{M}(S)$ is the classical moduli space $M^{g,n}$. If S is fully marked then $\mathcal{M}(S)$ is the classical Teichmüller space $T^{g,n}$. This space has a natural complex structure of complex dimension $3g - 3 + n$ for $g > 1$. For $g = 1$ (the torus case), $\dim T^{1,0} = 1$ (see §1.4.2) and $\dim T^{1,n} = n - 1$ for $n \geq 1$. For $g = 0$ (the sphere case), $\dim T^{0,n} = 0$ for $n \leq 3$ (by the Riemann-Koebe Uniformization Theorem and 3-transitivity of the Möbius group action) and $\dim T^{0,n} = n - 3$ for $n > 3$.

EXERCISE 2.11. *What is the complex modulus of the four punctured sphere?*

There is a natural projection (fogetting the marking) from $T^{g,n}$ onto $M^{g,n}$. The fibers of this projection are the orbits of the so called "Teichmüller modular group" acting on $T^{g,n}$ (it generalizes the classical modular group $\mathrm{PSL}(2,\mathbb{Z})$, see §1.4.2).

By the Riemann Mapping Theorem, the disk $\mathbb{D}$ does not have moduli. However, if we mark its boundary $\mathbb{T}$, then the space of moduli, $\mathcal{M}(\mathbb{D}, \mathbb{T})$, becomes infinitely dimensional! By definition, $\mathcal{M}(\mathbb{D}, \mathbb{T})$ is the space of all Beltrami differentials μ on $\mathbb{D}$ up to the action of the group of qc homeomorphisms $h : \mathbb{D} \to \mathbb{D}$ whose boundary restrictions are Möbius: $h|\mathbb{T} \in \mathrm{PSL}(2,\mathbb{R})$. It is called the *universal Teichmüller space*, since it contains all other deformation spaces. This space has several nice descriptions, which will be discussed later on. It plays an important role in holomorphic dynamics.

10.8. Dependence on parameters. It is important to know how the solution of the Beltrami equation depends on the Beltrami differential. It turns out that this dependence is very nice. Below we will formulate three statements of this kind (on continuous, smooth and holomorphic dependence).

PROPOSITION 2.16. *Let μ_n be a sequence of Beltrami differentials on $\mathbb{C}$ with uniformly bounded dilatation, converging a.e. to a differential μ. Consider qc solutions $h_n : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ and $h : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ of the corresponding Beltrami equations fixing $0, 1$ and ∞. Then the h_n converge to h uniformly on $\mathbb{C}$.*

PROOF. By Theorem 9.5, the sequence h_n is precompact. Take any limit map g of this sequence. By the argument of §10.5, its Beltrami differential is equal to μ. By uniqueness of the normalized solution of the Beltrami equation, $g = h$. The conclusion follows.   □

Consider a family of Beltrami differentials μ_t depending on parameters $t = (t_1, \ldots, t_n)$ ranging over a domain $U \subset \mathbb{R}^n$. This family is said to be *differentiable* at some $t \in U$ if there exist Beltrami differentials α_t^i of class $L^\infty(\mathbb{C})$ (but not necessarily in the unit ball of this space) such that for all sufficiently small $\epsilon = (\epsilon_1, \ldots, \epsilon_n) \in \mathbb{R}^n$, we have:

$$\mu_{t+\epsilon} - \mu_t = \sum_{i=1}^n \alpha_t^i \epsilon_i + |\epsilon| \, \beta(t, \epsilon),$$

where the norm $\|\beta_{t,\epsilon}\|_\infty$ stays bounded and $\beta_{t,\epsilon}(z) \to 0$ a.e. on $\mathbb{C}$ as $\epsilon \to 0$.

Assume additionally that the family μ_t is differentiable at all points $t \in U$, that the norms $\|\alpha_t^i\|$ are locally bounded, and that the $\alpha_t^i(z)$ continuously depend on t in the sense of the convergence a.e. Then the family μ_t is said to be *smooth*.

Let us now consider a family of qc maps $h_t : \mathbb{C} \to \mathbb{C}$ depending on parameters $t \in U$. Considering these maps as elements of the Sobolev space H, we can define differentiabilty and smoothness in the usual way. This family is differentiable at some point $t \in U$ if there exist vector fields v_t^i on $\mathbb{C}$ of Sobolev class H such that

$$h_{t+\epsilon} - h_t = \sum_{i=1}^n \epsilon_i v_t^i + |\epsilon| g_{t,\epsilon},$$

where $g_{t,\epsilon} \to 0$ in the Sobolev norm as $\epsilon \to 0$ (in particular $g_{t,\epsilon} \to 0$ uniformly on the sphere). If additionally the v_t^i depend continuously on t (as elements of H), then one says that h_t smoothly depends on t. Of course, in this case, any point $z \in \mathbb{C}$ smoothly moves as parameter t changes, i.e., $h_t(z)$ smoothly depends on t.

THEOREM 2.17. *If μ_t, $t \in U \subset \mathbb{R}^n$, is a smooth family of Beltrami differentials, then the normalized solutions $h_t : \mathbb{C} \to \mathbb{C}$ of the corresponding Beltrami equations smoothly depend on t.*

Let us finally discuss the holomorphic dependence on parameters. Let U be a domain in $\mathbb{C}^n$ and let $\mathcal{B}$ be a complex Banach space. A function $f : U \to \mathcal{B}$ is called *holomorphic* if for any linear functional $\phi \in \mathcal{B}^*$, the function $\phi \circ f : U \to \mathbb{C}$ is holomorphic. Beltrami differentials are elements of the complex Banach space L^∞, while qc maps $h : \mathbb{C} \to \mathbb{C}$ are elements of the complex Sobolev space H. So, it be careful here! makes sense to talk about holomorphic dependence of these objects on complex parameters $t = (t_1, \ldots, t_n) \in U$. Note that if h_t depends holomorphically on t, then any point $z \in \mathbb{C}$ moves holomorphically as t changes (in fact, holomorphic dependence on parameters is often understood in this weak sense).

THEOREM 2.18. *If the Beltrami differential μ_t holomorphically depends on parameters $t \in U$, then so do the normalized solutions $h_t : \mathbb{C} \to \mathbb{C}$ of the corresponding Beltrami equations.*

The proofs of the last two theorems can be found in [**AB**].

10.8.1. *Simple conditions.*

LEMMA 2.19. *Let $\mathcal{B}$ be a Banach space, and let $\{f_\lambda\}$, $\lambda \in \mathbb{D}_\rho$, be a uniformly bounded family of linear functionals on $\mathcal{B}$ such that for a dense linear subspace L of points $x \in \mathcal{B}$, the function $\lambda \mapsto f_\lambda(x)$*

is holomorphic in λ. Then $\{f_\lambda\}$ as an element of the dual space $\mathcal{B}^*$ depends holomorphically on λ.

PROOF. For $x \in L$, we have a power series expansion

$$f_\lambda(x) = \sum a_n(x)\lambda^n$$

convergent in $\mathbb{D}_\rho$. By the Cauchy estimate,

$$|a_n(x)| \leq \frac{C\|x\|}{\rho^n}, \quad x \in L,$$

where C is an upper bound for the norms $\|f_\lambda\|$, $\lambda \in \mathbb{D}_\rho$. Clearly, the $a_n(x)$ linearly depend on $x \in L$. Hence, a_n are bounded linear functionals on L; hence they admit an extension to bounded linear functionals on $\mathcal{B}$. Moreover, $\|a_n\| \leq C\rho^{-n}$. It follows that the power series $\sum a_n \lambda^n$ converges in the dual space $\mathcal{B}^*$ uniformly in λ over any disk $\mathbb{D}_r$, $r < \rho$. Hence it represents a holomorphic function $D_\rho \mapsto \mathcal{B}^*$, which, of course, coincides with $\lambda \mapsto f_\lambda$. $\square$

For further applications, let us formulate one simple condition of holomorphic dependence:

LEMMA 2.20. *Let $\rho > 0$ and let $U \subset \mathbb{C}$ be an open subset in $\bar{\mathbb{C}}$ of full measure. Let $\mu_\lambda \in L^\infty(\mathbb{C})$, $\lambda \in \mathbb{D}_\rho$, be a family of Beltrami differentials with $\|\mu_\lambda\|_\infty \leq 1$ whose restriction to U is smooth in both variables (λ, z) and is holomorphic in λ. Then $\{\mu_\lambda\}$ is a holomorphic family of Beltrami differentials.*

PROOF. Let us first assume that $U = \bar{\mathbb{C}}$. Then

$$\mu_\lambda(z) = \sum a_n(z)\lambda^n, \quad \lambda \in \mathbb{D}_\rho,$$

where the a_n are smooth functions on $\bar{\mathbb{C}}$, and the series converges uniformly over $\bar{\mathbb{C}} \times \mathbb{D}_r$ for any $r < \rho$. It follows that the series $\sum a_n \lambda^n$ in L^∞ converges uniformly over $\mathbb{D}_r$ and hence represents a holomorphic function $\mathbb{D}_r \to L^\infty$.

Let us now consider the general case; put $K = \bar{\mathbb{C}} \smallsetminus U$. Consider a sequence of smooth functions $\chi_l : \bar{\mathbb{C}} \to [0, 1]$ such that $\chi_l = 0$ on K and for any $z \in U$, $\chi_l(z) \to 1$ as $l \to \infty$.

Consider smooth Beltrami differentials $\mu_\lambda^l = \chi_l \mu_\lambda$. By the above consideration, they depend holomorphically on λ. Moreover, since K has zero area, $\chi_l \mu_\lambda \to \mu_\lambda$ a.e. as $l \to \infty$. Note also that $\|\mu_\lambda^l\|_\infty \leq 1$.

Take any smooth test function ϕ on $\bar{\mathbb{C}}$ and let

$$g_l(\lambda) = \int \mu_\lambda^l \phi dA; \quad g(\lambda) = \int \mu_\lambda \phi dA,$$

where dA is the (normalized) area element on $\bar{\mathbb{C}}$. The family $\{g_l\}$ is uniformly bounded: $|g_l(\lambda)| \leq \|\phi\|_\infty$ By the Lebesgue Bounded Convergence Theorem, $g_l(\lambda) \to g(\lambda)$ as $l \to \infty$

By the previous discussion, functions g_l are holomorphic functions on $\mathbb{D}_\rho$. By the Little Montel Theorem, this family is normal. Hence we can select a subsequence conveging to g uniformly on compact subsets of $\bar{\mathbb{C}}$. It follows that g is holomorphic on $\mathbb{D}_\rho$.

Since smooth functions are dense in L^1, Lemma 2.19 can be applied. It implies the assertion. $\square$

EXERCISE 2.12. *Let $f : S \to T$ be a holomorphic map between two Riemann surfaces, and let $\{\mu_\lambda\}$ be a holomorphic family of Beltrami differentials on T. Then $f^*(\mu_\lambda)$ is a holomorphic family of Beltrami differentials on S.*

11. Quasi-symmetric maps

DEFINITION 2.1. A map $h : X \to Y$ between two metric spaces is called *κ-quasi-symmetric* ("κ-qs") if for any triple of points a, b, c with $\mathrm{dist}(a, c) \leq \mathrm{dist}(a, b)$ we have: $\mathrm{dist}(h(a), h(c)) \leq \kappa \, \mathrm{dist}(h(a), h(b))$. A map is called quasi-symmetric ("qs") if it is κ-qs for some κ. The *dilatation* of a qs map is the smallest κ with this property.

EXERCISE 2.13. *A metric space is called geodesic if any two points in it can be joined with an isometric image of a real interval $[x, y]$. Assume that X is geodesic and $h : X \to Y$ is κ-qs. Then for any $L > 0$ there exists an $M = M(\kappa, L) > 0$ such that*

$$\mathrm{dist}(a, c) \leq L \, \mathrm{dist}(a, b) \Rightarrow \mathrm{dist}(h(a), h(c)) \leq M \, \mathrm{dist}(h(a), h(b)).$$

On the plane, the class of orientation preserving quasi-symmetric maps in fact coincides with the class of quasi-conformal maps. In one direction, it is a simple consequence of the Compactness Theorem:

PROPOSITION 2.21. *Any K-quasi-conformal map $h : \mathbb{C} \to \mathbb{C}$ is $\kappa(K)$-quasi-symmetric in the Euclidean metric of the plane.*

PROOF. Otherwise there would exist a sequence of K-qc maps $h_n : \mathbb{C} \to \mathbb{C}$ and a sequence of triples of points a_n, b_n, c_n in $\mathbb{C}$ such that

$$|a_n - c_n| \leq |a_n - b_n| \quad \text{but} \quad |h_n(a_n) - h_n(c_n)|/|h_n(a_n) - h_n(b_n)| \to \infty. \tag{11.1}$$

Consider two sequences of affine maps S_n and T_n such that

$$S_n(0) = a_n, \; T_n(h_n(a_n)) = 0 \quad \text{and} \quad S_n(1) = b_n, \; T_n(h_n(b_n)) = 1.$$

Then the normalized maps $H_n = T_n \circ h_n \circ S_n$ fix 0 and 1. By the Compactness Theorem 2.12, they are uniformly bounded on the unit

disk $\mathbb{D}$. On the other hand, (11.1) implies that the points $x_n = S_n^{-1} c_n$ belong to $\mathbb{D}$, while $H_n(x_n) = T_n(h_n(c_n)) \to \infty$ - contradiction. $\qquad \square$

In particular, if we consider a quasi-conformal map $h : \mathbb{C} \to \mathbb{C}$ preserving the real line $\mathbb{R}$, it restricts to a quasi-symmetric map on the latter. Remarkably, the inverse is also true:

THEOREM 2.22 (Ahlfors-Boerling Extension). *Any κ-qs orientation preserving map $h : \mathbb{R} \to \mathbb{R}$ extends to a $K(\kappa)$-qc map $H : \mathbb{C} \to \mathbb{C}$.*

PROOF. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Note that in the Ahlfors-Boerling extension is obviously affinely equivariant (that is, commutes with the action of the complex affine group $z \mapsto az + b$).

It looks at first glance that the class of 1D quasi-symmetric maps is a good analogue of the class of 2D quasi-conformal maps. However, this impression is superficial: two-dimensional qc maps are fundamentally better than one-dimensional qs maps. For instance, qc maps can be glued together without any loss of dilatation (Lemma 2.10), while qs maps cannot:

EXERCISE 2.14. *Consider a map $h : \mathbb{R} \to \mathbb{R}$ equal to id on the negative axis, and equal to $x \mapsto x^2$ on the positive one. This map is not quasi-symmetric, though its restrictions to the both positive and negative axes are.*

Another big defficiency of one-dimensional qs maps is that they can well be singular (and typically are in the dynamical setting - see **??**), while 2D qc maps are always absolutely continuous (Proposition 9.1).

These advantages of qc maps makes them much more efficient tool for dynamics than one-dimensional qs maps. This is a reason why complexification of one-dimensional dynamical systems is so powerful.

Let us now state an Extension Lemma in an annulus which will be usefull in what follows:

LEMMA 2.23 (Interpolation). *Let us consider two round annuli $A = \mathbb{A}[1, r]$ and $\tilde{A} = \mathbb{A}[1, \tilde{r}]$, with $0 < \epsilon \leq \mod A \leq \epsilon^{-1}$ and $\epsilon \leq \mod \tilde{A} \leq \epsilon^{-1}$. Then any κ-qs map $h : (\mathbb{T}, \mathbb{T}_r) \to (\tilde{\mathbb{T}}, \tilde{\mathbb{T}}_{\tilde{r}})$ admits a $K(\kappa, \epsilon)$-qc extension to a map $H : A \to \tilde{A}$.*

PROOF. Since A and $\tilde{A}$ are ϵ^2-qc equivalent, we can assume without loss of generality that $A = \tilde{A}$. Let us cover A by the upper half-plane, $\theta : \mathbb{H} \to A$, $\theta(z) = z^{\frac{-\log ri}{\pi}}$, where the covering group generated by the dilation $T : z \mapsto \lambda z$, with $\lambda = e^{\frac{2\pi^2}{\log r}}$. Let $\bar{h} : (\mathbb{R}, 0) \to (\mathbb{R}, 0)$ be the

lift of h to $\mathbb{R}$ such that $\bar{h}(1) \in [1, \lambda) \equiv I_\lambda$ and $\bar{h}(1) \in (-\lambda, -1]$ (note that $\mathbb{R}_+$ covers $\mathbb{T}_r$, while $\mathbb{R}_-$ covers $\mathbb{T}$). Moreover, since $\deg h = 1$, it commutes with the deck transformation T.

A direct calculation shows that the dilatation of the covering map θ on the fundamental intervals I_λ and $-I_\lambda$ is comparable with $(\log r)^{-1}$. Hence $\bar{h}$ is $C(\kappa, r)$-qs on this interval. By equivariance it is $C(\kappa, r)$-qc on the rays $\mathbb{R}_+$ and $\mathbb{R}_-$.

It is also quasi-symmetric near the origin. Indeed, by the equivariance and normalization,

$$(1 + \lambda)^{-1}|J| \leq |\bar{h}(J)| \leq (1 + \lambda)|J|$$

for any interval J containing 0, which easily implies quasi-symmetry.

Since the Ahlfors-Börling extension is affinely equivariant, the map $\bar{h}$ extends to a $K(\kappa, r)$-qc map $\bar{H} : \mathbb{H} \to \mathbb{H}$ commuting with T. Hence $\bar{H}$ descends to a $K(\kappa, r)$-qc map $H : A \to A$. $\qquad\square$

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Note that the Gluing Lemma makes a difference between complex qc and real qs maps which is crucial for the pull-back argument.

Let D be a simply connected domain conformally equivalent to the hyperbolic plane $\mathbb{H}^2$. Given a family of subsets $\{S_k\}_{k=1}^n$ in D, let us say that a family of disjoint annuli $A_k \subset D \smallsetminus \cup S_i$ is *separating* if A_k surrounds S_k but does not surround the S_i, $i \neq k$. The following lemma is used in the present paper uncountably many times:

Moving Lemma. • *Let $a, b \in D$ be two points on hyperbolic distance $\rho \leq \bar{\rho}$. Then there is a diffeomorphism $\phi : (D, a) \to (D, b)$, identical near ∂D, with dilatation $\mathrm{Dil}(\phi) = 1 + O(\rho)$, where the constant depends only on $\bar{\rho}$.*

• *Let $\{(a_k, b_k)\}_{k=1}^n$ be a family of pairs of points which admits a family of separating annuli A_k with $\mathrm{mod}\, A_k \geq \mu$. Then there is a diffeomorphism $\phi : (D, a_1, \ldots a_n) \to (D, b_1, \ldots, b_n)$, identical near ∂D, with dilatation $\mathrm{Dil}(\phi) = 1 + O(e^{-\mu})$.*

PROOF. As the statement is conformally equivalent, we can work with the unit disk model of the hyperbolic plane, and can also assume that $a = 0$, $b > 0$. Also, it is enough to prove the statement for sufficiently small ρ.

There is a smooth function $\psi : [0, 1] \to [b, 1]$ such that $\psi(x) \equiv b$ near 0, $\psi(x) \equiv 0$ near 1, and $\psi'(x) = O(\rho)$, with a constant depending only on $\bar{\rho}$.

Let us define a smooth map $\phi : (\mathbb{D}, 0) \to (\mathbb{D}, b)$ as $z \mapsto z + \psi(|z|)$. Then

$$\partial\phi(z) = 1 + \psi'(|z|)\frac{\bar{z}}{2|z|} = 1 + O(\rho), \quad \bar{\partial}\phi(z) = \psi'(|z|)\frac{z}{2|z|} = O(\rho). \tag{11.2}$$

Thus

$$\mathrm{Jac}(f) = \partial\phi(z)|^2 - |\bar{\partial}\phi(z)|^2 = 1 + O(\rho).$$

Hence for sufficiently small $\rho > 0$, f is a local orientation preserving diffeomorphism. As $f : \partial\mathbb{D} \to \partial\mathbb{D}$, f is a proper map. Hence it is a diffeomorphism.

Finally, (11.2) yields that the Beltrami coefficient $\mu_f = O(\rho)$, so that the dilatation $\mathrm{Dil}(f) = 1 + O(\rho)$. $\qquad\square$

Let $Q \subset \mathbb{C}$, $h : Q \to \mathbb{C}$ be a homeomorphism onto its image. It is called *quasi-symmetric (qs)* if for any three points $a, b, c \in Q$ such that $q^{-1} \leq |a-b|/|b-c| \leq q$, we have: $\kappa(q)^{-1} \leq |h(a) - h(b)|/|h(b) - h(c)| \leq \kappa(q)$. It is called κ-quasi-symmetric if $\kappa(1) \leq \kappa$. It follows from the Compactness Lemma that any K-qc map is κ-quasi-symmetric, with a κ depending only on K.

Let us discuss quasi-symmetric maps of the circle $\mathbb{T} = \{z : |z| = 1\}$. Given an interval $J \subset \mathbb{T}$, let $|J|$ denote its length. An orientation preserving map $h : \mathbb{T} \to \mathbb{T}$ is called κ-*quasi-symmetric* (κ-qs) if for any two adjacent intervals $I, J \subset \mathbb{T}$, $|hI|/|hJ| \leq \kappa$.

Let $\mathbb{T}_r = \{z : |z| = r\}$, $\mathbb{T} \equiv \mathbb{T}_1$. Let $\mathbb{A}(r, R) = \{z : r < |z| < R\}$. Similar notations are used for a closed annulus $\mathbb{A}[r, R]$ (or semi-closed one).

proclaim Ahlfors-Börling Extension Theorem. Any κ-quasi-symmetric map $h : \mathbb{T} \to T$ extends to a $K(\kappa)$-qc map $H : \mathbb{C} \to \mathbb{C}$. Vice versa: The restriction of any K-qc map $H : (\mathbb{A}(r^{-1}, r), \mathbb{T}) \to (U, \mathbb{T})$ (where $U \subset \mathbb{C}$) to the circle $\kappa(K, r)$-quasi-symmetric.

Let us note that in the upper half-plane model, the Ahlfors-Börling extension of a qs map $\mathbb{R} \to \mathbb{R}$ is affinely equivariant (that is, commutes with the action of the complex affine group $z \mapsto az + b$).

11.1. Quasicircles. Let us start with an intrinsic geometric definition of quasicircles:

DEFINITION 2.2. A Jordan curve $\gamma \subset \mathbb{C}$ is called a κ-*quasicircle* if for any two points $x, y \in \gamma$ there is an arc $\delta \subset \gamma$ bounded by these points such that

$$\mathrm{diam}\,\delta \leq \kappa|x - y|.$$

A curve is called a quasicircle if it is a κ-quasicircle for some κ. The best possible κ in the above definition is called the *dilatation* of the quasicircle. A Jordan disk is called $(\kappa$-$)quasidisk$ if it is bounded by a $(\kappa$-$)$quasicircle.

EXERCISE 2.15. *Let D be a κ-quasidisk, $\partial D = \gamma$. Show that*

$$\sup_{z \in D} \operatorname{dist}(z, \gamma) \geq c \operatorname{diam} D$$

for some constant $c > 0$ depending only on κ.

On the other hand, quasicircles can also be characterized as qc images of the circle (which explains the importance of this class of curves). Recall from §**??** that $r_{D,a}$ denote the inner radius of a pointed disk (D, a).

THEOREM 2.24. *Let (D, a) a pointed κ-quasidisk, and let $\phi : (\mathbb{D}, 0) \to (D, a)$ be the normalized Riemann mapping. Assume that $r_{D,a} \geq c \operatorname{diam} D$, where $c > 0$. Then ϕ admits a K-qc extension to the whole complex plane, where K depends only on κ and c.*

Vice versa, let (D, a) be a Jordan disk such that there exists a K-qc map $h : (\mathbb{C}, \mathbb{D}, 0) \to (\mathbb{C}, D, a)$. Then D is a κ-quasidisk and $r_{D,a} \geq c \operatorname{diam} D$, where the constants κ and $c > 0$ depend only on K.

Recall the definition of the inner and the outer radia, $r_{D,a}$ and $R_{D,a}$ of a pointed domain (D, a). Let $\mathcal{QD}_{\kappa,r}$, $r > 0$, denote the space of pointed κ-quasidisks $(D, 0)$ with $r \leq r_{D,0} \leq R_{D,0} \leq 1/r$, endowed with the Carathéodory topology.

PROPOSITION 2.25. *The space $\mathcal{QD}_{\kappa,r}$ is compact.*

PROOF. Consider a quasidisk $(D, 0) \in \mathcal{QD}_{\kappa,r}$. By Theorem 2.24, the normalized Riemann mapping $h : (\mathbb{D}, 0) \to (D, 0)$ admits a K-qc extension to the whole complex plane $\mathbb{C}$, where K depends only on κ and r. Moreover, $r \leq |h(1)| \leq 1/r$. By the Compactness Theorem (see Exercise 2.8), this family of qc maps is compact in the uniform topology on $\mathbb{C}$. Since uniform limits of κ-quasidisks are obviously κ-quasidisks, the conclusion follows. $\square$

A set is called "0-symmetric" if it is invariant under the reflection with respect to the origin.

EXERCISE 2.16. *Let γ be a 0-symmetric κ-quasicircle. Then the eccentricity of γ around 0 is bounded by $2\kappa + 1$.*

12. Removability

12.1. Conformal vs quasiconformal. Similarly to the notion of qc removability introduced in §9.3 we can define conformal removability:

DEFINITION 2.3. A compact subset $X \subset \mathbb{C}$ is called *conformally removable* if for any open sets $U \supset X$ in $\mathbb{C}$, any homeomorphic embedding $h : U \hookrightarrow \mathbb{C}$ which is conformal on $U \smallsetminus X$ is conformal/qc on U.

It is classical that isolated points and smooth Jordan curves are conformally removable. By §9.3 of Ch. 2, they are qc removable as well. In fact, these two properties are equivalent:

PROPOSITION 2.26. *Conformal removability is equivalent to qc removability.*

Thus, we can unambiguously call a set "removable".

12.2. Removability and area. The Measurable Riemann Mapping Theorem yields:

PROPOSITION 2.27. *Removable sets have zero area.*

PROOF. Assume that $m(X) > 0$. Then there exists a non-trivial Beltrami differential μ supported on X. Let $h : \mathbb{C} \to \mathbb{C}$ be a solution of the corresponding Beltrami equation. Then h is conformal outside X but is not conformal on X. $\qquad\qquad\square$

The reverse is false:

EXAMPLE 2.1.

12.3. Divergence property.

DEFINITION 2.4. Let us say that a compact set $X \subset \mathbb{C}$ satisfies the *divergence property* if for any point $z \in X$ there exists a nest of annuli $A^n(z)$ around z such that

$$\sum A^n(z) = \infty.$$

Without loss of generality we can assume (and we will always do so) that each annulus in this definition is bounded by two Jordan curves.

LEMMA 2.28. *Compact sets satisfying the divergence property are Cantor.*

PROOF. Consider any connected component X_0 of X, and let $z \in X_0$. Then the annuli $A^n(z)$ are nested around X_0. By Corollary 1.27 of the Grötzsch Inequality, X_0 is a single point. $\square$

LEMMA 2.29. *Let $X \subset \mathbb{C}$ be a compact set satisfying the divergence property. Then for any neighborhood $U \supset X$, any qc embedding $h : U \smallsetminus X \hookrightarrow \mathbb{C}$ admits a homeomorphic extension through X.*

PROOF. Let $h : U \smallsetminus X \hookrightarrow \mathbb{C}$ be a K-qc embedding. If $X \subset U' \Subset U$ then $h(U')$ is bounded in $\mathbb{C}$. So, without loss of generality we can assume that $h(U)$ is bounded in $\mathbb{C}$.

For $z \in X$, let us consider the nest of annuli $h(A^n(z))$. Since h is quasiconformal,

$$\sum \operatorname{mod} h(A^n(z)) \geq K^{-1} \sum \operatorname{mod} A^n(z) = \infty.$$

Let $\Delta^n(z)$ be the bounded component of $\mathbb{C} \smallsetminus h(A^n(z))$, and let

$$\Delta^\infty(z) = \bigcap_n D^n(z).$$

By Corollary 1.27 of the divergence property, $\Delta^\infty(z)$ is a single point $\zeta = \zeta(z)$. Let us extend h through X by letting $h(z) = \zeta$.

This extension is continuous. Indeed, let $D^n(z)$ be the bounded component of $\mathbb{C} \smallsetminus A^n(z)$. Then by Corollary 1.27, $\operatorname{diam} D^n(z) \to 0$, so that $D^n(z)$ is a base of (closed) neighborhoods of z. But

$$\operatorname{diam} h(D^n(z)) = \operatorname{diam} \Delta^n(z) \to 0,$$

which yields continuity of h at z.

Switching the roles of (U, X) and $(h(U), h(X))$, we conclude that h^{-1} admits a continuous extension through $h(X)$. Hence the extension of h is homeomorphic. $\square$

It is worthwhile to note that, in fact, general homeomorphisms extend through Cantor sets:

EXERCISE 2.17. *(i) Let us consider two Cantor sets X and $\tilde{X}$ in $\mathbb{C}$ and their respective neighborhoods U and $\tilde{U}$. Then any homeomorphism $h : U \smallsetminus X \to \tilde{U} \smallsetminus \tilde{X}$ admits a homeomorphic extension through X.*

(ii) It was essential to assume that both sets X and $\tilde{X}$ are Cantor! For any compact set $X \subset \mathbb{C}$, give an example of an embedding $h : C \smallsetminus X \hookrightarrow \mathbb{C}$ which does not admit a continuous extension through X.

LEMMA 2.30. *Compact sets satisfying the divergence property have zero area.*

We will show now that sets satisfying the divergence property are removable, and even in the following stronger sense:

THEOREM 2.31. *Let $X \subset \mathbb{C}$ be a compact set satisfying the divergence property. Then for any neighborhood $U \supset X$, any conformal/qc embedding $h : U \smallsetminus X \hookrightarrow \mathbb{C}$ admits a conformal/qc extension through X.*

PROOF. Let $h : U \smallsetminus X \hookrightarrow \mathbb{C}$ be a K-qc embedding. By Lemma 2.29, h extends to an embedding $U \hookrightarrow \mathbb{C}$, which will be still denoted by h. Let us show that h belongs to the Sobolev class $H(U)$.

Since X is a Cantor set, it admits a nested base of neighborhoods U^n such that each U^n is the union of finitely many disjoint Jordan diks. Take any $\mu > 0$. By the Grützsch Inequality, for any $n \in \mathbb{N}$ there is $k = k(\mu, l) > 0$ such that $\mathrm{mod}(\partial U^{n+k}, \partial U^n) \geq \mu > 0$. Let χ_n be the solution of the Dirichlet problem in $U^n \smallsetminus U^{n+k}$ vanishing on ∂U^{n+k} and equal to 1 on ∂U^n. By Theorem 1.30, $D(\chi_n) \leq 1/\mu$.

Let us continuously extend χ to the whole plane in such a way that it vanishes on U^{n+k} and identically equal to 1 on $\mathbb{C} \smallsetminus U^n$. We obtain a piecewice smooth function $\chi : \mathbb{C} \to [0, 1]$, with the jump of the derivative on the boundary of the domains U^n and U^{n+k}.

Let $h_n = \chi_n h$. These are piecewise smooth functions with bounded Dirichlet integral. Indeed,

$$D(h_n) = \int (|\nabla \chi_n|^2 |h|^2 + |\chi_n|^2 |\nabla h|^2) dm \leq \mathrm{diam}(h(U))/\mu + C(K)m(h(U)),$$

where $C(K) = (1 + k^2)/(1 - k^2)$ comes from the area estimate (area estimate). By weak compactness of the unit ball in $L^2(U)$, we can select a converging subsequence $\partial h_n \to \phi$, $\bar{\partial} h_n \to \psi$. But $h_n \to h$ pointwise on $U \smallsetminus X$, so that by Lemma 2.30, $h_n \to h$ a.e. It follows that ϕ and ψ are distributional partial derivatives of h (see (9.2)).

Finally, if h is conformal on $U \smallsetminus X$ then by Weyl's Lemma it is conformal on U. □

Compactness in H of functions with bounded D.I. - formulate as a lemma?

CHAPTER 3

Elements of Teichmüller theory

1. Holomorphic motions

1.1. Definition. Let $(\Lambda, *)$ be a pointed complex manifold (can be infinite dimensional[1]) and let $X \subset \bar{\mathbb{C}}$ be an arbitrary subset of the Riemann sphere (can be non-measurable). A *holomorphic motion* **h** *over* $(\Lambda, *)$[2] is a family of injections $h_\lambda : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$, $\lambda \in \Lambda$, depending holomorphically on λ (in a weak sense that the finctions $z \mapsto h_\lambda(z)$ are holomorphic in λ for all $z \in X$) and such that $h_* = $ id. In this situation, we let $X_\lambda = h_\lambda(X_*)$.[3]

Holomorphic functions $\phi_z : \Lambda \to \bar{\mathbb{C}}$, $\lambda \mapsto h_\lambda(z)$, are called *orbits* of the holomorphic motion. Since the functions h_λ are injective, the orbits do not collide, or equivalently, their graphs $\Gamma_z \subset \Lambda \times \bar{\mathbb{C}}$ are disjoint. Thus, a holomorphic motion provides us with a family of disjoint holomorphic graphs over Λ. We refer to such a family as a *(trivial) holomorphic lamination*. Of course, the above reasoning can be reversed, so that, trivial holomorphic laminations give us an equivalent (dual) way of describing holomorphic motions.

A *regularity* of a holomorphic motion is the regularity of the maps h_λ on X. For instance, a holomorphic motion is called *continuous, qc, smooth* or *bi-holomorphic* if all the maps h_λ, $\lambda \in \Lambda$, have the corresponding regularity on X (to make sense of it in some cases we need extra assumptions on X, e.g., openness).

Notice that *a priori* we do not impose any regularity on the maps h_λ (not even measurability!). A remarkable property of holomorphic motions is that they automatically have nice regularity properties and that they automatically extend to motions of the whole Riemann sphere.

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[1]We will eventually deal with infinite dimensional parameter spaces, so we need to prepare the background in this generality. However, in the first reading the reader can safely assume that the space Λ is a one-dimensional disk (which is anyway the main case to consider).

[2]We will often make a point $*$ implicit in the notation and terminology.

[3]we will sometimes say briefly that "the sets X_λ move holomorphically" or "the set X_* moves holomorphically" without mentioning explicitly the maps h_λ

This set of properties are usually referred to as the λ-*lemma*. It will be the subject of the rest of this section.

While dealing with a holomorphic motion of a set X, Y, etc., we let $X_\lambda = h_\lambda(X)$, $Y_\lambda = h_\lambda(Y)$, etc. We will refer to the z-variable of a holomorphic motion as the *dynamical variable* (though in general, there is no dynamics in the z-plane). The λ-variable is naturally referred to as the *parameter*.

1.2. First λ-lemma: extension to the closure and continuity.

LEMMA 3.1. *A holomorphic motion* $\mathbf{h}$ *of any set* $X \subset \bar{\mathbb{C}}$ *extends to a continuous holomorphic motion of its closure* $\bar{X}$.

PROOF. If X is finite, there is nothing to prove, so assume it is infinite (or, at least, contains more than two points).

Let us show that the family of orbits ϕ_z, $z \in X$, of our holomorphic motion is normal. To this end, let us remove from X three points $z_i \in X$; let $X' = X \smallsetminus \{z_i\}$ and let ψ_i be the orbits of the points z_i. Since the orbits of a holomorphic motion do not collide, the family of orbits of points $z \in X'$ satisfies the condition of the Refined Montel Theorem (1.14) with exceptional functions ψ_i, and the normality follows.

Let Φ be the closure of the family of orbits in the space $\mathcal{M}(\Lambda)$ of meromorphic functions on Λ. By the Hurwitz Theorem (see §4.3) the graphs of these functions are disjoint, so they form a holomorphic lamination representing a holomorphic motion of $\bar{X}$.

Let us keep notation h_λ for the extended holomorphic motion, and notation ϕ_z, $z \in \bar{X}$, for its orbits.

Let us show that this motion is continuous. Let $\lambda \in \Lambda$, let $z_n \to z$ be a converging sequence of points in $\bar{X}$, and let $\phi_n \in \Phi$ and $\phi \in \Phi$ be their respective orbits. We want to show that $h_\lambda(z_n) \to h_\lambda(z)$, or equivalently $\phi_n(\lambda) \to \phi(\lambda)$. But otherwise, the sequence ϕ_n would have a limit point $\psi \in \mathcal{M}(\Lambda)$ such that $\psi(*) = \phi(*)$ while $\psi(\lambda) \neq \phi(\lambda)$, which would contradict to the laminar property of the family Φ. $\square$

1.3. Second λ-lemma: quasi-conformality. **************************************

Let $X \subset \mathbb{C}$ be a subset of the complex plane. A *holomorphic motion* of X over a Banach ball $(\mathcal{B}_1, 0)$ is a a family of injections $h_\lambda : X \to \mathbb{C}$, $\lambda \in \mathcal{B}_1$, with $h_0 = \mathrm{id}$, holomorphically depending on $\lambda \in \mathcal{B}_1$ (for any given $z \in X$). The graphs of the functions $\lambda \mapsto h_\lambda(z)$, $z \in X$, form a foliation $\mathcal{F}$ (or rather a lamination as it is partially defined) in $\mathcal{B}_1 \times \mathbb{C}$ with complex codimension 1 analytic leaves. This is a ""dual" viewpoint on holomorphic motions.

We will now state a basic fact about holomorphic motions usually referred to as "λ-lemma". It consists of two parts: extension and quasi-conformality which will be stated separately. The consecutively improving versions of the Extension Lemma appeared in [**?, ?, ?, ?, ?**]. The final result is due to Slodkowski:

λ-lemma (Extension). *A holomorphic motion $h_\lambda : X_* \to X_\lambda$ of a set $X_* \subset \mathbb{C}$ over a topological disc D admits an extension to a holomorphic motion $H_\lambda : \mathbb{C} \to \mathbb{C}$ of the whole complex plane over D.*

The point of the following simple lemma as compared with the previous deep one is smoothness of the extension and that the parameter space is allowed to be infinitely dimensional.

LEMMA 3.2 (Local extension). *Let us consider a compact set $Q \subset \mathbb{C}$ and a smooth holomorphic motion $\mathbf{h}$ of a neighborhood U of Q over a Banach domain $(\Lambda, 0)$. Then there is a smooth holomorphic motion $\mathbf{H}$ of the whole complex plane $\mathbb{C}$ over some neighborhood $\Lambda' \subset \Lambda$ of 0 whose restriction to Q coincides with $\mathbf{h}$.*

PROOF. We can certainly assume that $\operatorname{cl} U$ is compact. Take a smooth function $\phi : \mathbb{C} \to \mathbb{R}$ supported in U and let

$$H_\lambda = \phi\, h_\lambda + (1 - \phi)\, \mathrm{id}\,.$$

Clearly H is smooth in both variables, holomorphic in λ, and identical outside U. As $H_0 = \mathrm{id}$, $H_\lambda : \mathbb{C} \to \mathbb{C}$ is a diffeomorphism for λ sufficiently close to 0, and we are done. $\qquad\square$

Given two complex one-dimensional transversals $\mathcal{S}$ and $\mathcal{T}$ to the lamination $\mathcal{F}$ in $\mathcal{B}_1 \times \mathbb{C}$, we have a holonomy $\mathcal{S} \to \mathcal{T}$. We say that this map is locally quasi-conformal if it admits local quasi-conformal extensions near any point.

Given two points $\lambda, \mu \in \mathcal{B}_1$, let us define the hyperbolic distance $\rho(\lambda, \mu)$ in $\mathcal{B}_1$ as the hyperbolic distance between λ and μ in the one-dimensional complex slice $\lambda + t(\mu - \lambda)$ passing through these points in $\mathcal{B}_1$.

λ-lemma (quasi-conformality). *Holomorphic motion h_λ of a set X over a Banach ball $\mathcal{B}_1$ is transversally quasi-conformal. The local dilatation K of the holonomy from $p = (\lambda, u) \in \mathcal{S}$ to $q = (\mu, v) \in \mathcal{T}$ depends only on the hyperbolic distance ρ between the points λ and μ in $\mathcal{B}_1$. Moreover, $K = 1 + O(\rho)$ as $\rho \to 0$.*

PROOF. If the transversals are vertical lines $\lambda \times \mathbb{C}$ and $\mu \times \mathbb{C}$ then the result follows from the classical λ-lemma [**?**] by restricting the motion to the complex line joining λ and μ.

Furthermore, the holonomy from the vertical line $\lambda \times \mathbb{C}$ to the transversal $\mathcal{S}$ is locally conformal at point p. To see this, let us select a holomorphic coordinates (θ, z) near p in such a way that $p = 0$ and the leaf via p becomes the parameter axis. Let $z = \psi(\theta) = \epsilon + \ldots$ parametrizes a nearby leaf of the foliation, while $\theta = g(z) = bz + \ldots$ parametrizes the transversal $\mathcal{S}$.

Let us do the rescaling $z = \epsilon\zeta, \theta = \epsilon\nu$. In these new coordinates, the above leaf is parametrized by the function $\Psi(\nu) = \epsilon^{-1}\psi(\epsilon\nu)$, $|\nu| < R$, where R is a fixed parameter. Then $\Psi'(\nu) = \psi'(\epsilon\nu)$ and $\Psi''(\nu) = \epsilon\psi''(\epsilon\nu)$. By the Cauchy Inequality, $\Psi''(\nu) = O(\epsilon)$. Moreover, ψ uniformly goes to 0 as $\psi(0) \to 0$. Hence $|\Psi'(0)| = |\psi'(0)| \leq \delta_0(\epsilon)$, where $\delta_0(\epsilon) \to 0$ as $\epsilon \to 0$. Thus $\Psi'(\nu) = \delta_0(\epsilon) + O(\epsilon) \leq \delta(\epsilon) \to 0$ as $\epsilon \to 0$ uniformly for all $|\nu| < R$. It follows that $\Psi(\nu) = 1 + O(\delta(\epsilon)) = 1 + o(1)$ as $\epsilon \to 0$.

On the other hand, the manifold $\mathcal{S}$ is parametrized in the rescaled coordinates by a function $\nu = b\zeta(1 + o(1))$. Since the transverse intersection persists, $\mathcal{S}$ intersects the leaf at the point $(\nu_0, \zeta_0) = (1, b)(1 + o(1))$ (so that R should be selected bigger than $\|b\|$). In the old coordinates the intersection point is $(\theta_0, z_0) = (\epsilon, b\epsilon)(1 + o(1))$.

Thus the holonomy from $\lambda \times \mathbb{C}$ to $\mathcal{S}$ transforms the disc of radius $|\epsilon|$ to an ellipse with small eccentricity, which means that this holonomy is asymptotically conformal. As the holonomy from $\mu \times \mathbb{C}$ to $\mathcal{T}$ is also asymptotically conformal, the conclusion follows. $\square$

Quasi-conformality is apparently the best regularity of holomorphic motions which is satisfied automatically.

2. Moduli and Teichmüller spaces of punctured spheres

2.1. Definitions. Let us consider the Riemann sphere with a tuple of n marked points $\mathcal{P} = (z_1, \ldots, z_n)$ (or, equivalently, n punctures). The punctures are considered to be "colored", or, in other words, the set $\mathcal{P}$ is ordered. Two such spheres $(\mathbb{C}, \mathcal{P})$ and $(\mathcal{C}, \mathcal{P}')$ are considered to be equivalent if there is a Möbius transformation $\phi : (\mathbb{C}, \mathcal{P}) \to (\mathbb{C}, \mathcal{P}')$ (preserving the colors of the punctures, i.e., $\phi(z_i) = z_i'$. The space of equivalence classes is called the *moduli space* $\mathcal{M}_n$.

If $n \leq 3$ then the moduli space $\mathcal{M}_n$ is a single point. If $n \geq 4$, we can place the last three points to $(0, 1, \infty)$ by means of a Möbius transformation. With this normalization $(\mathbb{C}, \mathcal{P}) \sim (\mathbb{C}, \mathcal{P}')$ if and only if $\mathcal{P} = \mathcal{P}'$, and we see that

$$\mathcal{M}_n = \{(z_1, \ldots, z_{n-3}) : \ z_i \neq 0, 1; \ z_i \neq z_j\}.$$

This turns $\mathcal{M}_n$ into an $(n - 3)$-dimensional complex manifold.

Let us fix some reference normalized tuple $\mathcal{P}_\mathsf{o} = (a_1, \ldots a_{n-3}, 0, 1, \infty)$. Then we can trivially redefine $\mathcal{M}_n$ as the space of normalized homeomorphisms $h : (\mathbb{C}, \mathcal{P}_\mathsf{o}) \to \mathbb{C}$ up to equivalence: $h \sim h'$ if $h(\mathcal{P}_b ase) = h'(\mathcal{P}_\mathsf{o})$.

Let us now refine this equivalence relation by declaring that $h \simeq h'$ if h is homotopic (or, equivalently, isotopic) to h' rel $\mathcal{P}_\mathsf{o}$, and let $[h]$ stand for the corresponding equivalence classes. It inherits the quotient topology from the space of homeomorphisms (endowed with the uniform topology). This quotient space is called the *Teichmüller space* $\mathcal{T}_n$. Since the equivalence relation $\simeq$ is obviously stronger than $\sim$ we have a natural projection $\pi : \mathcal{T}_n \to \mathcal{M}_n$.

2.2. Spiders. The homotopy class $[h]$ can be nicely visualised as the punctured sphere marked with a "spider". A *spider* $\mathcal{S}$ on the punctured sphere $(\mathbb{C}, \mathcal{P})$ is a family of disjoint paths σ_i connecting z_i to ∞, $i = 1, \ldots n-1$. We let $[\mathcal{S}]$ be the class of isotopic spiders (rel $\mathcal{P}$).

LEMMA 3.3. *There is a natural one-to-one correspondence between points of $\mathcal{T}_n$ and classes of isotopic spiders, $(\mathbb{C}, \mathcal{P}, [\mathcal{S}])$.*

PROOF. Let us fix a reference spider $(\mathbb{C}, \mathcal{P}_\mathsf{o}, \mathcal{S}_\mathsf{o})$. Then to each homeomorphism $h \in \mathcal{T}_n$ we can assossiate a spider $\mathcal{S} = h(\mathcal{S}_\mathsf{o})$. Isotopy h_t rel $\mathcal{P}_\mathsf{o}$ induces isotopy of the corresponding spiders rel $\mathcal{P}$. Hence we obtain a map $[h] \mapsto [\mathcal{S}]$.

Vice versa, let us have a spider $(\mathbb{C}, \mathcal{P}, \mathcal{S})$. Then there exists a homeomorphism $h : (\mathbb{C}, \mathcal{P}_\mathsf{o}, \mathcal{S}_\mathsf{o}) \to (\mathbb{C}, \mathcal{P}, \mathcal{S})$. If $(\mathbb{C}, \mathcal{P}, \mathcal{S}')$ is an isotopic spider then the isotopy $\mathcal{S}_t$ rel $\mathcal{P}$, $0 \le t \le 1$, lifts to an isotopy h_t rel $\mathcal{P}_0$. Given any parametrizing homeomorphism $h' : \mathcal{S}_\mathsf{o} \to \mathcal{S}'$, we can isotopy h_1 so that it will coincide with h' on $\mathcal{S}_\mathsf{o}$. Since two homeomorphisms of a topological disk coinciding on the boundary are isotopic rel the boundary, we are done. $\square$

2.3. Infinitesimal theory. A tangent vector to the moduli space $\mathcal{M}_n$ at point $z = (z_1 \ldots, z_{n-3}, 0, 1, \infty)$ can be represented as a tuple

$$v = (v(z_1), \ldots v(z_{n-3}))$$

of tangent vectors to $\mathbb{C}$ at points z_i. Since the natural projection $\mathcal{T}_n \to \mathcal{M}_n$ is a covering, tangent vectors to $\mathcal{T}_n$ can be represented in the same way.

Any such tuple of vectors admits an extension to a smooth vector field v vanishing at points $(0, 1, \infty)$ (such vector field will be called "normalized"). So, we can view the tangent space to $\mathcal{M}_n$ (and $\mathcal{T}_n$) as the space Vect of smooth normalized vector fields modulo equivalence relation: $v \sim w$ if $v(z_i) = w(z_i)$, $i = i, \ldots, n-3$.

With this in mind, we can give a nice description of the cotangent space to $\mathcal{M}_n$ and $\mathcal{T}_n$. Let us consider the space $\mathcal{Q} = \mathcal{Q}(\bar{\mathbb{C}} \setminus \mathcal{P})$ of integrable quadratic differentials $\phi = \phi(z)dz^2$ on $\bar{\mathbb{C}} \setminus \mathcal{P}$. Such differentials must have at most simple poles at the punctures (at ∞ it amounts to $\phi(z) = O(1/|z^3|)$).

EXERCISE 3.1. *Show that this space of quadratic differentials has complex dimension $n - 3$.*

It turns out that it is not an accident that $\dim \mathcal{Q} = \dim \mathcal{M}_n$.

PROPOSITION 3.4. *The space $\mathcal{Q}(\bar{\mathbb{C}} \setminus \mathcal{P})$ of quadratic differentials is naturally identified with the cotangent space to $\mathcal{M}_n$ (and $\mathcal{T}_n$). The pairing between a cotangent vector $\phi \in \mathcal{Q}$ and a tangent vector $v \in \text{Vect}$ is given by the formula:*

$$< \phi, v >= \frac{1}{2\pi i} \int \int \phi \, \bar{\partial}v. \tag{2.1}$$

PROOF. Let us first note that this pairing is well defined. Indeed, as we saw in §2, $\bar{\partial}v$ can be interpreted as a Beltrami differential, and the product $\phi \, \bar{\partial}v$ as an area form. Moreover, this area form is integrable since ϕ is integrable and ∂v is bounded.

Let us now calculate this integral. Since ϕ is holomorphic, we have:

$$\phi \, \partial_{\bar{z}}v \, dz \wedge d\bar{z} = \partial_{\bar{z}}(\phi \, v) \, dz \wedge d\bar{z} = -\bar{\partial}(\phi \, v \, dz) = -d(\phi \, v \, dz).$$

Let $\gamma_\epsilon(z_i))$ be the ϵ-circles centered at finite points of $\mathcal{P}$, $i = 1, \ldots, n-1$, and let Γ_ϵ be the ϵ^{-1}-circle centered at 0 (where all the circles are anti-clockwise oriented), and let D_ϵ be the domain of $\mathbb{C}$ bounded by these circles. Then by the Stokes formula

$$-\frac{1}{2\pi i} \int \int_{D_\epsilon} d(\phi \, vdz) = \frac{1}{2\pi i} \sum \int_{\gamma_\epsilon(z_i)} \phi \, vdz - \frac{1}{2\pi i} \int_{\Gamma_\epsilon} \phi \, vdz$$

But near any $z_i \in \mathbb{C}$ we have:

$$\phi \, v = \frac{\lambda_i v(z_i)}{z - z_i} + O(1),$$

where $\lambda_i = \text{Res}_{z_i} \phi$. Hence

$$\frac{1}{2\pi i} \int_{\gamma_\epsilon(z_i)} \phi \, v \, dz \to \lambda_i v(z_i) \text{ as } \epsilon \to 0.$$

Note that these integrals asymptotically vanish at $z_{n-2} = 0$ and $z_{n-1} = 1$ since v vanishes at these points. The integral over Γ_ϵ asymptotically vanishes as well since $\phi(z) = O(|z|^{-3})$ while $v(z) = o(|z|^2)$ near ∞ (as the vector field v/dz vanishes at ∞).

Finally, we obtain:

$$\frac{1}{2\pi i} \int \int \phi \, \partial_{\bar{z}} v \, dz \wedge d\bar{z} = \sum_{i=1}^{n-3} \lambda_i v(z_i)$$

So, the pairing (2.1) depends only on the values of v at the points $z_1, \ldots, z_{n-3}$, and hence defines a functional on tangent space $\mathrm{T}\mathcal{M}_n$. This gives an isomorphism between $\mathcal{Q}$ and the cotangent space $\mathrm{T}^*\mathcal{M}_n$ since $(\lambda_1, \ldots, \lambda_{n-3})$ are clobal coordinates on the both spaces (compare Exercise 3.1).

$\square$

2.4. General Teichmüller spaces.

2.4.1. *Marked Riemann surfaces.* The previous discussion admits an extension to an arbitrary qc class $\mathcal{QC}$ of Riemann surfaces that we will outline in this section. Take some *base* Riemann surface $S_0 \in \mathcal{QC}$ (without boundary), and let $\bar{S}_0$ be the ideal boundary compactification of S_0. Given another Riemann surface $S \in \mathcal{QC}$ (with compactification $\bar{S}$), a *marking* of S is a choice of a qc homeomorphism $\phi : \bar{S}_0 \to \bar{S}$ (*parametrization* by S_0) up to the following equivalence relation. Two parametrized surfaces (S, ϕ) and (S', ϕ') are equivalent if there is a conformal isomorphism $h : S \to S'$ that makes the following diagram *homotopically commutative rel the ideal boundary* (i.e., there is a qc homeomorphism $\tilde{\phi} : S_0 \to S$ homotopic to ϕ rel $\partial \bar{S}_0$ such that $h \circ \tilde{\phi} = \phi'$). A *marked Riemann surfaces* is an equivalence class $\tau = [S, \phi]$ of this relation. The space of all marked Riemann surfaces is called the *Teichmüller space* $\mathcal{T}(S_0)$.

REMARK 3.1. Fixing a set Δ_0 of generators of $\pi_1(S_0)$ and parametrizations of the boundary components of $\partial \bar{S}_0$ by the standard circle $\mathbb{T}$, we naturally endow any marked Riemann surface $[S, \phi]$ with a set of generators of $\pi_1(S)$ (up to an inner automolrphism of $\pi_1(S)$) and with a parametrization of the components ∂S by $\mathbb{T}$. Thus, we obtain a marked surface in the sense of §1.1.4.

2.4.2. *Representation variety.* Let us now uniformize the base Riemann surface S_0 by a Fuchsian group Γ_0. The *(Fuchsian) representation variety* $\mathrm{Rep}(\Gamma_0)$ is the space of faithful[4] Fuchsian representations $i : \Gamma_0 \to \mathrm{PSL}(2, \mathbb{R})$ up to conjugacy in $\mathrm{PSL}(2, \mathbb{R})$ endowed with the *algebraic* topology. In this topology $i_n \to i$ if after a possible replacement of the i_n with conjugate representations, we have: $i_n(\gamma) \to i(\gamma)$ for any $\gamma \in \Gamma_0$.

---

[4]i.e., injective

LEMMA 3.5. *There is a natural embedding $e : \mathcal{T}(S_0) \to \mathrm{Rep}(S_0)$.*

PROOF. Let $\phi : S_0 \to S$ be a qc parametrization of some Riemann surface $S \in \mathcal{QC}$, and let Γ be a Fuchsian group uniformizing S. Then ϕ lifts to an equivariant qc homeomorphism $\Phi : (\mathbb{H}, \Gamma_0) \to (\mathbb{H}, \Gamma)$, so there is an isomorphism $i : \Gamma_0 \to \Gamma$ such that $\Phi \circ \gamma_0 = \gamma \circ \Phi$ for any $\gamma_0 \in \Gamma_0$ and $\gamma = i(\gamma_0)$.

If we replace Φ with another lift $T \circ \Phi$, where $T \in \Gamma$, then i will be replaced with a conjugate representation $\gamma_0 \mapsto T^{-1} \circ i(\gamma_0) \circ T$.

If we replace ϕ with a homotopic parametrization $\tilde{\phi} : S_0 \to S$ then the induced representation $\Gamma_0 \to \Gamma$ will not change. Indeed, a homotopy ϕ_t connecting ϕ to $\tilde{\phi}$ lifts to an equivariant homotopy $\Phi_t : (\mathbb{H}, \Gamma_0) \to (\mathbb{H}, \Gamma)$ inducing a path of representations $i_t : \Gamma_0 \to \Gamma$. Then for any $\gamma_0 \in \Gamma_0$, the image $i_t(\gamma_0) \in \Gamma$ moves continuously with t. Since Γ is discrete, $i_t(\gamma)$ cannot move at all.

If we further replace $\tilde{\phi}$ with $h \circ \tilde{\phi}$, where $h : S \to S'$ is a conformal isomorphism then the representation $i : \Gamma_0 \to \Gamma$ will be replaced with a conjugate by $T : \mathbb{H} \to \mathbb{H}$ where $T \in \mathrm{PSL}(2, \mathbb{R})$ is a lift of h.

Thus, we obtain a well defined map $e : \mathcal{T}(S_0) \to \mathrm{Rep}(S_0)$ that associates to a marked surface $[S, \phi]$ the induced representation $i : \Gamma_0 \to \Gamma$ up to conjugacy in $\mathrm{PSL}(2, \mathbb{R})$.

Let us now show that e is injective. Let $\phi : S_0 \to S$ and $\phi' : S_0 \to S'$ be two parametrizastions whose lifts Φ and Φ' to $\mathbb{H}$ induce two representations i and i' of Γ_0 that are conjugate by $T \in \mathrm{PSL}(2, \mathbb{R})$. Then Φ and $\Psi = T^{-1} \circ \Phi$ are two equivariant homeomorphisms $(\mathbb{H}, \Gamma_0) \to (\mathbb{H}, \Gamma)$ that induce the same representation $i : \Gamma_0 \to \Gamma$. We need to show that they are equivariantly homotopic.

To this end let us consider the following diagram encoding equivariance of Φ and Ψ:

Let $\delta(x)$ be the hyperbolic geodesic connecting $\Phi(x)$ to $\Psi(x)$. Since γ is a hyperbolic isometry, it isometrically maps $\delta(x)$ to $\delta(\gamma_0 x)$. Let $t \mapsto \Phi_t(x)$ be a uniform motion along $\delta(x)$ from $\Phi(x)$ to $\Psi(x)$ with such a speed that at time $t = 1$ we reach the destination (in other words, $\Phi_t(x)$ is the point on $\delta(x)$ on hyperbolic distance $t \, \mathrm{dist}_{\mathrm{hyp}}(\Phi(x), \Psi(x))$ from $\Phi(x)$). Then $\gamma(\Phi_t x) = \Phi_t(\gamma_0 x)$, and we obtain a desired equivariant homotopy. $\square$

2.4.3. *Teichmüller metric.* Let us endow the space $\mathcal{T}(S_0)$ with the following *Teichmüller metric.* Given two marked surfaces $\tau = [S, \phi]$ and $\tau' = [S', \phi']$, we let $\mathrm{dist}_\mathrm{T}(\tau, \tau')$ be the infimum of dilatations of qc maps $h : S \to S'$ that make diagram (**??**) homotopically commutative.

LEMMA 3.6. dist_T *is a metric.*

PROOF. Triangle ineaquality for dist_T follows from submultiplicativity of the dilatation under composition. So, dist_T is a pseudo-metric. Let us show that it is a metric, Indeed, if $\mathrm{dist}_T(\tau, \tau') = 0$ then there exists a sequence $h_n : S \to S'$ of qc maps in the right homotopy class with $\mathrm{Dil}(h_n) \to 0$. Let $H_n : \mathbb{H} \to \mathbb{H}$ be the lifts of the h_n that induce the same isomorphism between Γ and Γ'. Then the H_n is a sequence of qc maps with uniformly bounded dilatation whose extensions to $\mathbb{R} = \partial \mathbb{H}$ all coincide. Now Compactness Theorem 2.12 implies that the H_n uniformly converge to an equivariant conformal isomorphism $T : (\mathbb{H}, \Gamma_0) \to (\mathbb{H}, \Gamma)$. It descends to a conformal isomorphism $h : S \to S'$ in the samehomotopy class as the h_n. $\square$

EXERCISE 3.2. *Show that the embedding $e : \mathcal{T}(S_0) \to \mathrm{Rep}(\Gamma_0)$ is continuous. (from the Teichmúller metric to the algebraic topology).*

3. Bers Embedding

3.1. Schwarzian derivative and projective structures.
3.1.1. *Definition.* The fastest way to define the Schwarzian derivative Sf is by means of a mysterious formula:

$$Sf = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2. \tag{3.1}$$

However, there is a bit longer but much better motivated approach.

Let us try to measure how a function f at a non-critical point z deviates from a Möbius transformation. Möbius transformations depend on three complex parameters. So, one expects to find a unique Möbius transformation A_z that coincides with f to the second order. Then

$$f(\zeta) - A_z(\zeta) \sim \frac{b}{6}(\zeta - z)^3$$

near z, and we let $Sf(z) = b/f'(z)$.

REMARK 3.2. Division by $f'(z)$ ensures scaling invariance of the Schwarzian derivative: $S(\lambda f) = Sf$. Coefficient $1/6$ provides a convenient normalization suggested by the Taylor formula: it makes $Sf = f'''$ for a normalized map $f(\zeta) = \zeta + O(|\zeta - z|^3)$.

The best Möbius approximation to f is easy to write down explicitly. Let $f(\zeta) = a_0 + a_1(\zeta - z) + a_2(\zeta - z)^2 + \ldots$ near z with $a_1 = f'(z) \neq 0$. Then

$$A_z(\zeta) = a_0 + \frac{a_1(\zeta - z)}{1 - \beta(\zeta - z)} \quad \text{with} \quad \beta = \frac{a_2}{a_1},$$

the 3d Taylor coefficient of $f - A_z$ is $(a_3 - a_2^2/a_1)$, and (3.1) follows.

3.1.2. *Chain rule.*

LEMMA 3.7. *Let f be a holomorphic function on a domain U. Then $Sf \equiv 0$ on U if and only if f is a restrictin of a Möbius map to U.*

PROOF. Sufficiency is obvious from the definition: If f is a Möbius map then $A_z = f$ at any point z, and $Sf(z) = 0$.

Vice versa, assume $Sf \equiv 0$ on U. Then f is a solution of a 3d order analytic ODE

$$f''' = \frac{3}{2} \frac{(f'')^2}{f'}$$

on $U \smallsetminus C_f$, where C_f is the critical set of f. Such a solution is uniquely determined by its 2-jet[5] at any point $z \in U \smallsetminus C_f$. Hence $f = A_z$. □

Similarly, one can prove:

EXERCISE 3.3. *Let f and g be two holomorphic functions on a domain U. Then $Sf \equiv Sg$ on U if and only if $f = A \circ g$ for some Möbius map A.*

LEMMA 3.8 (Chain Rule).

$$S(f \circ g) = (Sf \circ g) \cdot (g')^2 + Sg. \tag{3.2}$$

PROOF. Since the Schwarzian derivative is translationally invariant on both sides (i.e., $S(T_1 \circ f \circ T_2) = Sf$ for any translations T_1 and T_2), it is sufficient to check (3.2) at the origin and to assume that $g(0) = f(0) = 0$. Furthermore, by Exercise 3.3, postcomposition of f with a Möbius transformation would not change either side of (3.2). In this way, we can bring f to a normalized form:

$$f(\zeta) = \zeta + \frac{Sf(0)}{6} \zeta^3 + \dots \tag{3.3}$$

and then painlessly check (3.2) by composing (3.3) with the 3-get of g. □

In particular, for a Möbius transformation A, we have:

$$S(f \circ A) = (Sf \circ A) \cdot (A')^2, \tag{3.4}$$

which coincides with the transformation rule for quadratic differentials. It suggests that the Schwarzian should be viewed not as a function but rather as a quadratic differential $Sf(z)dz^2$. This point of view is not quite right on Riemann surfaces, but it becomes exactly correct on projective surfaces.

---

[5]Recall that a n-jet of a function f at z is its Taylor approximant of order n at z.

3.1.3. *Projective surfaces.* A *projective structure* on a Riemann surface S is an atlas of holomorphic local charts with Möbius transit maps. A surface endowed with a projective structure is called a *projective surface.* Projective morphisms are defined naturally, so that we can refer to isomorphic projective surfaces.

Of course, the Riemann sphere $\bar{\mathbb{C}}$ has a natural projective structure, and any domain $U \subset \bar{\mathbb{C}}$ inherits it. If we have a group Γ of Möbius transformations acting properly discontinuously and freely on U then the quotient Riemann surface $V = U/\Gamma$ inherits a unique projective structure that makes the quotient map $\pi : U \to V$ projective. In particular, any Riemann surface S is endowed with the *Fuchsian* projective structure coming from the uniformization $\pi : \mathbb{H} \to V$.

Given a meromorphic function f on a projective surface V, the Chain Rule (3.4) tells us that the local expressions $Sf(z)dz^2$ determine a global quadratic differential on V.

EXERCISE 3.4. *Check carefully this assertion.*

More generally, let us consider two projective structures f and g on a Riemann surface V given by atlases $\{f_\alpha\}$ and $\{g_\beta\}$ respectively. Then the Chain Rule (more specifically, Exercises 3.3 and 3.4) tell us that the local expressions $S(f_\alpha \circ g_\beta^{-1})(z)\,dz^2$ determine a global quadratic differential on V endowed with the g-structure. This differential is denoted $S\{f, g\}$. It measures the distance between f and g.

In particular, given a holomorphic map $f : V \to W$ between two projective surfaces, we obtain a quadratic differential $S\{f^*(W), V\}$ on[6] V. Writing f in projective local coordinates $(\zeta = f(z))$, we obtain the familiar expression, $Sf(z)\,dz^2$, for this differential.

4. Appendix: Elements of infinite dimensional complex analysis

4.1. Holomorphic maps in complex Banach spaces. Let $\mathcal{B}$ and $\mathcal{D}$ be separable complex Banach spaces, and let $\mathcal{U} \subset \mathcal{B}$ be an open set in $\mathcal{B}$. A map $f : \mathcal{U} \to \mathcal{D}$ is called *holomorphic* if it is continuous and for any complex line $L \subset \mathcal{B}$ and any $\mathbb{C}$-linear functional $\phi : \mathcal{D} \to \mathbb{C}$, the function $\phi \circ f \,|\, L$ is holomorphic.

Any holomorphic map is smooth.

4.2. Basic properties. Given a Banach space $\mathcal{B}$, let $\mathcal{B}_r(x)$ stand for the ball of radius r centered at x in $\mathcal{B}$, and $\mathcal{B}_r \equiv \mathcal{B}_r(x)$.

---

[6]Here we notationally identify surfaces with their projective structures

Cauchy Inequality. *Let $f : (\mathcal{B}_1, 0) \to (\mathcal{D}_1, 0)$ be a complex analytic map between two unit Banach balls. Then $\|Df(0)\| \leq 1$. Moreover, for $x \in \mathcal{B}_1$,*

$$\|Df(x)\| \leq \frac{1}{1 - \|x\|}.$$

PROOF. Take a vector $v \in \mathcal{B}$ with $\|v\| = 1$ and a linear functional ψ on $\mathcal{D}$ with $\|\psi\| = 1$. Let is consider an analytic function $\phi : \mathbb{D}_1 \to \mathbb{D}_1$, $\phi(\lambda) = \psi(f(\lambda v))$. As $|\phi(\lambda)| < 1$, the usual Cauchy Inequality yields: $|\phi'(0)| = |\psi(Df(0)v)| \leq 1$. Since this holds for any normalized v and ψ, the former estimate follows by the Hahn-Banach Theorem.

The latter one follows from the former by restricting f to the ball $\mathcal{B}_{1-\|x\|}(x)$. $\qquad\square$

The Cauchy Inequality yields:

Schwarz Lemma. *Let $r < 1/2$ and $f : (\mathcal{B}_1, 0) \to (\mathcal{D}_r, 0)$ be a complex analytic map between two Banach balls. Then the restriction of f onto the ball $\mathcal{B}_r$ is contracting: $\|f(x) - f(y)\| \leq q\|x - y\|$, where $q = r/(1 - r) < 1$.*

PROOF. By the Cauchy Inequality, $\|Df(x)\| \leq q$ for $x \in \mathcal{B}_r$. Integrating this along the interval $[x, y]$, we obtain the desired. $\qquad\square$

Let us consider a family Φ of meromorphic functions on a Banach neighborhood $\mathcal{U} \subset \mathcal{B}$. To define normality of the family, we should be a little careful since $\mathcal{B}$ is not locally compact. So, let us say that the family Φ is *normal* if it is *locally equicontinuous*.

LEMMA 3.9. *A normal family of meromorphic functions on a Banach neighborhood $\mathcal{U}$ is sequentially pre-compact.*

PROOF. Let ϕ_n be a sequence of functions in our family. We will keep the same notation for all further subsequences.

Take some countable dense subset $X \subset U$. By the diagonal procedure, we can select a subsequence pointwise converging on X. Since the family Φ is locally equicontinuous, ϕ_n converges locally uniformly to some continuous function ϕ (by a standard argument). By the classical complex analysis, this function is holomorphic on every complex line, and hence holomorphic on $\mathcal{U}$. $\qquad\square$

Montel Theorem. *If a family of meromorphic functions $\phi_n : \mathcal{U} \to \bar{\mathbb{C}}$ on a Banach neighborhood $\mathcal{U} \subset \mathcal{B}$ does not assume three values then it is normal.*

PROOF. Since normality is a local property, we can assume that $\mathcal{U}$ is the Banach ball $\mathcal{B}_1$. Let us endow $\mathcal{B}_1$ with a hypebolic distance by letting $\rho(x, y)$ be the hyperbolic distance between x and y in the disk $L \cap \mathcal{B}_1$ of the compex line L passing through x and y. By the classical Schwarz Lemma (§**??**), the functions ϕ_n are contracting from ρ to the hyperbolic metric ρ' of the thrice punctured sphere. Since ρ restricted to any ball $\mathcal{B}_r$, $r < 1$, is equivalent to the Banach metric, while ρ' dominates the spherical metric, the local equicontinuity follows. $\square$

This result implies the Refined Montel Theorem on Banach spaces in the same way as in the one-dimensional case (§**??**).

4.3. Submanifolds and their intersections. A subset $\mathcal{X} \subset \mathcal{B}$ is called a (complex analytic) submanifold in $\mathcal{B}$ of dimension n (which can be infinite) if there is a Banach space $\mathcal{D}$ of dimension n such that for any $x \in \mathcal{X}$ there exist neighborhoods $x \in \mathcal{U} \subset \mathcal{B}$ and $0 \in \mathcal{V} \subset \mathcal{D}$ and a holomorphic map $h : (\mathcal{V}, 0) \rightarrow (\mathcal{B}, x)$ whose image is equal to $\mathcal{B} \cap \mathcal{X}$. Then the tangent space $T_x \mathcal{X}$ is defined as the image of $\mathcal{V}$ under the differential $Dh(0)$. The codimension of X at x is the codimension of $T_x \mathcal{X}$. Obviously, it is a locally constant function on $\mathcal{X}$.

Let $\mathcal{X}$ and $\mathcal{S}$ be two submanifolds in the Banach space $\mathcal{B}$ intersecting at point x. Assume that $\operatorname{codim} \mathcal{X} = \dim \mathcal{S} = 1$. Let us define the *intersection multiplicity* σ between $\mathcal{X}$ and $\mathcal{S}$ at x as follows. Select a local coordinate system (w, z) near x in such a way that $x = 0$ and $\mathcal{X} = \{z = 0\}$. Let us analytically parametrize $\mathcal{S}$ near 0: $z = z(t), w = w(t), z(0) = 0, w(0) = 0$. Then by definition, σ is the multiplicity of the root of $z(t)$ at $t = 0$.

Hurwitz Theorem. *Under the above circumstances, let us consider a submanifold $\mathcal{Y}$ of codimension 1 obtained by a small perturbation of $\mathcal{X}$. Then $\mathcal{S}$ has σ intersection points with $\mathcal{Y}$ near x counted with multiplicity.*

PROOF. Let us use the above local coordinates and parametrization. In these coordinates $\mathcal{Y}$ is a graph of a holomorphic function $z = \phi(w)$ which is uniformly small at some neighborhood of 0 (this is the meaning of $\mathcal{Y}$ being a small perturbation of $\mathcal{X}$). The intersection points of $\mathcal{Y}$ and $\mathcal{S}$ are the roots of the equation $z(t) = \phi(w(t))$. By the classical Hurwitz Theorem, this equation has exactly σ roots near the origin counted with multiplicities if ϕ is small enough. $\square$

As usual, a foliation of some analytic Banach manifold is called analytic (smooth) if it can be locally straightened by an analytic (smooth) change of variable.

Intersection Lemma. *Let $\mathcal{F}$ be a codimension one complex analytic foliation in a domain of a Banach space. Let $\mathcal{S}$ be a one-dimensional complex analytic submanifold intersecting a leaf $\mathcal{L}_0$ of the foliation at a point x with multiplicity σ. Then $\mathcal{S}$ has σ simple intersection points with any nearby leaf.*

PROOF. Let us select local coordinates (w, z) near x so that x corresponds to 0, and the leaves of the foliation near 0 are given by the equations $\mathcal{L}_\epsilon = \{z = \epsilon\}$. Let $z = z(t), w = w(t)$ be an analytic parametrization of $\mathcal{S}$, with $t = 0$ corresponding to $x = 0$. Then $z(t) = at^\sigma(1+0(t))$, $a \neq 0$, has root of multiplicity σ at 0. Clearly there is an analytic local chart $\tau = \tau(t)$ in which the curve is parametrized as exact power: $z(\tau) = \tau^\sigma$. Then for small $\epsilon \neq 0$, the equation $z(\tau) = \epsilon$ has σ simple roots near 0: $\tau_i = \epsilon^{1/\sigma}$. $\qquad\square$

COROLLARY 3.10. *Under the circumstances of the above lemma, $\mathcal{S}$ is transverse to $\mathcal{L}_0$ at x if and only if it has a single intersection point near x with all nearby leaves.*

Part 2

Complex quadratic family

Dynamical plane

12. Glossary of topological dynamics

This glossary collects some basic notions of dynamics. Its purpose is to fix terminology and notations and to comfort a reader who has no experience with dynamics.

Consider a continuous endomorphism $f : X \to X$ of a topological space X. The n-fold iterate of f is denoted by f^n, $n \in \mathbb{N}$. A *topological dynamical system* (with discrete positive time) is the $\mathbb{N}$-action generated by f, $n \mapsto f^n$. The *orbit* or *trajectory* of a point $x \in K$ is $\mathrm{orb}(x) = \{f^n x\}_{n \in \mathbb{N}}$. The subject of topological dynamics is to study qualitative behavior of orbits of a topological dynamical system.

Here is the simplest possible behavior: a point x is called *fixed* if $fx = x$. More generally, a point x is called *periodic* if it has a finite orbit, i.e., there exists a $p \in \mathbb{N}$ such that $f^p x = x$. The smallest p with this property is called the *period* of x. The orbit of x (consisting of p permutted points) is naturally called a *periodic orbit* or a *cycle* (of period p).

The asymptotic behavior of an orbit can be studied in terms of its limit set. The ω-limit set $\omega(x)$ of a point x is the set of all accumulation points of $\mathrm{orb}(x)$. If X is compact then $\omega(x)$ is a non-empty compact subset of X. We say that the orbit of x *converges to a cycle* (of a periodic point α) if $\omega(x) = \mathrm{orb}(\alpha)$.

A point x is called *recurrent* if $\omega(x) \ni x$. Existence of non-periodic recurrrent points is a feature of non-trivial dynamics.

Two dynamical systems $f : X \to X$ and $g : Y \to Y$ are called *topologically conjugate* (or topologically equivalent) if there exists a homeomorphism $h : X \to Y$ such that $h \circ f = g \circ h$, i.e., the following commutative diagram holds:

$$
\begin{array}{ccc}
X & \xrightarrow{\ f\ } & X \\
h \downarrow & & \downarrow h \\
Y & \xrightarrow{\ g\ } & Y
\end{array}
$$

Classes of topologically equivalent dynamical systems (within an *a priori* specified family) are called *topological classes.* If X and Y are endowed with an extra structure (smooth, conformal, quasi-conformal etc.) respected by h, then f and g are called *smoothly/conformally/quasi-conformally conjugate* (or equivalent). The corresponding equivalence classes are called smooth/conformal/quasi-conformal classes.

Topological conjugacies respect all properties which can be formulated in terms of topological dynamics: orbits go to orbits, cycles go to cycles of the same period, ω-limit sets go to ω-limit sets, converging orbits go to converging orbits etc.

A homeomorphism $h : X \to X$ commuting with a dynamical system $f : X \to X$ (i.e., conjugating f to itself) is called an *automorphism* of f.

A continuous map which makes the above diagram commutative is called *equivariant* (with respect to the actions of f and g). A *surjective* equivariant map is called a *semi-conjugacy* between f and g. In this case g is also called a *quotient* of f.

It will be very convenient to extend the above terminology to *partially defined* maps. Let f and g be partially defined maps on the spaces X and Y respectively (i.e., f maps its domain $\mathrm{Dom}(f) \subset X$ to X, and similarly does g). Let $A \subset X$. A map $h : A \to Y$ is called equivariant (with respect to the actions of f and g) if for any $x \in A \cap \mathrm{Dom}(f)$ such that $fx \in A$ we have: $hx \in \mathrm{Dom}(g)$ and $h(fx) = g(hx)$. (Briefly speaking, the equivariance equation is satisfied whenever it makes sense.)

13. Holomorphic dynamics: basic objects

Below
$$f \equiv f_c : z \mapsto z^2 + c$$
unless otherwise is stated. Dynamical objects will be labelled by either f or c whatever is more convenient in a particular situation (for instance, $D_f(\infty) \equiv D_c(\infty)$ by default). Moreover, the label can be skipped altogether if f is not varied.

13.1. Critical points and values. First note that f^n is a branched covering of $\mathbb{C}$ over itself of degree 2^n. Its critical points and values have a good dynamical meaning:

EXERCISE 4.1. *The set of finite critical points of f^n is $\cup_{k=0}^{n-1} f^{-k}(0)$. We let*
$$C_f = \bigcup_{n=0}^{\infty} \bigcup_{k=0}^{n-1} f^{-k}(0)$$
be the set of critical points of iterated f.

The set of critical values of f^n is $\{f^k 0\}_{k=1}^n$. (There are much fewer critical values than critical points!)

Thus, f^n is an unbranced covering over the complement of $\{f^k 0\}_{k=1}^n$.

COROLLARY 4.1. *Let V be a topological disk which does not contain points $f^k 0$, $k = 1, 2, \ldots, n$. Then the inverse function f^{-n} has 2^n single-values branches f_i^{-n} which univalently map V onto pairwise disjoint topological disks U_i, $i = 1, 2, \ldots, 2^n$.*

These simple remarks explain why the forward orbit of 0 plays a very special role. We will have many occasions to see that this one orbit is responsible for the diversity of the global dynamics of f.

However, f has one more critical point overlooked so far:

13.2. Looking from infinity. Extend f to an endomorphism of the Riemann sphere $\bar{\mathbb{C}}$. This extension has a critical point at ∞ fixed under f. We will start exploring the dynamics of f from there. The first observation is that $\bar{\mathbb{C}} \smallsetminus \mathbb{D}_R$ is f-invariant for a sufficiently big R, and moreover $f^n z \to \infty$ as $n \to \infty$ for $z \in \bar{\mathbb{C}} \smallsetminus \mathbb{D}_R$. This can be expressed by saying that $\mathbb{C} \smallsetminus \mathbb{D}_R$ belongs to the *basin of infinity* defined as the set of all escaping points:

$$D_f(\infty) = \{z : f^n z \to \infty, \ n \to \infty\} = \bigcup_{n=0}^{\infty} f^{-n}(\mathbb{C} \smallsetminus \mathbb{D}_R).$$

PROPOSITION 4.2. *The basin of infinity $D_f(\infty)$ is a completely invariant domain containing ∞.*   *def*

PROOF. The only non-obvious statement to check is connectivity of $D_f(\infty)$. To this end let us show inductively that the sets $U_n = f^{-n}(\bar{\mathbb{C}} \smallsetminus \bar{\mathbb{D}}_R)$ are connected. Indeed, assume that U_n is connected while U_{n+1} is not. Consider a bounded component V of U_{n+1}. Then the restriction $f : V \to U_n$ is proper and hence surjective (see §6). In particular f would have a pole in V - contradiction. $\square$

Let $\bar{D}_f(\infty) = D_f(\infty) \cup \{\infty\}$.

13.3. Basic Dichotomy for Julia sets. We can now introduce the fundamental dynamical object, the *filled Julia set* $K(f) = \bar{\mathbb{C}} \smallsetminus D_f(\infty)$. Proposition 4.2 implies that $K(f)$ is a completely invariant compact subset of $\mathbb{C}$. Moreover, it is *full*, i.e., it does not separate the plane (since $D_f(\infty)$ is connected).

EXERCISE 4.2. (i) *The filled Julia set consists of more than one point.* (ii) *Each component of* int $K(f)$ *is simply connected.*

picture

The filled Julia set and the basin of infinity have a common boundary, which is called the *Julia set*, $J(f) = \partial K(f) = \partial D_f(\infty)$. Figure .... shows several pictures of the Julia sets $J(f_c)$ for different parameter values c. Generally, topology and geometry of the Julia set is very complicated, and it is hard to put a hold on it. However, there is the following rough classification:

THEOREM 4.3 (Basic Dichotomy). *The Julia set (and the filled Julia set) is either connected or Cantor. The latter happens if and only if the critical point escapes to infinity: $f^n(0) \to \infty$ as $n \to \infty$.*

PROOF. As in the proof of Proposition 4.2, let us consider the increasing sequence of domains $U_n = f^{-n}(\bar{\mathbb{C}} \smallsetminus \bar{\mathbb{D}}_R)$ exhausting the basin of infinity. Assume first that the critical point does not escape to ∞. Then $f : U_{n+1} \to U_n$ is a branched double covering with the only branched point at ∞. By the Riemann-Hurwitz formula, if U_n is simply connected then U_{n+1} is simply connected as well. We conclude inductively that all the domains U_n are simply connected. Hence their union, $D_f(\infty)$, is simply connected as well, and its complement, $K(f)$, is connected. But the boundary of a full connected compact set is connected. Hence $J(f)$ is connected.

Assume now that the critical point escapes to infinity. Then 0 belongs to some domain U_n. Take the smallest n with this property. Adjust the radius R in such a way that the orbit of 0 does not pass through $\mathbb{T}_R = \partial U_0$. Then $0 \notin \partial U_{n-1}$, and hence ∂U_{n-1} is a Jordan curve. Let us consider the complimentary Jordan disk $D \equiv D^0 = \mathbb{C} \smallsetminus \bar{U}_{n-1}$. Since $f(0) \in U_{n-1}$, f is unbranched over D. Hence $f^{-1}D = D_0^1 \cup D_1^1$, where the $D_i^1 \Subset D$ are disjoint topological disks conformally mapped onto D.

Take now the f-preimages of $D_0^1 \cup D_1^1$ in D_0^1. We obtain two Jordan disks D_{00}^2 and D_{01}^2 with disjoint closures conformally mapped by f onto D_0^1 and D_1^1 repsectively. Similar disks, D_{10}^2 and D_{11}^2, we find in D_1^1 (see Figure ....).

picture

Iterating this procedure, we will find that $f^{-n}D$ is the union of 2^n Jordan disks $D_{i_0 i_1 \dots i_n}^n$ such that $D_{i_0 \dots i_n}^n$ is compactly contained in $D_{i_0 \dots i_{n-1}}^{n-1}$ and is conformally mapped by f onto $D_{i_1 \dots i_n}^{n-1}$.

Since $D_0^1 \cup D_1^1$ is compactly contained in D, the branches of the inverse map, $f^{-1} : D_i^1 \to D_{ij}^2$, are uniformly contracting in the hyperbolic metric of D (by the Schwarz-Pick Lemma). Since the domains $D_{i_0 i_1 \dots i_n}^n$ are obtained by iterating these branches, they uniformly exponentially shrink as $n \to \infty$. Hence the filled Julia set $K(f) = \cap f^{-n}D$ is a Cantor set. Of course, the Julia set $J(f)$ coincides with $K(f)$ in this case. $\square$

The Basic Dichotomy is the first example of how the behavior of the critical point influences the global dynamics. In fact, at least on the philosophical level, the dynamics is completely determined by the behavior of this single point. We will see many confirmations of this principle.

EXERCISE 4.3. (i) $J(f)$ *is connected if and only if* $K(f)$ *is connected.* (ii) *Both sets do not contain isolated points, and are always uncountable.*

13.4. Bernoulli shift. When the Julia set is Cantor, there is an explicit symbolic model for the dynamics of f on it. Consider the space $\Sigma \equiv \Sigma_2^+$ of one-sided sequences $(i_0 i_1 \dots)$ of zeros and ones. Supply it with the weak topology (convergence in this topology means that all coordinates eventually stabilize). We obtain a Cantor set. Define the shift β on this space as the map of forgetting the first coordinate,

$$\beta : (i_0 i_1 \dots) \mapsto (i_1 i_2 \dots).$$

It is called the *(one-sided) Bernoulli shift* with two states.

EXERCISE 4.4. *Show that:*
- *For any open set* $U \subset \Sigma$, *there exists an* $n \in \mathbb{N}$ *such that* $\beta^n(U) = \Sigma$;
- β *is topologically transitive;*
- *Periodic points of* β *are dense in* Σ.

EXERCISE 4.5. *Show that the only non-trivial automorphism of the one-sided Bernoulli shift with two states is induced by the relabeling* $0 \leftrightarrow 1$.

If some endomorphism $f : X \to X$ of a compact space is topologically conjugate to a one-sided Bernoulli shift with two states, then X can be partitioned into two pieces X_0 and X_1 corresponding to sequences which begin with 0 and 1 respectively. This partition is called a *Bernoulli generator* for f. The statement of Exercise 4.5 is equivalent to saying that a Bernoulli generator is unique. For a Cantor Julia set $J(f_c)$, the Bernoulli generator was constructed in the course of the proof of Theorem 4.3:

EXERCISE 4.6. *If* $J(f)$ *is a Cantor set, then the restriction of* f *onto* $J(f)$ *is topologically conjugate to the one-sided Bernoulli shift with two states.*

13.5. Real dichotomy. In the case of real parameter values c, the Bernoulli coding of $J(f_c)$ becomes particularly nice:

EXERCISE 4.7. *Consider a quadratic polynomial $f_c : z \mapsto z^2 + c$ with a real c. Let $J \equiv J(f_c)$.*

- *If $c < -2$ then J is a Cantor set on the real line. In this case the Bernoulli generator for f_c consists of*

$$J_0 = J \cap \{z : \Re z < 0\} \ and \ J_1 = J \cap \{z : \Re z > 0\}.$$

picture

- *If $c > 1/4$ then J is a Cantor set disjoint from the real line. In this case the Bernoulli generator for f_c consists of*

$$J_0 = J \cap \{z : \operatorname{Im} z > 0\} \ and \ J_1 = J \cap \{z : \operatorname{Im} z < 0\}.$$

The boundary parameter values $c = 1/4$ and $c = -2$ play a special role in one-dimensional dynamics (both real and complex).

The former map ($c = 1/4$) is specified by the property that it has a multiple fixed point $\alpha = \beta = 1/2$, i.e., $f_c(\alpha) = \alpha$, $f'_c(\alpha) = 1$. The Julia set of this map is a Jordan curve depicted on Figure ... (see §?? for an explanation of some features of this picture). It is called *cauliflower*, and the map $f_c : z \mapsto z^2 + 1/4$ itself is sometimes called the *cauliflower map*.

The latter map ($c = -2$) is specified by the property that the second iterate of the critical point is fixed under f_c: $0 \mapsto -2 \mapsto 2 \mapsto 2$ (see Figure ...). This map is called *Chebyshev* or *Ulam-Neumann*. The Julia set of this map is unusually simple:

EXERCISE 4.8 (Chebyshev map). *Let $f \equiv f_{-2} : z \mapsto z^2 - 2$.*

- *The interval $I = [-2, 2]$ is completely invariant under f, i.e., $f^{-1}I = I$.*
- *$J(f) = I$. (To show that all points in $\mathbb{C} \smallsetminus I$ escape to ∞, use Montel's Theorem.)*
- *Consider the the sawlike map*

$$g : [-1, 1] \to [-1, 1], \quad g : x \mapsto 2|x| - 1.$$

Show that the map $h : x \mapsto 2 \sin \frac{\pi}{2} x$ conjugates g to $f|I$.
- *The map $f|I$ is nicely semi-conjugate to the one-sided Bernoulli shift $\sigma : \Sigma \to \Sigma$. Namely, there exists a natural semi-conjugacy $h : \Sigma \to I$ such that card $f^{-1}x = 1$ for all $x \in I$ except countable many points (preimages of the fixed point $\beta = 2$ under iterates of f). For these special points, card $f^{-1}(x) = 2$.*

Let us finish with a statement which will complete our discussion of the Basic Dichotomy for real parameter values:

EXERCISE 4.9.   (i) *For $c \in (-\infty, 1/4)$, the map f_c has two real fixed points $\alpha_c < \beta_c$. (We have already observed that these*

> *two points collide at $1/2$ when $c = 1/4$.) Point β_c is always repelling.*
> (ii) *For $c \in [-2, 1/4]$, the interval $I_c = [-\beta_c, \beta_c]$ is invariant under f_c, and it is the maximal f_c-invariant interval on the real line.*
> (iii) *For $c \in [-2, 1/4]$, the critical point is non-escaping and hence the Julia set $J(f_c)$ is connected.*

The above fixed points, α_c and β_c, will be called α- and β-fixed points respectively. As one can see from the second item of the above Exercise, they play quite a different dynamical role. In §**??** a similar classification of the fixed points will be given for any quadratic polynomial with connected Julia set.

Let us summarize Exercises 4.7 and 4.9:

PROPOSITION 4.4. *For real c, the Julia set $J(f_c)$ is connected if and only if $c \in [-2, 1/4]$.*

13.6. Fatou set. The Fatou set is defined as the complement of the Julia set:

$$F(f) = \bar{\mathbb{C}} \setminus J(f) = D_f(\infty) \cup \operatorname{int} K(f).$$

Since $K(f)$ is full, all components of $\operatorname{int} K(f)$ are simply connected. Only one of them can contain the critical point. Such a component (if exists) is called *critical*.

Let U be one of the components of $\operatorname{int} K$. Since $\operatorname{int} K$ is invariant, it is mapped by f to some other component V. Moreover, $f(\partial U) \subset \partial V$ since the Julia set is also invariant. Hence $f : U \to V$ is proper, and thus surjective. Moreover, since V is simply connected, $f : U \to V$ is either a conformal isomorphism (if U is not critical), or is a double branched covering (if U is critical).

The Fatou set can be also characterized as *the set of normality* (and was actually classically defined in this way):

PROPOSITION 4.5. *The Fatou set $F(f)$ is the maximal set on which the family of iterates f^n is normal.*

PROOF. On $D_f(\infty)$, the iterates of f locally uniformly converge to ∞, while on $\operatorname{int} K(f)$ they are uniformly bounded. Hence they form a normal family on $F(f)$. On the other hand, if $z \in J(f)$, then the orbit of z is bounded while there are nearby points escaping to ∞. Hence the family of iterates is not normal near z.   □

13.7. Postcritical set. Let $O_f = \operatorname{cl}\{f^n(0)\}_{n=1}^{\infty}$ stand for the *postcritical set* of f. It is forward invariant and contains the critical value c of f. The map f is tremendously contracting near the critical point 0,

and under iteration this contraction propagates through the postcritical set. The following lemma is the first indication that otherwise, the map f tends to be expanding:

LEMMA 4.6. *Let $c \neq 0$. Then the complement $\mathbb{C} \smallsetminus O_f$ is hyperbolic[1].
Let Ω be a component of $\mathbb{C} \smallsetminus O_f$ that intersects $f^{-1}(O_f) \smallsetminus O_f$. Then
f on Ω is strictly expanding with respect to this hyperbolic metric, i.e,
for any $z \in \Omega \smallsetminus f^{-1}(O_f)$, $\|Df(z)\|_{\mathrm{hyp}} > 1$.*

PROOF. If $\mathbb{C} \smallsetminus O_f$ is not hyperbolic, then O_f consists of a single
point, c. But then $f(c) = c$ and hence $c = 0$.

Let ρ and ρ' be the hyprebolic metrics on $\mathbb{C} \smallsetminus O_f$ and $\mathbb{C} \smallsetminus f^{-1}(O_f)$
respectively. Since the map $f : \mathbb{C} \smallsetminus f^{-1}O_f \to \mathbb{C} \smallsetminus O_f$ is a covering, it
is a local isometry from ρ' to ρ.

Let Ω' be the component of $\mathbb{C} \smallsetminus f^{-1}(O_f)$ containing z. Since O_f
is forward invariant, $\Omega' \subset \Omega$, and by the assumption of the lemma,
Ω' is strictly smaller than Ω. By the Schwarz Lemma, the natural
emebdding $i : \Omega' \to \Omega$ is strictly contracting from ρ' to ρ. Thus, the
inverse map $i^{-1}|\Omega'$ is strictly expanding from ρ to ρ', and we conclude
that the composition $f \circ i^{-1} : \Omega' \to \mathbb{C} \smallsetminus O_f$ is strictly expanding with
respect to ρ.   □

13.8. Higher degree polynomials. The above basic definitions
and results admit a straightforward extension to higher degree polynomials

$$f : z \mapsto a_0 z^d + a_1 z^{d-1} + \cdots + a_d, \quad d \geq 2, \ a_0 \neq 1.$$

The following point should be kept in mind though: the Basic Dichotomy is not valid any more in the higher degree case. Instead, there
is the following partial description of the topology of the Julia set:

• The Julia set $J(f)$ (and the filled Julia set $K(f)$) is connected if
and only all the critical points c_i are non-escaping to ∞, i.e., $c_i \in K(f)$.

• If all the critical points escape to ∞, then $J(f)$ is a Cantor set.

EXERCISE 4.10. *Work out the basic dynamical definitions and results in the case of higher degree polynomials.*

However, the Basic Dichotomy is still valid in the case of *unicritical*
polynomials, that is, the ones that have a single critical point. (Note
that any such polynomial is affinely conjugate to $z \mapsto z^d + c$.)

In the theory of quadratic maps f_c, higher degree polynomials still
appear as the iterates of f_c. It is useful to know that they have the
same Julia set:

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[1]Meaning that each component of $\mathbb{C} \smallsetminus O_f$ is hyperbolic.

EXERCISE 4.11. *Show that $K(f^n) = K(f)$ for any polynomial f.*

14. Periodic motions

Poincaré said that ...

14.1. Rough classification of periodic points by the multiplier. Consider a periodic point α of period p. The local dynamics near its cycle $\boldsymbol{\alpha} = \{f^n\alpha\}_{n=0}^{p-1}$ depends first of all on its *multiplier* $\sigma = (f^p)'(z)$. The point (and its cycle)[2] is called *attracting* if $|\sigma| < 1$, A particular case of an attracting point is a *superattracting* one when $\sigma = 0$. In this case, the critical point 0 belongs to the cycle $\boldsymbol{\alpha}$. (When we want to emphasize that an attracting periodic point is not superattracting, we call it *simply* attracting.)

A periodic point is called *repelling* if $|\sigma| > 1$, and *neutral* if $\sigma = e^{2\pi i\theta}$, $\theta \in \mathbb{R}/\mathbb{Z}$. In latter case, θ is called the *rotation number* of α. Local dynamics near a neutral cycle depends delicately on the arithmetic of the rotation number. A neutral point is called *parabolic* if the rotation number is rational, $\theta = r/q$, and is called *irrational* othewise. An irrational periodic point can be of *Siegel* and *Cremer* type, to be defined below.

Let us consider these case one by one.

14.2. Attracting cycles. Let $\boldsymbol{\alpha}$ be an attracting cycle. The orbits of all nearby points uniformly converge to $\boldsymbol{\alpha}$ and, in particular, are bounded. It follows that attracting cycles belong to $F(f)$. The rate of convergence is exponential in the simply attracting case and superexponential in the superattracting case.

For a simply attracting periodic point α, we say that a smooth (open) disk $P \ni \alpha$ is a *petal* of α if $f|P$ is univalent and $f(P) \Subset P$. (For instance, one can take a small round disk $\mathbb{D}(\alpha, \epsilon)$ as a petal.) Then the annulus $A = \bar{P} \smallsetminus f^p(P)$ is called a *fundamental annulus* of α.

In the superattracting case, a *petal* is a smooth disk $P \ni \alpha$ such that $f^p : P \to f^p(P)$ is a branched covering of degree d (with a single critical point at α), and $f(P) \Subset P$. (For instance, one can let P be the component of $f^{-p}(\mathbb{D}(\alpha, \epsilon))$ containing α.) The corresponding *fundamental annulus* is $\bar{P} \smallsetminus f^p(P)$.

The *basin of attraction* of an attracting cycle $\boldsymbol{\alpha}$ is the set of all points whose orbits converge to $\boldsymbol{\alpha}$:

$$D(\boldsymbol{\alpha}) = D_f(\boldsymbol{\alpha}) = \{z : \ f^n z \to \boldsymbol{\alpha} \text{ as } n \to \infty.\}$$

---

[2]All the terminology introduced for periodic points applies to their cycles, and vice versa.

EXERCISE 4.12. *Show that the basin $D(\boldsymbol{\alpha})$ is a completely invariant union of components of* $\operatorname{int} K(f)$.

The union of components of $D(\boldsymbol{\alpha})$ containing the points of $\boldsymbol{\alpha}$ is called the *immediate basin* of attraction of the cycle $\boldsymbol{\alpha}$. We will denote it by $D^0 = D_f^0(\boldsymbol{\alpha})$. The component of $D^0(\boldsymbol{\alpha})$ containing α will be denoted $D^0(\alpha) = D_f^0(\alpha)$.

EXERCISE 4.13. (i) *The immediate basin of an attracting cycle consists of exactly p components, where p is the period of α.*

(ii) *Show that it can be constructed as follows. Let P_0 be a petal of α and let P_n be defined inductively as the component of $f^{-p}(P_n)$ containing α. Then $P_0 \subset P_1 \subset P_2 \subset \ldots$, and*

$$D^0(\alpha) = \bigcup_{n=0}^{\infty} P_n.$$

We will now state one of the most important facts of the classical holomorphic dynamics:

THEOREM 4.7. *The immediate basin of attraction $D_f^0(\boldsymbol{\alpha})$ of an attracting cycle $\boldsymbol{\alpha}$ contains the critical point 0. Moreover, if $\boldsymbol{\alpha}$ is simply attracting then the critical orbit $\operatorname{orb}(0)$ crosses any fundamental annulus A.*

REMARK 4.1. Of cource, the assertion is trivial when $\boldsymbol{\alpha}$ is superattracting as $0 \in \boldsymbol{\alpha}$ in this case.

PROOF. Otherwise f^p would conformally map each component D of the immediate basin onto itself. Hence it would be a hyperbolic isometry of D, despite the fact that $|f'(\alpha)| < 1$.

To prove the second assertion (which would also give another proof of the first one), let us consider a petal P_0 containing some point α of $\boldsymbol{\alpha}$, and let us define P_n inductively as the component of $f^{-p}(P_{n-1})$ containing α. (compare with Exercise 4.13 above). Then $P_0 \subset P_1 \subset P_2 \subset \ldots$. If non of these domains contains a critical point of f^p, then the all the maps $f^p : P_n \to P_{n-1}$ are isomorphisms and all the P_n all be topological disks. Hence their union, P_∞, is a topological disk as well, and $f^p : P_\infty \to P_\infty$ is an automorphism. Hence it is a hyperbolic isometry contradicting the fact that α is attracting.

Hence some P_n contains a critical point of f^p. Take the first such n (obviously, $n \geq 1$). Then $P_{n-1} \smallsetminus P_n$ contains a critical value of f^p, which is contained in $\operatorname{orb}(0)$. Applying further iterates of f^p, we will bring it to the fundamental annulus. $\square$

COROLLARY 4.8. *A quadratic polynomial can have at most one attracting cycle. If it has one, all other cycles are repelling.*

PROOF. The first assertion is immediate. For the second one, notice that under the circumstances, the postcritical set O_f is a discrete set accumulating on the attracting cycle $\boldsymbol{\alpha}$. Hence it does not divide the complex plane, and $0 \in \mathbb{C} \smallsetminus O_f$. Applying Lemma 4.6, we conclude that $|\sigma(\beta)| = \|Df^q(\beta)\|_{\text{hyp}} > 1$ for any other periodic point β of period q. $\qquad\square$

Of course, the period of this cycle can be arbitrary big. A quadratic polynomial is called *hyperbolic* if it either has an attracting cycle, or if its Julia set is Cantor. (The unifying property is that for hyperbolic maps, orb(0) coverges to an attracting cycle in the Riemann sphere.) For instance, polynomials $z \mapsto z^2$, $z \mapsto z^2 - 1$,... (see Figure ...) are hyperbolic. Though dynamically non-trivial, it is a well understood class of quadratic polynomials (see §**??**).

14.3. Parabolic cycles.
14.3.1. *Leau-Fatou Flower.* Let us consider a parabolic germ

$$f : z \mapsto e^{2\pi r/q}z + az^2 + \dots$$

with rotation number r/q near the origin.

EXERCISE 4.14. *The first non-vanishing term of the expansion $f^q(z) = z + b_k z^k + \dots$ has order $k = ql + 1$ for some $l \in \mathbb{N}$.*

In the case $l = 1$, the parabolic germ f is called *non-degenerate*.

An open Jordan disk P is called an *attracting petal* for f if:

- $0 \in \partial P$;
- $f^q(P) \subset P$ and $f^q | P$ is univalent;
- 0 is the only point where ∂P and $\partial(f^q P)$ touch;
- If $z \in P$ then $f^{qn} z \to 0$ as $n \to \infty$[3].

Given such a petal, the set $\bar{P} \smallsetminus \overline{f(P)}$ is called an attracting *fundamental crescent.*

We say that a petal P has *wedge* γ at 0 if both local branches of the boundary $\partial P \smallsetminus \{\alpha\}$ have tangent lines at 0 that meet at angle γ.

Two attracting petals are called *equivalent* if they overlap[4].

---

[3]It is convenient to impose this condition, though in fact it can be derived from the the other properties.

[4]At the moment, it is not evident that this is an equivalence relation, but the following theorem shows that it is.

THEOREM 4.9. *There is a choice of disjoint lq petals P_i (one in each class) with wedge $2\pi/ql$ at 0 such that the flower $\Phi = \bigcup P_i$ is invariant under rotation by $2\pi/ql$ and under f. The orbits is $z \in \Phi$ converge to 0 locally uniformly. Vice versa, if some orbit $\operatorname{orb}(z)$ converges to 0 without direct landing at 0 then eventually it lands in the flower Φ.*

PROOF. The proof will be split in several cases. The main analysis happens in the following one:

The germ f is non-degenerate with zero rotation number. Thus $f : z \mapsto z + az^2 + \dots$, $a \neq 0$. Conjugating f by complex scaling $\zeta = az$ we make $a = 1$.

Let us move the fixed point to ∞ by inversion $Z = -\dfrac{1}{z}$. It brings f to the form

$$F : Z \mapsto Z + 1 + O(\frac{1}{Z}) \qquad (14.1)$$

near ∞. It is obvious from this asymptotical expression that any right half-plane

$$Q_t = \{Z : \Re Z > t\}$$

with $t > 0$ sufficiently big is invariant under F, and in fact

$$F(Q_t) \subset Q_{t+1-\epsilon}, \qquad (14.2)$$

where $\epsilon = \epsilon(t) \to 0$ as $t \to \infty$. So, such a half-plane provides us with a petal with wedge π at ∞. Moreover,

$$\Re(F^n Z) \geq \Re Z + (1 - \epsilon)n, \qquad (14.3)$$

so the orbits in Q_t converge to ∞ locally uniformly.

Vice versa, if $F^n Z \to \infty$ without direct landing at it, then due to asymptotical expression (14.1) we eventually have $\Re(F^{n+1}Z) \geq \Re(F^n Z) + 1 - \epsilon$. Hence $\Re(F^n Z) \to +\infty$ and $\operatorname{orb} z$ eventually lands in the halh-plane Q_t.

Now we would like to enlarge Q_t to a petal P with wedge 2π at ∞. To this end let us consider two logarithmic curves

$$\Gamma_\pm = \{Y = \pm C\log(t - X + 1) + R)\}, \quad X \leq t, \text{ where } Z = X + iY.$$

If R is big enough then $\Gamma_\pm$ lie in the domain where the asymptotics (14.1) applies. If C is big enough then the half-slope of these curves is bigger (in absolute value) than the slope of $F(Z) - Z$. It follows that F moves the curves $\Gamma_\pm$ to the right, and the region P bounded by these curves and the segment of the vertical line $\Re Z = t$ in between is mapped univalently into itself. This is the desired petal.

Let f *be a general parabolic germ has zero rotation number.* Thus $f : z \mapsto z + bz^{k+1} + \dots$ with $k \geq 1$, $b \neq 0$. Again, conjugating f by a complex scaling $\zeta = \lambda z$, where $\lambda^k = b$, we make $b = 1$.

Let us now use a non-invertible change of variable $\zeta = z^k$. A formal calculation shows that it conjugates f to a multi-valued germ $g : \zeta \mapsto \zeta + \zeta^2 + O(|\zeta|^{2+1/k})$, where the residual term is given by a power series in $\zeta^{1/k}$. Making now a change of variable $Z = -1/z$ we come up with a multi-valued germ $G : Z \mapsto Z + 1 + O(1/Z^{1/k})$ near ∞. Let us take any single-valued branch of this germ in the slit plane $\mathbb{C} \smallsetminus \mathbb{R}_0$. Then the same considerations as in the non-degenerate case show that G has a petal P with wedge 2π at ∞. Lifting this petal to the z-plane, gives us k petals of f with wedge $2\pi/k$. $\qquad\square$

Let us now consider a parabolic periodic point α with period p and rotation number r/q. As the following exercise implies, $\alpha \in J(f)$:

EXERCISE 4.15. *Show that* $(f^{pqn})'(\alpha) \to \infty$.

The basin of attraction of a parabolic cycle $\boldsymbol{\alpha}$ is defined as follows:

$$D_f(\boldsymbol{\alpha}) = \{z : f^n z \to \boldsymbol{\alpha} \text{ as } n \to \infty \text{ but } f^n z \notin \boldsymbol{\alpha} \text{ for any } n \in \mathbb{N}^*.\}$$

It turns out that with this definition, $D_f(\boldsymbol{\alpha})$ is a completely invariant union of components of $\operatorname{int} K$. Moreover, among these components there are pl components cyclically permuted by f, while all others are preimages of these. The union of these pl components is called the the *immediate basin* of attraction of $\boldsymbol{\alpha}$. It will also be denoted as $D_f^0(\boldsymbol{\alpha})$.

As in the attracting case, the immediate basin of a parabolic cycle also must contain the critical point.

As in the hyperbolic case, we now conclude:

COROLLARY 4.10. *A quadratic polynomial can have at most one parabolic cycle. If it has one, all other cycles are repelling.*

Such a quadratic polynomial is naturally called *parabolic*.

14.4. Repelling cycles. Let us now consider a repelling cycle $\boldsymbol{\alpha} = \{f^k \alpha\}_{k=0}^{p-1}$. Nearby points escape (exponentially fast) from a small neighborhood of $\boldsymbol{\alpha}$, which implies that the family of iterates f^n is not normal near $\boldsymbol{\alpha}$. Hence repelling periodic points belong to the Julia set. In fact, as we are about to demonstrate, they are dense in the Julia set, so that the Julia can be alternatively defined as the closure of repelling cycles. It gives us a view of the Julia set "from inside".

But first, let us now show that almost all cycles are repelling:

LEMMA 4.11. *A quadratic polynomial may have at most two non-repelling cycles.*

PROOF. Let α_{o} be a neutral periodic point of period p with multiplier σ_{o} of a quadratic polynomial $f_{\mathsf{o}} : z \mapsto z^2 + c_{\mathsf{o}}$. Due to Lemma 4.10, we can assume that $\sigma_{\mathsf{o}} \neq 1$. Then by the Implicit Function Theorem, the equation $f^p(z) = z$ has a local holomorphic solution $z = \alpha_c$ assuming value α_{o} at c_{o}. The multiplier of this periodic point, $\sigma_c = (f^p)'(\alpha_c)$ is also a local holomorphic function of c. In fact, it is a global algebraic function. So, if it was locally constant then it would be globally constant, and the map $f_0 : z \mapsto z^2$ would have a neutral cycle. Since this is not the case, the multiplier is not constant, and hence near c_{o} it assumes all values in some neighborhood of σ_{o}. In particular, it assumes values $|\sigma| < 1$. Moreover, if near c_{o}

$$\sigma(c) = \sigma_0 + a(c - c_{\mathsf{o}})^k + \dots, \quad a \neq 0,$$

then the set $\{c : |\sigma(c)| < 1\}$ is the union of k sectors that asymptotically occupy $1/2$ of the area of a small disk $\mathbb{D}(c_{\mathsf{o}}, \epsilon)$. It follows that if we take three of such multiplier functions, then two of them must have overlapping sectors, so that the corresponding two cycles can be made simultaneously attracting, contradincting Theorem 4.7. □

THEOREM 4.12. *The Julia set is the closure of repelling cycles.*

PROOF. Let us first show that any point of the Julia set can be approximated by a periodic point. Let $z \in J(f)$ be a point we want to approximate. Since the Julia set does not have isolated points, we can assume that z is not the critical value. Then in a small neighborhood $U \ni z$, there exist two branches of the inverse function, $\phi_1 = f_1^{-1}$ and $\phi_2 = f_2^{-1}$. Since the family of iterates is not normal in U, one of the equations, $f^n z = z$, $f^n z = \phi_1(z)$, or $f^n z = \phi_2(z)$, has a solution in U for some $n \geq 1$ (by the Refined Montel Theorem (1.14)). If it is an equation of the first series, we find in U a periodic point of period n. Otherwise, we find a periodic point of period $n + 1$.

Since by Lemma 4.11, almost all periodic points are repelling, we come to the desired conclusion. □

14.5. Siegel cycles. Irrational periodic points may or may not belong to the Julia set (depending primarily on the Diophantine properties of its rotaion number). Irrational periodic points lying in the Fatou set are called *Siegel*, and those lying in the Julia set are called *Cremer*. The component of $F(f)$ containing a Siegel point is called a *Siegel disk*. Local dynamics on a Siegel disk is quite simple:

PROPOSITION 4.13. *Let U be a Siegel disk of period p containing a periodic point α with rotation number θ. Then $f^p|U$ is conformally conjugate to the rotation of $\mathbb{D}$ by θ.*

PROOF. Consider the Riemann map $\phi : (U, \alpha) \to (\mathbb{D}, 0)$. Then $g = \phi \circ f^p \circ \phi^{-1}$ is a holomorphic endomorphism of the unit disk fixing 0, with $|g'(0)| = |\lambda| = 1$. By the Schwarz Lemma, $g(z) = \lambda z$. $\qquad \square$

We will see later on that *a quadratic polynomial can have at most one non-repelling cycle* (see theorem 4.29). If it has one, it can be non-contradictory classified as either hyperbolic, or parabolic, or Siegel, or Cremer.

14.6. Periodic components. The notions of a periodic component of $F(f)$ and its cycle are self-explanatory. It is classically known that such a component is always associated with a non-repelling periodic point:

THEOREM 4.14. *Let* $\mathbf{U} = \{U_i\}_{i=1}^p$ *be a cycle of periodic components of* int $K(f)$. *Then one of the following three possibilities can happen:*
- **U** *is the immediate basin of an attracting cycle;*
- **U** *is the immediate basin of a parabolic cycle* $\boldsymbol{\alpha} \subset \partial\mathbf{U}$ *of some period* $q|p$;
- **U** *is the cycle of Siegel disks.*

Proof Take a component U of the cycle $\mathbf{U}$, and let $g = f^p$. By the Schwarz-Pick Lemma, $g|U$ is either a conformal automorphism of U, or it strictly contracts the hyperbolic metric dist$_h$ on U. In the former case, it is either elliptic, or otherwise. If g is elliptic then U is a Siegel disk. Otherwise the orbits of g converge to the boundary of U.

Let us show that if an orbit $\{z_n = g^n z\}$, $z \in U$, converges to ∂U, then it converges to a g-fixed point $\beta \in \partial U$. Join z and $g(z)$ with a smooth arc γ, and let $\gamma_n = f^n \gamma$. By the Schwarz-Pick Lemma, the hyperbolic length of the arcs γ_n stays bounded. Hence they uniformly escape to the boundary of U. Moreover, by the relation between the hyperbolic and Euclidean metrics (Lemma 1.19), the Euclidean length of the γ_n shrinks to 0. In particular,

$$|g(z_n) - z_n| = |z_{n+1} - z_n| \to 0 \qquad (14.4)$$

as $n \to \infty$. By continuity, all limit points of the orbit $\{z_n\}$ are fixed under g. But g being a polynomial has only finitely many fixed points. On the other hand, (14.4) implies the ω-limit set of the orbit $\{z_n\}$ is connected. Hence it consists of a single fixed point β.

Moreover, the orbit $\{\zeta_n\}$ of any other point $\zeta \in U$ must converge to the same fixed point β. Indeed, the hyperbolic distance between z_n and ζ_n stays bounded and hence the Euclidean distance between these points shrink to 0.

Thus either U is a Siegel disk, or the g-orbits in U converge to a g-fixed point β, or the map $g : U \to U$ strictly contracts the hyperbolic metric and its orbits do not escape to the boundary ∂U. Let us show that in the latter case, g has an attracting fixed point α in U.

Take a g-orbit $\{z_n\}$, and let $d_n = \mathrm{dist}_h(z_0, z_n)$. Since g is strictly contracting,

$$\mathrm{dist}_h(z_{n+1}, z_n) \le \rho(d_n)\, \mathrm{dist}_h(z_n, z_{n-1}),$$

where the contraction factor $\rho(d_n) < 1$ depends only on $\mathrm{dist}_h(z_n, z_0)$. Since the orbit $\{z_n\}$ does not escape to ∂U, this contraction factor is bounded away from 1 for infinitely many moments n, and hence $\mathrm{dist}_h(z_{n+1}, z_n) \to 0$. It follows that any ω-limit point of this orbit in U is fixed under g.

By strict contraction, g can have only one fixed point in U, and hence any orbit must converge to this point. Strict contraction also implies that this point is attracting.

We still need to prove the most delicate property: in the case when the orbits escape to the boundary point $\beta \in \partial U$, this point is parabolic. In fact, we will show that $g'(\beta) = 1$. Of course, this point cannot be either repelling (since it attracts some orbits) or attracting (since it lies on the Julia set). So it is a neutral point with some rotation number $\theta \in [0, 1)$. The following lemma will complete the proof.

LEMMA 4.15 (Necklace Lemma). *Let $f : z \mapsto \lambda z + a_2 z^2 + \dots$ be a holomorphic map near the origin, and let $|\lambda| = 1$. Assume that there exists a domain $\Omega \subset \mathbb{C}^*$ such that all iterates f^n are well-defined on Ω, $f(\Omega) \cap \Omega \ne \emptyset$, and $f^n(\Omega) \to 0$ as $n \to \infty$. Then $\lambda = 1$.*

PROOF. Consider a chain of domains $\Omega_n = f^n \Omega$ convergin to 0. Without loss of generality we can assume that all the domains lie in a small neighborhood of 0 and hence the iterates $f^n | \Omega$ are univalent. Fix a base point $a \in \Omega$ such that $f(a) \in \Omega$, and let

$$\phi_n(z) = \frac{f^n(z)}{f^n(a)}.$$

These functions are univalent, normalized by $\phi_n(a) = 1$, and do not have zeros. By the Koebe Distortion Theorem (the version given in Exercise 1.28,b), they form a normal family. Moreover, any limit function ϕ of this family is non-constant since $\phi(fa) = \lambda \ne 1 = \phi(a)$. Hence the derivatives $\phi_n'|\Omega$ are bounded away from 0 and $\mathrm{dist}(1, \partial\Omega_n) \ge \epsilon > 0$ for all $n \in \mathbb{N}$. It follows that

$$\mathrm{dist}(f^n a, \partial\Omega_n) \ge \epsilon\, r_n, \quad n \in \mathbb{N},$$

where $r_n = |f^n a|$. On the other hand, f acts almost as the rotation by θ near 0, where $\theta = \arg \lambda \in (0, 1)$. Since this rotation is recurrent and $\theta \neq 0$, there exists an $l > 0$ such that

$$\operatorname{dist}(f^{n+l}a, f^n a) = o(r_n) \quad \text{as } n \to \infty$$

The last two estimates imply that $\Omega_{n+l} \cap \Omega_n \neq \emptyset$ for alll sufficiently big n.

Hence the chain of domains $\Omega_n, \ldots, \Omega_{n+l}$ closes up, and their union form a "necklace" around 0. Take a Jordan curve γ in this necklace, and let D be the disk bounded by γ. Then $f^n(\gamma) \to 0$ as $n \to \infty$. By the Maximum Principle, $f^N(D) \Subset D$ for some N. By the Schwarz Lemma, $|\lambda| < 1$ – contradiction. $\qquad \square$

15. Quasi-conformal deformations

15.1. Idea of the method.

15.1.1. *Pullbacks.* Consider a K-quasi-regular branched covering $f : S \to S'$ between Riemann surfaces (see §7.4). Then any conformal structure μ on S' can be *pulled back* to a structure $\nu = \mathbf{f}^*(\mu)$ on S. Indeed, quasi-regular maps are differentiable a.e. on S with non-degenerate derivative so that we can let $\nu(z) = (Df(z)^{-1})_*(\mu)$ for a.e. $z \in S$. This structure has a bounded dilatation:

$$\frac{\|\nu\|_\infty + 1}{\|\nu\|_\infty - 1} \leq K \frac{\|\mu\|_\infty + 1}{\|\mu\|_\infty - 1}.$$

If f is holomorphic then in any conformal local charts near z and $f(z)$ we have:

$$f^* \mu(z) = \frac{\overline{f'(z)}}{f'(z)} \mu(fz)$$

(since the critical points of f are isolated, this expression makes sence a.e.). An obvious (either from this formula or geometrically) but crucial remark is that *holomorphic pull-backs preserve dilatation of conformal structures.*

15.1.2. *Qc surgeries and deformations.* Consider now a qr map $f : \mathbb{C} \to \mathbb{C}$ *preserving* some conformal structure μ on $\bar{\mathbb{C}}$. By the Measurable Riemann Mapping Theorem, there is a qc homeomorphism $h_\mu : \mathbb{C} \to \mathbb{C}$ such that $(h_\mu)_*(\mu) = \sigma$. Then $f_\mu = h_\mu \circ f \circ h_\mu^{-1}$ is a quasi-regular map preserving the standard structure σ on $\bar{\mathbb{C}}$. By Weil's Lemma, f_μ is holomorphic outside its critical points. Since the isolated singularities are removable, f_μ is holomorphic everywhere, so that it is a rational endormorphism of the Riemann sphere. Of course, $\deg(f_\mu) = \deg(f)$. Since h_μ is unique up to post-composition with a

Möbius map, $f = f_\mu$ is uniquely determined by μ up to conjugacy by a Möbius map.

Thus, a qc invariant view of a rational map of the Riemann sphere is a *quasi-regular endomorphism* $f : (S^2, \mu) \to (S^2, \mu)$ *of a qc sphere* S^2 *which preserves some conformal structure* μ. This provides us with a powerful tool of holomorphic dynamics: the method of qc surgery. The recepie is to cook by hands a quasi-regular endomorphism of a qc sphere with desired dynamical properties. If it admits an invariant conformal structure, then it can be realized as a rational endomorphism of the Riemann sphere.

It may happen that f itself is a rational map preserving a non-trivial conformal structure μ. Then f_μ is called a *qc deformation of f*. If f is polynomial, then let us normalize h_μ so that it fixes ∞. Then $f_\mu^{-1}(\infty) = \infty$ and hence the deformation f_μ is polynomial as well. If $f : z \mapsto z^2 + c$ is quadratic then let us additionally make h_μ fix 0. Then 0 is a critical point of f_μ, so that

$$f_\mu(z) = t(\mu)z^2 + b(\mu), \quad t \in \mathbb{C}^*. \tag{15.1}$$

Composing h_μ with complex scaling $z \mapsto t(\mu)z$, we turn this quadratic polynomial to the normal form $z \mapsto z^2 + c(\mu)$.

Assume now that $\mu = \mu_\lambda$ depends holomorphically on parameter λ. By Theorem 2.18, the map $h_\lambda \equiv h_{\mu(\lambda)}$ is also holomorphic in λ. However, the inverse map h_λ^{-1} is not necessarilly holomorphic in λ.

EXERCISE 4.16. *Give an example.*

It is a miracle that despite it, the deformation $f_\lambda \equiv f_{\mu(\lambda)}$ is still holomorphic in λ!

LEMMA 4.16. *Let $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$, where f and f_λ are holomorphic functions and h_λ is a holomorphic motion (of an appropriate domain). Then f_λ holomorphically depends on λ.*

PROOF. Taking $\partial_{\bar\lambda}$-derivative of the expression $f_\lambda \circ h_\lambda = h_\lambda \circ f_0$, we obtain:

$$0 = \partial_{\bar\lambda} h_\lambda \circ f_0 = f_\lambda' \circ \partial_{\bar\lambda} h_\lambda + \partial_{\bar\lambda} f_\lambda \circ h_\lambda = \partial_{\bar\lambda} f_\lambda \circ h_\lambda.$$

$\square$

COROLLARY 4.17. *Consider a quadratic map $f : z \mapsto z^2 + c_0$. Let μ_λ be a holomorphic family of f-invariant Beltrami differentials on $\mathbb{C}$. Normalize the solution $h_\lambda : \mathbb{C} \to \mathbb{C}$ of the corresponding Beltrami equiation so that the qc deformation $f_\lambda = h_\lambda \circ f \circ h_\lambda^{-1}$ has a normal form $f_\lambda : z \mapsto z^2 + c(\lambda)$. Then the parameter $c(\lambda)$ depends holomorphically on λ.*

PROOF. Consider first the solution $H_\lambda : \mathbb{C} \to \mathbb{C}$ of the Beltrami equation which fixes 0 and 1. It conjugates f to a quadratic polynomial of form (15.1). By Lemma 4.16, its coefficients $t(\lambda)$ and $b(\lambda)$ depend holomorphically on λ. The complex rescaling $T_\lambda : z \mapsto t(\lambda)z$ reduces this polynomial to the normal form with $c(\lambda) = t(\lambda)b(\lambda)$, and we see that $c(\lambda)$ depends holomorphically on λ as well. $\qquad\square$

15.2. Sullivan's No Wandering Domains Theorem.

16. Remarkable functional equations

Study of certain functional equations was one of the main motivations for the classical work in holomorphic dynamics. By means of these equations the local dynamics near periodic points of different types can be reduced to the simplest normal form. But it turns out that the role of the equations goes far beyond local issues: global solutions of the equations play a crucial role in understanding the dynamics.

We will start with the local analysis and then globalize it (though sometimes one can go the other way around). For the local analysis we put the fixed point at the origin and consider a holomorphic map

$$f : z \mapsto \sigma z + a_2 z^2 + \dots \tag{16.1}$$

near the origin.

16.1. Attracting points and linearizing coordinates. Let us start with the simplest case of an attracting fixed point. In turns out that such a map can always be linearized near the origin:

THEOREM 4.18. *Consider a holomorphic map (16.1) near the origin. Assume $0 < |\sigma| < 1$. Then there exists an f-invariant Jordan disk $V \ni 0$, an $r > 0$, and a conformal map $\phi : (V, 0) \to \mathbb{D}_r$ with $\phi'(0) = 1$ satisfying the equation:*

$$\phi(fz) = \sigma\phi(z) \tag{16.2}$$

The above properties determine uniquely the germ of ϕ at the origin.

The above function ϕ is called the *linearizing coordinate* for f near 0 or the *Königs function*. The linearizing equation 16.2 is also called the *Schroeder equation*. It locally conjugates f to its linear part $z \mapsto \sigma z$.

PROOF. The linearizer ϕ can be given by the following explicit formula:

$$\phi(z) = \lim_{n \to \infty} \sigma^{-n} f^n z. \tag{16.3}$$

To see that the limit exists (uniformly near the origin), let $z_n = f^n z$, $z_0 \equiv z$, notice that $z_n = O(|z\sigma|^n)$ uniformly near the origin, and take the ratio of the two consecutive terms in (16.3):

$$\frac{\sigma^{-n-1} z_{n+1}}{\sigma^{-n} z_n} = \sigma^{-1} \frac{\sigma z_n (1 + O(|z_n|))}{z_n} = 1 + O(|z\sigma^n|).$$

Hence

$$\phi(z) = z \prod_{n=0}^{\infty} \frac{\sigma^{-n-1} z_{n+1}}{\sigma^{-n} z_n} = 1 + O(|z|)$$

uniformly near the origin, and the conclusion follows.

Obviously, ϕ is a linearizer. Its uniqueness follows from the exercise below.   □

EXERCISE 4.17. *Show that if a holomorphic germ f near the origin commutes with the linear germ $z \mapsto \sigma z$, $0 < |\sigma| < 1$, then f is itself linear.*

REMARK 4.2. We see that the conjugacy ϕ is constructed by going forward by the iterates of f and then returning back by the iterates of the corresponding linear map. This method of constructing a conjugacy between two maps will be used on several other occassions, see (16.7) and (**??**).

16.2. Global leaf of a repelling point. Taking the local inverse of f, we conclude that repelling maps are also locally linearizable:

COROLLARY 4.19. *Consider a holomorphic map (16.1) near the origin. Assume $|\sigma| > 1$. Then there exist Jordan disks $V \ni V' \ni 0$ such that $f(V') = V$, an $r > 0$, and a conformal map $\phi : (V, 0) \to \mathbb{D}_r$ with $\phi'(0) = 1$ satisfying the equation:*

$$\phi(fz) = \sigma\phi(z), \quad z \in V'. \tag{16.4}$$

The above properties determine uniquely the germ of ϕ at the origin.

Assume now that $f : \bar{\mathbb{C}} \to \bar{\mathbb{C}}$ is a polynomial with a repelling fixed point a. Let us consider the inverse linearizing function $\psi : (\mathbb{D}_r, 0) \to (V, a)$, $\psi = \phi^{-1}$. It satisfies the functional equation

$$\psi(\sigma z) = f(\psi(z)), \quad z \in V'. \tag{16.5}$$

It allows us to extend ψ holomorphically to the disk $\mathbb{D}_{|\sigma|r}$ by letting $\psi(\zeta) = f(\psi(\zeta/\lambda))$ for $\zeta \in \mathbb{D}_{|\sigma|r}$. Repeating this procedure, we can consecutively extend f to the disks $\mathbb{D}_{|\sigma|^n r}$, $n = 1, 2, \ldots$, so that in the end f we obtain an entire function $\psi : \mathbb{C} \to \mathbb{C}$ satisfying (16.5).

We will now construct in a dynamical way the Riemann surface of the inverse (multivalued) function $\phi = \psi^{-1}$. The construction below is

a special case of a general *natural extension* or *inverse limit* construction. Let us consider the space of inverse orbits of f converging to the fixed point 0:

$$\mathcal{L} = \{\hat{z} = (z_{-n})_{n=0}^{\infty} : \ f(z_{-n-1}) = z_{-n}, \ z_{-n} \to 0\}.$$

Define $\pi_{-n} : \mathcal{L} \to \mathbb{C}$ as the natural projections $\hat{z} \mapsto z_{-n}$. Let $z \equiv z_0$ and $\pi \equiv \pi_0 : \hat{z} \mapsto z$. The map f lifts to an invertible map $\hat{f} : \mathcal{L} \to \mathcal{L}$, $\hat{f}(\hat{z}) = (fz_{-n})_{n=0}^{\infty}$ such that $\hat{f}^{-1}(\hat{z}) = (z_{-n})_{n=1}^{\infty}$. Moreover, the projection π is equivariant: $\pi \circ \hat{f} = f \circ \pi$.

For a neighborhood U of z let $\hat{U} = \hat{U}(\hat{z}) = (U_{-n})_{n=0}^{\infty}$, where U_{-n-1} is defined inductively as the component of $f^{-1}(U_{-n})$ containing z_{-n-1}. We call $\hat{U}$ the *pullback* of U along $\hat{z}$. Let us call a pullback $\hat{U}$ *regular* if the maps $f : U_{-n} \to U_{-n-1}$ are eventually univalent. Since $z_{-n} \to 0$, $z_{-n} \in V$ for all $n \geq N$. Selecting U so small that $U_{-N} \subset V$, we see that $U_{-n} \subset V$ for all $n \geq N$, and hence all the maps $f : U_{-n-1} \to U_{-n}$ are univalent for $n \geq N$. Thus, $\hat{U}$ is regular for a sufficiently small U.

We define topology on $\mathcal{L}$ by letting all the regular pullbacks $\hat{U}(\hat{z})$ be the basis of neighborhoods of $\hat{z} \in \mathcal{L}$. Moreover, if $f : U_{-n-1} \to U_{-n}$ are univalent for $n \geq N$, the projection $\pi_{-N} : \hat{U} \to U_{-N}$ is homeomorphic, and we take it as a local chart on $\hat{\mathcal{L}}$. Transition maps between such local charts are given by iterates of f, so that, they turn $\mathcal{L}$ into a Riemann surface.

EXERCISE 4.18. *Show that the projections $\pi_{-n} : \mathcal{L} \to \mathbb{C}$ are holomorphic. Show that the critical points of π are the orbits $\hat{z} = (z_{-n})_{n=0}^{\infty}$ passing through a critical point of f (such orbits are called critical). Find the degree of branching of π at $\hat{z}$.*

Let $\hat{a} = (a\,a\,\dots) \in \mathcal{L}$ be the fixed point lift of a. The following statement shows that $\hat{\mathcal{L}}$ is the indeed the Riemann surface for ϕ:

PROPOSITION 4.20. *The maps ψ and ϕ lift to mutually inverse conformal isomorphisms $\hat{\psi} : (\mathbb{C}, 0) \to (\mathcal{L}, \hat{a})$ and $\hat{\phi} : \mathcal{L} \to \mathbb{C}$ conjugating $z \mapsto \sigma z$ to $\hat{f}$ and such that $\pi \circ \hat{\psi} = \psi$.*

PROOF. For $u \in \mathbb{C}$, we let $\hat{\psi}(u) = (\psi(u/\sigma^n))_{n=0}^{\infty}$.

Vice versa, if $\hat{z} = (z_{-n})_{n=0}^{\infty}$ then eventually $z_{-n} \in V$, so that the local linearizer ϕ is well defined on all z_{-n}, $n \geq N$. Let now $\hat{\psi}(\hat{z}) = \sigma^n \psi(z_{-n})$ for any $n \geq N$. It does not depend on the choice of n since $\phi|V$ conjugates f to $z \to \sigma z$.

We leave to the reader to check all the properties of these maps. $\quad\square$

LEMMA 4.21. *Let $\mathcal{C}_f = \pi^{-1}(\bar{C}_f)$. Then the map $\mathcal{L} \smallsetminus \mathcal{C}_f \to \mathcal{L} \smallsetminus \bar{C}_f$ is a covering.*

PROOF. Let $z \in \mathbb{C} \smallsetminus \bar{C}_f$ and let $U \subset \mathbb{C} \smallsetminus \bar{C}_f$ be a little disk around z. Then

$$\pi^{-1}(U) = \bigcup_{\hat{z} \in \pi^{-1}z} \hat{U}(\hat{z}),$$

and each $\hat{U}$ projects univalently onto U.   □

Let $\hat{K}(f) = \pi^{-1}(K(f))$.

COROLLARY 4.22. *Assume $K(f)$ is connected. Let U be a component of $\mathcal{L} \smallsetminus \hat{K}(f)$. Then U is simply connected, so that, the projection $\pi : U \to D_f(\infty)$ is a universal covering.*

PROOF. Since $K(f)$ is connected, $\bar{C}_f \subset K(f)$. By Lemma 4.21, $U \to D_f(\infty)$ is a covering map. Since $D_f(\infty)$ is conformally equivalent to $\mathbb{D}^*$, U is either conformally equivalent to $\mathbb{D}^*$ or is simply connected. But in the former case U would be a neighborhood of ∞ in $\mathcal{L} \approx \mathbb{C}$, so that, $\hat{K}(f)$ would be bounded. It is impossible since $\hat{K}(f)$ is $\hat{f}$-invariant, where by Proposition 4.20 $\hat{f}$ is conjugate to $z \mapsto \sigma z$ with $|\sigma| > 1$.   □

16.3. Superattractng points and Böttcher coordinates.

THEOREM 4.23. *Let $f : z \mapsto z^d + a_{d+1}z^{d+1} + \ldots$ be a holomorphic map near the origin, $d \geq 2$. Then there exists an f-invariant Jordan disk $V \ni 0$, $r \in (0,1)$, and a conformal map $\phi : (V,0) \to (\mathbb{D}_r, 0)$ satisfying the equation:*

$$\phi(fz) = \phi(z)^d. \tag{16.6}$$

The above properties determine uniquely the germ of ϕ at the origin, up to postcomposition with rotation $z \mapsto e^{2\pi i/(d-1)}z$ (so, it is unique in the quadratic case $d = 2$). Moreover, it can be normalized so that $\phi'(0) = 1$.

The map ϕ is called the *Böttcher function*, or the *Böttcher coordinate* near 0. Equation (16.6) is called the *Böttcher equation*. In the Böttcher coordinate the map f assumes the normal form $z \mapsto z^d$.

PROOF. The Böttcher function can be given by the following explicit formula:

$$\phi(z) = \lim_{n \to \infty} \sqrt[d^n]{f^n z}, \tag{16.7}$$

where the value of the d^nth root is selected so that it is tangent to the id at ∞. Obviously, this finction, if exists, satisfied the Böttcher equation. So, we only need to check that the limit exists.

Let $z_n = f^n z$, where $z_0 \equiv z$. Then

$$\frac{\sqrt[d^{n+1}]{z_{n+1}}}{\sqrt[2^n]{z_n}} = \frac{\sqrt[d^{n+1}]{z_n^2(1+O(z_n))}}{\sqrt[d^n]{z_n}} = \sqrt[d^{n+1}]{(1+O(z_n))} = 1 + O\left(\frac{z_n}{d^{n+1}}\right).$$

Hence

$$\phi(z) = \lim_{n\to\infty} \sqrt[d^n]{z_n} = z \prod_{n=0}^{\infty} \frac{\sqrt[d^{n+1}]{z_{n+1}}}{\sqrt[2^n]{z_n}} = z \prod_{n=0}^{\infty} \left(1 + O\left(\frac{z_n}{d^{n+1}}\right)\right) = z(1+O(z)),$$

where the last product is convergant uniformly at a superexponential rate.

Finally, uniqueness of the Böttcher function follows from the exercise below. $\square$

EXERCISE 4.19. *Let $d \geq 2$. Show that there are no holomorphic germs commuting with $g : z \mapsto z^d$ near the origin, except except rotations $z \mapsto e^{2\pi i/(d-1)} z$.*

16.4. Böttcher vs Riemann. Let us now consider a quadratic polynomial f_c near ∞. Since ∞ is a superattracting fixed point of f of degree 2, the map f_c near ∞ can be reduced in the Böttcher coordinate to the map $z \mapsto z^2$ (Theorem 4.23). Thus, there is a Jordan disk $V = V_c \subset \mathbb{C}$ whose complement $\mathbb{C} \setminus V$ is f_c–invariant, some $R > 1$, and a conformal map $\phi_c : \mathbb{C} \setminus V \to \mathbb{C} \setminus \mathbb{D}_R$ satisfying the Böttcher equation:

$$\phi_c(f_c z) = \phi_c(z)^2. \tag{16.8}$$

Moreover, $\phi_c(z) \sim z$ as $z \to \infty$.

We will now globalize the Böttcher function.

16.4.1. *Connected case.*

THEOREM 4.24. *Let $f_c : z \mapsto z^2 + c$ be a quadratic polynomial with connected Julia set. Then the Böttcher function admits an analytic extension to the whole basin of ∞. Moreover, it conformally maps $D_c(\infty)$ onto the complement of the unit disk.*

PROOF. We will skip label c from the notations. Let, as usual, $f_0(z) = z^2$.

Let $U^n = \bar{\mathbb{C}} \setminus f^{-n}\bar{V}$. Then $U^0 \subset U^1 \subset U^2 \subset \dots$ and $\cup U^n = D_f(\infty)$. Since the filled Julia set $K(f)$ is connected, the domains U^n are topological disks and the maps $f : U^{n+1} \to U^n$ are double coverings branched point at ∞ (recall the proof of Theorem 4.3).

Let $\Delta^n = \bar{\mathbb{C}} \setminus \bar{\mathbb{D}}_{R^{1/2^n}}$. By Lemma 1.23, the Böttcher function $\phi : U^0 \to \Delta^0$ admits a lift $\Phi : U^1 \to \Delta^1$ such that $f_0 \circ \Phi = \phi \circ f$. But the Böttcher equation tells us that $\phi : U^0 \to \Delta^0$ is a lift of its

restriction $\phi : f(U^0) \to f_0(\Delta^0)$. If we select Φ so that $\Phi(z) = \phi(z)$ at some finite point $z \in U^0$, then these two lifts must coincide on U^0: $\Phi|U^0 = \phi$. Thus, Φ is the analytic extension of ϕ to U^1. Obviously, it satisfies the Böttcher equation as well.

In the same way, the Böttcher function can be consecutively extended to all the domains U^n and hence to their union, $D_f(\infty)$. $\square$

Thus, the Böttcher function gives the uniformization of $\mathbb{C} \smallsetminus K(f)$ by the unit disk. Given the intricate fractal structure of the Julia set, this is quite remarkable that its complement can be uniformized in this explixit way!

One can also go the other way around and costruct the Böttcher function by means of uniformization:

EXERCISE 4.20. *Let $f = f_c$ be a quadratic polynomial with connected Julia set. Then the basin of infinity $\bar{D}_f(\infty)$ is a conformal disk. Uniformize it by the complement of the unit disk; $\psi : (\mathbb{D}, \infty) \to (D_f(\infty), \infty)$, normalized at ∞ so that $\psi(z) \sim \lambda z$ with $\lambda > 0$. Prove (without using the Böttcher theorem) that ψ conjugates $f_0 : z \mapsto z^2$ on $\mathbb{C} \smallsetminus \mathbb{D}$ to f on the basin of ∞ (and that $\lambda = 1$).*

EXERCISE 4.21. *Prove that $D_c(\infty)$ is the maximal domain of analyticity of the Böttcher function.*

Let us finish with a curious consequence of Theorem 4.24. The *capacity* of a connected compact set $K \subset \mathbb{C}$ rel ∞ is defined as $1/R$, where R is the radius of the disk $\mathbb{D}_R$ such that the domain $\mathbb{C} \smallsetminus K(f_c)$ can be conformaly mapped onto $\mathbb{C} \smallsetminus \bar{\mathbb{D}}_R$ by a map tangent to the id at ∞.

COROLLARY 4.25. *Let $f_c : z \mapsto z^2 + c$. Then the capacity of the filled Julia set $K(f_c)$ is equal to 1.*

16.4.2. *Cantor case.* In the disconnected case the Böttcher function ϕ_c cannot be any more extended to the whole basin of ∞, as it branches at the critical point 0. However, ϕ_c can still be extended to a big invariant region Ω_c containing 0 on its boundary.

THEOREM 4.26. *Let $f_c : z \mapsto z^2 + c$ be a quadratic polynomial with disconnected Julia set. Then the Böttcher function ϕ_c admits the analytic extension to a domain Ω_c bounded by a "figure eight" curve branched at the critical point 0. Moreover, ϕ_c maps Ω_c conformally onto the complement of some disk $\bar{\mathbb{D}}_R$ with $R > 1$.*

PROOF. Again, we skip the label c.

Since $0 \in D_f(\infty)$, the orb(0) lands at the domain V of the Böttcher function near ∞. By shrinking V, we can make $f^n 0 \in \partial V$ for some $n > 0$. Then there are no obstructions for consecutive extensions of ϕ to the domains $U^k = \bar{\mathbb{C}} \smallsetminus f^{-k}\bar{V}$, $k = 0, 1, \ldots, n$ (in the same way as in the connectef case). All these domains are bounded by real analytic curves except the last one, U^n, which is bounded by a figure eight curve branched at 0. This is the desired domain Ω. $\square$

For $c \in M$, we let $\Omega_c \equiv D_c(\infty)$.

16.4.3. *Böttcher position of the critical value.* Since the critical value $c \in \partial U^{n-1}$ belongs to the domain of ϕ_c, the expression $\phi_c(c)$ is well-defined (provided the Julia set $J(f_c)$ is disconnected). It gives the Böttcher position of the critical value as a function of the parameter c. This function will play a crucial role in what follows.

16.5. External rays and equipotentials. The map $f_0 : z \mapsto z^2$ on $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$ has two invariant foliations: foliation by the straight rays going to ∞ and foliation by round circles centered at the origin. (Note that the first foliation is dynamically defined: see the hint to Exercise 4.19.) We will label the rays by their angles $\theta \in \mathbb{R}/2\pi\mathbb{Z}$ and the circles by their "heights" $t = \log r \in \mathbb{R}_+$. Note that $f(\mathcal{R}^\theta) = \mathcal{R}^{2\theta}$ and $f(\mathcal{E}^t) = \mathcal{E}^{2t}$.

By means of the Böttcher function, these two foliations can be transferred to the domain $\Omega_c \subset D_c(\infty)$, supplying us with the foliation by *external rays* and *equipotentials.* They are naturally labeled by the *external angles* and *equipotential heights* respectively. Let $\mathcal{R}^\theta$ stand for the external ray of angle θ and $\mathcal{E}^t$ stand for the equipotential of height t. (We will use notation $\mathcal{R}^\theta(t)$ for the point on the ray $\mathcal{R}^\theta$ whose equipotential level is equal to t.)

If $K(f_c)$ is connected then $\Omega_c = D_c(\infty)$, so that the whole basin of infinity is foliated by the external rays and equipotentials.

In the disconnectedf case, we can pull the two foliations in Ω_c back by the iterates of f to obtain to singular foliations on the whole basin of ∞. They have singularities at the critical points of iterated f, i.e., at 0 and all its preimages under the iterates of f.

In this context *external rays* will be understood as the non-singular leaves of these foliaitons that go to ∞ (i.e., the maximal non-singular extensions of the rays in Ω_c). Countably many rays land at the preimages of 0. All other rays are properly embedded into the basin; they will be called *proper rays.* Two (improper) rays landing at the critical point 0 will be called the *critical rays.* The particularly important ray going through the critical value will be called the *principal ray* (its

external angle will be also called *principal*). Of cource, it contains the (coinciding) images of the critical rays.

The figure-eight that bounds Ω_c will be called the *critical equipotential*.

16.6. Green function. The *Green function* of a quadratic polynomial $f = f_c$ is defined as follows:

$$G_c(z) = \log |\phi_c(z)|, \qquad (16.9)$$

where ϕ_c is the Böttcher function of f_c. The Green function is harmonic wherever the Böttcher function is defined (since the Böttcher function never vanishes) and has a logarithmic singularity at ∞:

$$G(z) = \log |z| + o(1).$$

In the connected case, (16.9) defines the Green function in the whole basin $D(\infty)$. In the disconnected case definition (16.9) can be used only in the domain Ω. However, in either case the Green function satisfies the equation:

$$G(fz) = 2G(z). \qquad (16.10)$$

This equation can be obviously used in order to extend the Green function harmonically to the *whole basin of* ∞. Let us summarize simple properties of this extension:

EXERCISE 4.22. *a) In the connected case the Green function does not have critical points. In the disconnected case, its critical points coincide with the critical points of iterated f.*

b) Equipotentials are the level sets of the Green function, while external rays (and their preimages) are its gradient curves.

c) The Brolin formula holds:

$$G(z) = \lim_{n \to \infty} \frac{1}{2^n} \log |f^n z|, \quad z \in D(\infty). \qquad (16.11)$$

d) Extention of the Green function by 0 through the filled Julia set $K(f)$, gives a continuous subharmonic function on the whole complex plane.

From the physical point of view, one should imagine that the filled Julia set K is a conductor of electric charge put in the electric field of the unit charge at ∞. Let the charges in K settle down in the equilibrium state (according to the "harmonic distribution" on the Julia set). Then the Green function is the electric potential in the space $\mathbb{R}^2$ created jointly by these charges on K and the charge at ∞. (That is why the name "equipotentials").

EXERCISE 4.23. *Assume that the Julia set $J(f)$ is connected. Endow its basin $D(\infty)$ with the hyprbolci metric ρ. Then for any external ray $\mathcal{R}^\theta$ we have:*

$$\rho(z, \zeta) = \left| \log \frac{G(z)}{G(\zeta)} \right|, \quad z, \zeta \in \mathcal{R}^\theta.$$

16.7. Parabolic points and Écale-Voronin cylinders.

17. Quadratic-like maps

17.1. The concept.

17.1.1. *Definition and first properties.* The notion of a quadratic-like map is a fruitful generalization of the notion of a quadratic polynomial.

DEFINITION 4.1. A *quadratic-like map* $f : U \to U'$ is a holomorphic double branched covering between two conformal disks U and U' such that $U \Subset U'$. The annulus $A = U' \smallsetminus \bar{U}$ is called the *fundamental annulus* of f.

define

By the Riemann-Hurwitz Theorem, any quadratic-like map has a single critical point, which is of course non-degenerate. We normalize f so that the critical point sits at 0 (unless otherwise is explicitly stated). Note that any quadratic polynomial $f = f_c$ restricts to a quadratic-like map $f : f^{-1}(\mathbb{D}_R) \to \mathbb{D}_R$ whose range is a round disk with sufficiently big radius R.

need?

Technical Conventions: In what follows we will consider only *even* quadratic-like maps, i.e, such that $f(z) = f(-z)$ for all $z \in U$, with 0-*symmetric* domains U and U'. Moreover, we will assume that both domains are bounded by *piecewice smooth Jordan curves*.

The notion of a quadratic-like map does not fit to a canonical dynamical framework, where the phase space is assumed to be invariant under the dynamics. In the quadratic- like case, some orbits *escape* through the fundamental annulus (i.e., $f^n z \in A$ for some $n \in \mathbb{N}$), and we cannot iterate them any further. However, there are still a plenty of non-escaping points, which form a dynamically significant object. The set of all non-escaping points is called the *filled Julia set* of f and is denoted in the same way as for polynomials:

$$K(f) = \{z : f^n z \in U, \ n = 0, 1, \ldots.\}$$

By definition, the *Julia set* of f is the boundary of the filled Julia set: $J(f) = \partial K(f)$. Dynamical features of quadratic-like maps are very similar to those of quadratic maps (in §17.3 we will see a good reason for it):

EXERCISE 4.24. *Check that all dynamical properties of quadratic polynomials established in in §§13 - 14 are still valid for quadratic-like maps. In particular,*

(i) The filled Julia set $K(f)$ is a completely invariant full compact subset of U.

(ii) Basic dichotomy: $J(f)$ and $K(f)$ are either connected or Cantor; the former holds if and only if the critical point is non-escaping: $0 \in K(f)$.

(iii) Any periodic component of $\operatorname{int} K(f)$ is either in the immediate basin of an attracting/parabolic cycle, or is a Siegel disk.

(iv) f can have at most one attracting cycle.

17.1.2. *Adjustments.* In fact, the notion of a quadratic-like map with the fixed domain is too rigid. We want to allow some adjustment of the domains which does not effect the essential dynamics of the map. Let us say that a quadratic-like map $g : V \to V'$ is an *adjustment* of another quadratic-like map $f : U \to U'$ if $V \subset U$, $g = f|V$, and $\partial V' \subset \bar{U}' \smallsetminus U$. (In particular, we can restrict f to $V = f^{-1}U$, provided $f(0) \in U$.)

EXERCISE 4.25. *(i) Show that adjustments do not change the Julia set.*

(ii) Consider a topological disk $V' \subset U'$ containing the critical value $f(0)$ and such that $\partial V' \subset \bar{U}' \smallsetminus \bar{U}$. Let $V = f^{-1}V'$. Then the restriction $f : V \to V'$ is a quadratic-like map.

An appropriate adjustment allows one to improve the geometry of a quadratic-like map:

LEMMA 4.27. *Consider a quadratic-like map $f : U \to U'$ with*

$$\operatorname{mod}(U' \smallsetminus \bar{U}) \geq \mu > 0 \tag{17.1}$$

and $f(0) \in U$. Then there is an adjustment $g : V \to V'$ such that:

(i) The new domains V and V' are bounded by real analytic κ-quasicircles γ and γ' with κ depending only on μ. Moreover, these curves have a bounded (in terms of μ) eccentricity around the origin.

(ii) $\operatorname{mod}(V' \smallsetminus \bar{V}) \geq \mu/2 > 0$.

(iii) g admits a decomposition

$$g = h \circ f_0, \tag{17.2}$$

where $f_0(z) = z^2$ and h is a univalent function on $W = f_0(V)$ with distortion bounded by some constant $C(\mu)$.

PROOF. Let us uniformize the fundamental annulus A of f by a round annulus, $\phi : \mathbb{A}(1/r, r) \to A$, where $r \geq e^{\mu/2} \equiv r_0$. Then $\gamma' = \phi(\mathbb{T})$ is the equator of A. Consider the disk V' bounded by γ', and let $V = f^{-1}V'$. Since $f(0) \in V'$, V is a conformal disk and the restriction $f : V \to V'$ is a quadratic-like adjsutment of f (see Exerecise 4.25).

Restrict ϕ to the annulus $\mathbb{A}(1/r_0, r_0)$. Take an arc $\alpha = [a, b]$ on $\mathbb{T}$ of length at most $\delta = (1 - 1/r_0)/2$. By the Koebe Distortion and 1/4 Theorems in the disk $\mathbb{D}_{2\delta}(u)$,

$$|\phi(b) - \phi(a)| \geq \frac{\delta}{2}|f'(a)|; \quad l(\phi(\alpha)) \leq K(r_0)\,|f'(a)|,$$

where l stands for the arc length. Hence $\gamma' = \phi(\mathbb{T})$ is a quasi-circle with the dilatation depending only on r_0.

Applying the same argument to the uniformization of $f^{-1}A$, we conclude that its equator $\gamma = \partial V$ is a quasicircle with bounded dilatation as well.

Since γ and γ' are 0-symmetric κ-quasicircle, the eccentricity of these curves around 0 is bounded by some constant $C(\kappa)$ (see Exercise 2.16). This proves (i).

Property (ii) is obvious since $\mathrm{mod}(V' \smallsetminus \bar{V}) \geq \mathrm{mod}\,\mathbb{A}(1, r_0) = \log r_0$.

Since g is assumed to be even, it admits decomposition (17.2). Moreover, h admits a univalent extension to the disk $\tilde{W} = f_0(U)$, and

$$\mathrm{mod}(\tilde{W} \smallsetminus W) = 2\,\mathrm{mod}(U \smallsetminus V) \geq \mu/2.$$

The Koebe Distortion Theorem (in the invariant form 1.17) completes the proof. $\square$

If some map g admits decomposition (17.2), we say that "it is a quadratic map up to bounded distortion".

17.1.3. *Quadratic-like germs.* Let us say that two quadratic-like maps f and $\tilde{f}$ represent the same *quadratic-like germ* if there is a sequence of quadratic-like maps $f = f_0, f_1, \ldots, f_n = \tilde{f}$, such that f_{i+1} is obtained by an ajustment of f_i or the other way around. We will not make notational differences between maps and germs.

According to Exercise 4.25, a quadratic-like germ f have a well-defined Julia set $J(f)$ (the notations for the dynamical objects of the germs will be the same as for the maps).

We will usually consider quadratic-like maps/germs up to *affine conjugacy* or *rescaling*. Thus, we allow ourselves to replace $f(z)$ by $\lambda^{-1}f(\lambda z)$ with some $\lambda \in \mathbb{C}^*$. This allows us to *normalize* f in different

convenient ways. For example, we can select the normal form

$$f(z) = c + z^2 + \dots \tag{17.3}$$

with the second order Taylor coefficient at the origin equal to 1.

Let us refine Lemma 4.27 a bit:

LEMMA 4.28. *Let $f : U \to U'$ be a quadratic-like map with connected Julia set satisfying (17.1). Then the germ of f can be represented with a quadratic-like map $g : V \to V'$ satisfying the following properties:*

(i) *The same as in Lemma 4.27;*

(ii) $\min(\mu/2, 1/4) \leq \mod(V' \smallsetminus V) \leq 1$;

(iii) *If f is normalized by (17.3) then*

$$\rho \leq r_V \leq R_{V'} \leq 1/\rho$$

for some constant $\rho \in (0,1)$ depending only on μ.

PROOF. Let $U^n = f^{-n}U'$ and let $A^n = U^{n-1} \smallsetminus U^n$. Since the Julia set is connected, the restrictions $f : U^n \to U^{n-1}$ are quadratic-like maps obtained by consecutive adjustments of $f : U \to U'$. Hence they represent the same germ. Since $\mod A^n = \mod A^1/2^{n-1}$, we can select n in such a way that $\tilde{\mu} \equiv \min(\mu, 1/2) \leq \mod A^n \leq 1$. Let us now adjust $f|U^n$ once more as in Lemma 4.27. We obtain a quadratic-like map $g : V \to V'$ representing the same germ and satisfying properties (i)-(ii). Moreover, both domains have eccentricity bounded by some $e = e\mu)$.

Assume now that f is normalized by (17.3), so is g. Then in representation (17.2), $g = h \circ f_0$, the univalent map $h : (W, 0) \to (V', c)$ is also normalized: $h'(0) = 1$. Since $W = f_0(V)$,

$$0 < C^{-1} r_W \leq r_{V',c} \leq R_{V'c} \leq C R_W$$

for some constant $C = C(\mu)$ depending only on μ. Hence

$$C^{-1} r_V^2 \leq r_{V',c} \leq R_{V',c} \leq C R_V^2. \tag{17.4}$$

But since $V' \supset V$, we have: $R_{V',c} \geq R_V/2$. By the right-hand side of (17.4), $R_V \geq 1/2C$. Since V has a bounded eccentricity, the inner radius r_V is also bounded away from 0: $r_V \geq 1/2Ce$.

On the other hand, if $r_V = L >> 1$ then the left-hand side of (17.4) (and bounded eccentricity of V) implies that the annulus $V' \smallsetminus V$ contains the round annulus whose inner radius is of order L and the outer radius is of order L^2, so that $\mod(V' \smallsetminus V) \geq \gamma \log L$, where $\gamma = \gamma(\mu) > 0$. Since the modulus of $V' \smallsetminus V$ is bounded, we conclude that L is bounded as well. $\square$

17.2. Uniqueness of a non-repelling cycle. We will now give the first illustration of how useful the notion of a quadratic-like map is. It exploits the flexibility of this class of maps: small perturbations of a quadratic-like map are still quadratic-like (on a slightly adjusted domain):

EXERCISE 4.26. *Let $f : U \to U'$ be a quadratic-like map with the fundamental annulus A. Take a Jordan curve $\gamma' \subset A$ generating $\pi_1(A)$, and let V' be the domain bounded by γ'. Let ϕ be a bounded holomorphic function on U with $\|\phi\|_\infty < \text{dist}(\gamma, \partial U')$. Let $g = f + \phi$ and $V = g^{-1}V'$. Then $g : V \to V'$ is a quadratic-like map. (Hint: Take a Jordan curve Γ close to ∂U with winding number 1 around the origin and, look at the curve $g : \Gamma \to \mathbb{C}$, and apply the Argument Principle.)*

THEOREM 4.29. *Any quadratic-like map (in particular, any quadratic polynomial) has at most one non-repelling cycle.*

PROOF. Assume that a quadratic-like map $f : U \to U'$ has two non-repelling cycles $\boldsymbol{\alpha} = \{\alpha_k\}_{k=0}^p$ and $\boldsymbol{\beta} = \{\beta_k\}_{k=0}^q$. Let μ and ν be their multipliers. Take two numbers a and b to be specified below.

Using the Interpolation formulas, find a polynomial ϕ (of degree $2p + 2q - 1$) vanishing at points α_k and β_k, such that $\phi'(\alpha_0) = a$, $\phi'(\beta_0) = b$, while the derivatives at all other points α_k and β_k $(k > 0)$ vanish.

Let $g = f + \epsilon\phi$, where $\epsilon > 0$. Then $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ are periodic cycles for g with multipliers

$$\lambda' = \lambda + a\epsilon \prod_{k>0} f'(\alpha_k) \quad \text{and} \quad \mu' = \mu + b\epsilon \prod_{k>0} f'(\beta_k)$$

respectively. Since $|\lambda| \le 1$ and $|\mu \le 1$, parameters a and b can be obviously selected in such a way that $|\lambda'| < 1$ and $|\mu'| < 1$ for all sufficiently small $\epsilon > 0$. Thus the cycles $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ become attracting under g. But for a sufficiently small ϵ, g is a quadratic-like map on a slightly adjusted domain containing both cycles (see Exercise 4.26). As such, it is allowed to have at most one attracting cycle (Exercise 4.24) - contradiction. $\square$

This result together with Theorem 4.14 immediately yields:

COROLLARY 4.30. *A quadratic polynomial can have at most one cycle of components of $\text{int } K(f)$.*

17.3. Straightening Theorem. If the reader tried to extend the basic dynamical theory from quadratic polynomials to quadratic-like maps, quite likely he was stuck with the No Wandering Domains Theorem. The only known proof of this theorem crucially uses the fact that

a polynomial of a given degree depends on finitely many parameters. The flexibility offered by the infinitely dimensional space of quadratic-like maps looks at this moment like a big disadvantage. It turns out, however, that the theorem is still valid for quadratic-like maps, and actually there is no need to prove it independently (as well as to repeat any other pieces of the topological theory). In fact, quadratic-like maps do not exibit any new features of topological dynamics, since all of them are topologically equivalent to polynomials (restricted to appropriate domains)!

The proof of this theorem was historically the first application of the so called *quasi-conformal surgery* technique. The idea of this technique is to cook by hands a quasi-regular map with desired dynamical properties which topologically looks like a polynomial. If you then manage to find an invariant conformal structure for this map, then by the Measurable Riemann Mapping Theorem it can be realised as a true polynomial.

To state the result precisely, we need a few definitions. Two quadratic-like maps f and g are called *topologically conjugate* if they become such after some adjustments of their domains. Thus there exist adjustments $f : U \to U'$ and $g : V \to V'$ and a homeomorphism $h : (U', U) \to (V', V)$ such that the following diagram is commutative:

$$
\begin{array}{ccc}
U & \xrightarrow{\ f\ } & U' \\
h \downarrow & & \downarrow h \\
V & \xrightarrow{\ g\ } & V'
\end{array}
$$

In case when one of the maps is a global polynomial, we allow to take any quadratic-like restriction of it.

If the homeomorphism h in the above definition can be selected quasi-conformal (respectively conformal or affine) then the maps f and g are called quasi-conformally (respectively conformally or affinely) conjugate. Two quadratic-like maps are called *hybrid* equivalent if they are qc conjugate by a map h with $\bar{\partial}h = 0$ a.e. on the filled Julia set $K(f)$.

Remark. The last condition implies that h is conformal on the int $K(f)$. On the Julia set $J(f)$ it gives an extra restriction only if $J(f)$ has positive measure (and so far there are no examples of Julia sets of positive measure).

The equivalence classes of topologically (respectively qc, hybrid etc.) conjugate quadratic-like maps are called topological (respectively qc, hybrid etc.) classes.

THEOREM 4.31. *Any quadratic-like map g is hybrid conjugate to a quadratic polynomial f_c. If $J(f)$ is connected then the corresponding polynomial f_c is unique.*

This polynomial f_c is called the *straightening* of g.

COROLLARY 4.32. *If g is a quadratic-like map, then:*

(i) *There are no wandering components of* int $K(g)$;
(ii) *Repelling periodic points are dense in $J(g)$;*
(iii) *If all periodic points of g are repelling then $K(g)$ is nowhere dense.*

Remark. If $J(g)$ is a Cantor set, then the straightening is not unique. Indeed, by **??**, all quadratic polynomials f_c, $c \in \mathbb{C} \smallsetminus M$, are qc equivalent. Since their filled Julia sets have zero measure, they are actually hybrid equivalent. Hence all of them are going to be the "straightenings" of g. We will see however that sometimes there is a preferred choice (see §**??**).

Existence of the straightening will be proven in the next section, while uniqueness will be postponed until the end of §18.

17.4. Construction of the straightening. The idea is to "mate" g near $K(g)$ with $f_0 : z \mapsto z^2$ near ∞.

First let us adjust $g : U \to U'$ by Lemma 4.27 so that U and U' are bounded by real analytic curves. Take some $r > 1$. Consider two closed disks: the disk $\bar{U}'$ endowed with the map $g : \bar{U} \to \bar{U}'$ and the disk $\bar{\mathbb{C}} \smallsetminus \mathbb{D}_r$ endowed with the map $f_0 : \bar{\mathbb{C}} \smallsetminus \mathbb{D}_r \to \bar{\mathbb{C}} \smallsetminus \mathbb{D}_{r^2}$. Think of them as two hemi-spheres $S_+^2 \equiv U'$ and $S_-^2 \equiv \bar{\mathbb{C}} \smallsetminus \mathbb{D}_r$ (see Fugure ...) and glue them together by an orientation preserving diffeomorphism $h : \bar{U}' \smallsetminus U \to \mathbb{A}[r, r^2]$ between the closed fundamental annnuli *respecting the boundary dynamical relation*, i.e., such that

$$h(gz) = f_0(hz) \quad \text{for } z \in \partial U. \tag{17.5}$$

EXERCISE 4.27. *Construct such a diffeomorphism. To this end first consider any diffeomorphism $h_1 : \partial U' \to \mathbb{T}_{r^2}$, then lift it to a diffeomorphism $h_2 : \partial U \to \mathbb{T}_r$ satisfying (17.5), and finally interpolate in between h_1 and h_2.*

In this way we obtain a smooth oriented sphere

$$S^2 = \bar{S}_+^2 \sqcup_h \bar{S}_-^2 \equiv \bar{U}' \sqcup_h (\mathbb{C} \smallsetminus \mathbb{D}_r)$$

with the atlas of two local charts given by the identical maps $\phi_+ : S_+^2 \to U'$ and $\phi_- : S_-^2 \to \bar{\mathbb{C}} \smallsetminus \bar{\mathbb{D}}_r$. Moreover, the hemi-shperes S_+^2 and

S_-^2 are bounded by smooth Jordan curves. For instance,

$$\gamma \equiv \partial S_-^2 = \phi_+^{-1} h^{-1}(\mathbb{T}_2) = \phi_+^{-1} \partial U.$$

Define now a map $F : S^2 \to S^2$ by letting

$$F(z) = \begin{cases} \phi_+^{-1} \circ g \circ \phi_+(z) & \text{for } z \in \phi_+^{-1}\bar{U} \\ \phi_-^{-1} \circ f_0 \circ \phi_-(z) & \text{for } z \in \bar{S}_-^2 \end{cases}$$

(It is certainly quite a puritan way of writing since the maps ϕ_- and ϕ_+ are un fact identical.) By (17.5), these two formulas match on $\gamma \equiv \partial S_-^2 = \phi_+^{-1} \partial U$. Hence F is a continuous endomorphism of S^2. Moreover, it is a double branched covering of the sphere onto itself (with two simple branched points at "0"$\equiv \phi_-^{-1}(0)$ and "∞"$\equiv \phi_+^{-1}(\infty)$).

EXERCISE 4.28. *Check the last statement.*

Since F is holomorphic in the local charts $\phi_\pm$, it is a smooth quasi-regular map on $S^2 \smallsetminus \gamma$. By Lemma 2.10, F is quasi-regular on the whole sphere.

EXERCISE 4.29. *Show that the gluing diffeomorphism h can be chosen in such a way that the map F is smooth.*

use t for h; include a proof of this exercise with bounds for the extension

We will now construct an F-invariant conformal structure μ on S^2 (with a bounded dilatation with respect to the qc structure of the *smooth* sphere S^2). Start in a neighborhood of ∞: $\mu|S_-^2 = (\phi_-)^*\sigma$. Since σ is f_0-invariant, $\mu|S_-^2$ is F-invariant. Since ϕ_- admits a smooth extension to $\gamma = \partial S_-^2$, it has a bounded dilatation. Hence $\mu|S_-^2$ has a bounded dilatation as well.

Next, pull-back this structure from the fundamental annulus $A = S_+^2 \cap S_-^2$ to its preimages $A_n = F^{-n}A$, $\mu|A_n = (F^n)^*(\mu|A)$. (We do not bother to define the structure on the union of smooth curves, $\cup \partial A_n$, since it is a set of measure zero.) Since F is holomorphic in the local chart ϕ_+ (namely, equal to g), all these structures have the same dilatation as $\mu|A$. Hence they form a single F-invariant measurable conformal structure with bounded dilatation on $S^2 \smallsetminus \phi_+^{-1}K(g)$.

Finally, let $\mu = (\phi_+)^*\sigma$ on $\phi_+^{-1}K(g)$. We obtain an F-invariant measurable conformal structure μ with bounded dilatation on the whole sphere S^2. By the Measurable Riemann Mapping Theorem, there exists a qc map $H : (S^2, \mu) \to \mathbb{C}$ normalised so that $H(0) = 0$, $H(\infty) = \infty$ and $H\phi_-^{-1}(z) \sim z$ as $z \to \infty$. Then the map $f = H \circ F \circ H^{-1}$ is a quadratic polynomial (see §**??**) with the critical point at the origin and asymptotic to z^2 at ∞. Hence $f = f_c : z \mapsto z^2 + c$ for some c.

EXERCISE 4.30. *Show that $K(f) = H(\phi_+^{-1} K(g))$.*

The qc map $H \circ \phi_+^{-1}$ conjugates $g : U \to U'$ to a quadratic-like restriction of f. Moreover, restricting it to $K(g)$, we see that

$$(H \circ \phi_+^{-1})_* \sigma = H_* \mu = \sigma,$$

so that H is a hybrid conjugacy between g and a restriction of f. Thus f is a straightening of g.

17.4.1. *Comments on the straightening construction.* Note first that the map $B \equiv \phi_- \circ H^{-1}$ in the above construction is the Böttcher coordinate for f on $\Omega \equiv H(S_-^2)$. Indeed:

- B is conformal on Ω since both ϕ_- and H transfer the conformal structure $\mu|S_-^2$ to σ, and
- B conjugates $f_0 : z \mapsto z^2$ to f.

Since $B(\partial\Omega) = \mathbb{T}_r$, $\partial\Omega = E_r$ is the equipotential of radius r for f. Thus *we have conjugated $f : U \to U'$ to $f : D_r \to D_{r^2}$ where D_r is the disk bounded by the equipotential E_r of radius r.*

extend the tubing up to the critical point and improve correspondingly the above statement

Second, note that the above construction of f was uniquely determined by the choice of the gluing diffeomorphism $h : \bar{U}' \smallsetminus U \to \mathbb{A}[r, r^2]$ satisfying (17.5). Such a diffeomorphism will be called *tubing*. Thus *tubing determines the straightening uniquely.* In fact, in the case of connected Julia set, the straightening is independent even of the choice of tubing (see the next section).

Finally, let us dwell on an important issue of a bound on the dilatation of the qc homeomorphism conjugating g to f.

LEMMA 4.33. *Let $g : U \to U'$ be a quadratic-like map with $\mathrm{mod}(U' \smallsetminus U) \geq \delta > 0$. Then g is hybrid conjugate to a straightening f_c by a K-qc map, where the dilatation K depends only on δ.*

PROOF. Let us first adjust g according to Lemma 4.27 (keeping the same notations for the domains U and U').

Let us now follow the proof of the Straightening Theorem. Look at the conformal structure $\mu = (\phi_-)^* \sigma$ on the fundamental annulus A in the local chart ϕ_+, i.e., consider

$$\nu = (\phi_+)_*(\mu|A) = h^* \sigma.$$

Its dilatation is equal to the dilatation of h. The pull-backs of ν by the iterates of g (corresponding to the pull-backs of μ by the iterates F) do not change its dilatation. The final extension to the filled Julia

set $K(g)$ has zero dilatation. Thus *the dilatation of $\nu|U' = \phi_+(\mu|S_+^2)$ is equal to the dilatation of the tubing diffeomorphism h.*

The qc map $H \circ \phi_+^{-1}$ conjugating $g : U \to U'$ to $f_c : D_r \to D_{r^2}$ transfers $\nu|U'$ to σ. Hence its dilatation is also equal to $\mathrm{Dil}(h)$. Thus we only need to argue that h can be selected so that its dilatation depends only on δ.

Let us conformally uniformize the fundamental annulus $R = U' \smallsetminus \bar{U}$, $\Phi : \mathbb{A}(1, \rho) \to R$. Since the boundary curves of R are $\kappa(\delta)$-quasi-circles, Φ admits a $\kappa_1(\delta)$-quasi-symmetric extension to the boundary (**??**). Let us select the map $h : \partial U' \to \mathbb{T}_{r^2}$ on the outer boundary of R in such a way that $h \circ \Phi$ is the homothety of $\mathbb{T}_\rho$ onto T_{r^2}. Following the strategy of Exercise 4.27, lift h to the inner boundary ∂U via the covering map $g : \partial U \to \partial U'$. Since by Lemma 4.27 this covering is $\kappa(\delta)$-quasi-symmetric, $h : \partial U \to \mathbb{T}_r$ is $\kappa_2(\delta)$-quasi-symmetric and hence $h \circ \Phi : \mathbb{T}_1 \to \mathbb{T}_r$ is $\kappa_3(\delta)$-quasi-symmetric.

By Lemma **??**, $h \circ \Phi$ admits a qc extension to R with the dilatation depending only on $\kappa_3(\delta)$ and $\mathrm{mod}(R)/\log r$. Selecting r in such a way that the latter ratio is bounded (for instance, take $\log r = \mathrm{mod}\, R$), we obtain a map $h \circ \Phi$ with dilatation depending only on δ. Since $\mathrm{Dil}(h) = \mathrm{Dil}(h \circ \Phi)$, we are done. $\qquad\qquad\square$

18. Expanding circle maps

Before passing to the uniquenss part of the Straightening Theorem, let us dwell on an important relation between quadratic-like and circle maps.

18.1. Definition. Recall that $\mathbb{T} \subset \mathbb{C}$ stands for the unit circle (endowed with the induced real analytic structure and Riemannian metric). *Symmetry* with respect to $\mathbb{T}$ is understood in the sense of the anti-holomorphic reflection $\tau : z \mapsto 1/\bar{z}$.

Let us say that $g : \mathbb{T} \to \mathbb{T}$ is an *expanding circle map of class $\mathcal{E}$* if it satisfies the following properties:

(i) g is an orientation preserving double covering of the circle over itself;

(ii) g is real analytic;

(iii) g is expanding, i.e, there exist constants $C > 0$ and $\lambda > 1$ such that for any $z \in \mathbb{T}$,

$$\|Dg^n(z)\| \geq C\lambda^n, \quad n = 0, 1, \ldots. \qquad (18.1)$$

The simplest example is provided by the *quadratic circle map f_0* : $z \mapsto z^2$. A little more generally, we have the Blyaschke circle maps:

EXERCISE 4.31. *Let $g : \mathbb{D} \to \mathbb{D}$ be a holomorphic double covering of the unit disk over itself which has a fixed point in $\mathbb{D}$. By Exercise ??, g admits a continuous extension to the unit circle T. Show that this extension is an expanding circle map of class $\mathcal{E}$.*

Hint: By Exercise ??, g actually extends to the whole sphere. To show that it is expanding on $\mathbb{T}$, use the hyperbolic metric in $\bar{\mathbb{C}} \smallsetminus (\mathrm{orb}(a) \cup \mathrm{orb}(1/\bar{a})$, where $a \in \mathbb{D}$ is the critical point of g.

To state some results in adequately general form, we will also consider a bigger class $\mathcal{E}^1$ of C^1-smooth expanding circle maps and a class $\mathcal{E}^{1+\alpha}$ of C^1-smooth maps whose derivative satisfies the Hölder condition with exponent $\alpha \in (0, 1)$. (However, for applications to holomorphic dynamics we will only need real analytic maps, so that the reader can always assume it.)

EXERCISE 4.32. *(i) For any $g \in \mathcal{E}^1$, there exists a smooth Riemannian metric ρ on $\mathbb{T}$ such that*

$$\|Dg(z)\|_\rho \geq \lambda > 1 \text{ for all } z \in \mathbb{T}.$$

This metric is called Lyapunov. Hint: Consider $\rho = ...$

EXERCISE 4.33. *Show that any expanding circle map $g \in \mathcal{E}^1$ has a unique fixed point $\beta \equiv \beta_g \in \mathbb{T}$.*

Hint: Lifting g to the universal covering, you obtain an orientation preserving diffeomorphism $G : \mathbb{R} \to \mathbb{R}$ satisfying the properties: a)$G(x+1) = G(x) + 2$; b) all fixed points of G is repelling. Or, use the Lefschetz formula instead.

18.2. Symbolic model. Let us consider a symbolic sequence $\bar{k} = (k_0, k_1, \dots) \in \Sigma$ of zeros and ones. Each such a sequence represents some number

$$\theta(\bar{k}) = \sum_{n=0}^{\infty} \frac{k_n}{2^{n+1}} \in [0, 1]$$

in its diadic expansion. As everybody learns in the school (in the context of decimal expansions), all numbers except those of the form $m/2^n$ admit a unique diadic expansion. The numbers of the form $m/2^n$ with odd m admit exactly two diadic expansions:

$$\frac{k_0}{2} + \cdots + \frac{k_{n-2}}{2^{n-1}} + \frac{1}{2^n} = \frac{k_0}{2} + \cdots + \frac{k_{n-2}}{2^{n-1}} + \sum_{m=n+1}^{\infty} \frac{1}{2^m}.$$

Thus the corresponding symbolic sequences viewed as representations of numbers should be identified. If we consider the numbers mod 1, then we should also identify the sequence **0** of all zeros to the sequence

1 of all ones. Let us call these identifications on Σ "arithmetic" and the space Σ modulo these identifications *arithmetic quotient* of Σ. Of course, this quotient is in a natural one-to-one correspondence with the unit interval with identified endpoints, i.e., with the circle.

EXERCISE 4.34. *Show that the projection*

$$\pi_0 : \Sigma \to \mathbb{T}, \quad \bar{k} \mapsto \exp(2\pi i\, \theta(\bar{k}))$$

(continuously) semi-conjugates the Bernoulli shift $\sigma : \Sigma \to \Sigma$ *(see §13.4) to the circle endomorphism* $f_0 : z \mapsto z^2$. *Thus* $f_0 : \mathbb{T} \to \mathbb{T}$ *is topologically conjugate to the arithmetic quotient of the Bernoulli shift.*

It turns out that the same is true for all expanding circle maps $g \in \mathcal{E}^1$:

LEMMA 4.34. *Any circle expanding map* $f \in \mathcal{E}^1$ *is topologically conjugate to the arithmetic quotient of the Bernoulli shift.*

PROOF. Let $g \in \mathcal{E}^1$. Consider its fixed point β. It has a single perimage β^1 different from $\beta \equiv \beta^0$. These two points, β and β^0, divide the circle into two (open) intervals intervals, I_0^1 and I_1^1 (counting anti-clockwise starting from β). Moreover, g homeomorphically maps each I_k^1 onto $\mathbb{T} \smallsetminus \beta$. Hence each I_k^1 contains a preimage β_k^2 of β^1. This point divides I_k^1 into two open intervals, I_{k0}^2 and I_{k1}^2 (counting anti-clockwise). We obtain four intervals, I_{kj}^2, $k, j \in \{0, 1\}$ such that g homeomorphically maps each I_{kj}^2 onto I_k^1.

Continuing inductively, we see that

$$\mathbb{T} \smallsetminus g^{-n}\beta = \bigcup_{k_s \in \{0,1\}} I_{k_0\,k_1\,\ldots\,k_{n-1}}^n,$$

where:

(i) the anti-clockwise order of the intervals $I_{\bar{k}}^n$ (starting from β) corresponds to the lexicographic order on the symbolic strings $\bar{k} = (k_0\,k_1\,\ldots\,k_{n-1})$;

(ii) the map g homeomorphically maps $I_{\bar{k}}^n$ onto $I_{\sigma(\bar{k})}^{n-1})$, where the string $\sigma(\bar{k}) = (k_1 \ldots k_{n-1})$ is obtained from $\bar{k}$ by erasing the first symbol.

(iii) any interval $I_{\bar{k}}^n$ contains a point $\beta_{\bar{k}}^{n+1} \in g^{-(n+1)}\beta$ which divides it into two intervals $I_{\bar{k}\,0}^{n+1}$ and $I_{\bar{k}\,1}^{n+1}$ of the next level.

Thus g^n homeomorphically maps each interval $I_{\bar{k}}^n$ onto the punctured circle $\mathbb{T} \smallsetminus \{\beta\}$. Since g is expanding, the lengths of these intervals shrink exponentially fast:

$$|I_{\bar{k}}^n| \leq \frac{2\pi}{C}\lambda^{-n},$$

where $C > 0$ and $\lambda > 1$ are constants from (18.1). It follows that for any infinite sequence $\bar{k} = (k_0 k_1 \dots) \in \Sigma$ of zeros and ones, the closed intervals $\bar{I}^n_{k_0 \dots k_{n-1}}$ form a nest shrinking to a single point $z = \pi(\bar{k})$. Thus we obtain a map $\pi : \Sigma \to \mathbb{T}$.

Under this map, the cylinders of rank n are mapped to the intervals of rank n. Since the latter shrink, π is continuous.

The above property (ii) implies that π is equivariant. Thus g is a quotient of the Bernoulli shift.

We only need to describe the fibers of π. If z is not an iterated preimage of β, then it belongs to a *single* interval of any rank. Hence $\mathrm{card}(\pi^{-1}(z)) = 1$. Obviously the fiber $\pi^{-1}(\beta)$ consists of two extremal sequences, $(\mathbf{0})$ and $\mathbf{1}$. Otherwise $z = \beta^{n+1}_{k_0 \dots k_{n-1}} \in g^{-(n+1)}\beta$ for some $n \geq 0$ (except that for $n = 0$, the point β^1 does not have subsripts). Then it is a boundary point for exactly two intervals of each order $m \geq n + 1$. For $m = n + 1$, the corresponding symbolic sequences differ by the last symbol only: $(k_0 \dots k_{n-1} 0)$ and $(k_0 \dots k_{n-1} 1)$. For all further levels, we should add symbol 1 to the first sequence and symbol 0 to the second one. Thus:

$$\pi(k_0 \dots k_{n-1} 0\, 1\, 1\, 1 \dots) = z = \pi(k_0 \dots k_{n-1} 1\, 0\, 0\, 0 \dots),$$

which are exactly the arithmetic identifications on Σ. $\qquad\square$

Thus all expanding circle maps of class $\mathcal{E}^1$ are topologically the same:

PROPOSITION 4.35. *Any two expanding circle maps of class $\mathcal{E}^1$ are topologically conjugate by a unique orientation preserving circle homeomorphism. In particular, expanding circle maps do not admit non-trivial orientation preserving automorphisms.*

PROOF. Lemma 4.34 gives the same standard model for any expanding circle map of class $\mathcal{E}^1$. In this model, the anti-clockwise order on $\mathbb{T} \smallsetminus \{\beta\}$ corresponds to the lexicographic order on Σ. Hence the corresponding conjugacy h between two circle maps, g and $\tilde{g}$, is orientation preserving.

Such a conjugacy is unique. Indeed, it must carry the points of $g^{-n}(\beta)$ to $\tilde{g}^{-1}(\tilde{\beta})$ preserving their anti-clockwise order starting from the corresponding fixed points, β and $\tilde{\beta}$. Hence h is uniquely determined on the iterated preimages of β. Since these preimages are dense in $\mathbb{T}$ (by the previous lemma), h is uniquely determined on the whole circle. $\quad\square$

Remarks. 1. Expanding circle maps have one orientation reversing automorphism. In the case of $z \mapsto z^2$ it is just $z \mapsto \bar{z}$ (compare with Exercise 4.5).

2. The above discussion can be generalized in a straightforward way to expanding circle maps of degree $d > 2$. There is one difference though: if $d > 2$ then the group of orientation preserving automorphisms of g is not trivial any more but rather the cyclic group of order $d - 1$ (consider $z \mapsto z^d$).

18.3. Equivariant liftings. Let us describe a lifting construction which will find numerous applications in what follows.

Consider two open conformal annuli $\Omega \subset \Omega' \subset \mathbb{C}$ with a common inner boundary. Assume that $A = \overline{\Omega' \smallsetminus \Omega}$ is a (closed) annulus whose boundary components are smooth Jordan curves. Let $g : \Omega \to \Omega'$ be a holomorphic double covering map. A point $z \in \Omega$ is called *escaping* if $f^n z \in A$ for some $n \in \mathbb{N}$.

Consider also another map $\tilde{g} : \tilde{\Omega} \to \tilde{\Omega}'$ with the same properties (all corresponding objects for $\tilde{g}$ will be marked with "tilde").

LEMMA 4.36. *Under the circumstances just described, assume that all points of Ω and Ω' are escaping. Then any equivariant homeomorphism $H : A \to \tilde{A}$ admits a unique homeomorphic extension $h : \Omega' \to \tilde{\Omega}'$ conjugating g to $\tilde{g}$. If H is quasi-conformal then so is h, and $\mathrm{Dil}(h) = \mathrm{Dil}(H)$. Moreover, the Beltrami differential $\mu_h = h^* \sigma$ is obtained by pulling back the Beltrami differential $\mu_H = H^* \sigma$ by the iterates of g: $\mu_h | A^n = (g^n)^* \mu_H$.*

PROOF. Let $A^n = g^{-n} A$, and let Γ^n be the outer boundary of A^n (coinciding for $n \geq 1$ with the inner boundary of A^{n-1}). Consider an equivariant homeomorphism $H : A \to \tilde{A}$. This map admits a lift $H_1 : A^1 \to \tilde{A}^1$ such that $\tilde{g} \circ H_1 = H \circ g | A^1$. In fact, there are exacly two such lifts determined by a value of H_1 at a single point.

The restriction of H_1 to the outer boundary Γ^1 is a lift of $H : \Gamma^0 \to \tilde{\Gamma}^0$. But since H is equivariant on ∂A, its restriction to Γ^1 is also a lift of $H : \Gamma^0 \to \tilde{\Gamma}^0$. Hence the lift H_1 can be chosen in such a way that $H_1 | \Gamma^1 = H | \Gamma^1$. With this choice, H and H_1 glue together to an equivarinat homeomorphism $h_1 : A \cup A^1 \to \tilde{A} \cup \tilde{A}^1$. Now equivariance means that $\tilde{g} \circ h_1 | A^1 = h_1 \circ g | A^1$. In particular, h_1 is equivariant on the boundary of A^1, so that we can apply to it the above construction. It provides us with an equivariant extension $h_2 : A \cup A^1 \cup A^2 \to \tilde{A} \cup \tilde{A}^1 \cup \tilde{A}^2$ of h_1.

Proceeding in this way we will obtain a sequence of equivariant liftings $H_n : A^n \to \tilde{A}^n$ which glue together to equivariant homeomorphisms

$$h_n : \bigcup_{k=0}^{n} A^k \to \bigcup_{k=0}^{n} \tilde{A}^k$$

extending one another. Since all the points in Ω escape, the annuli A^k exhaust Ω', and similarly for $\tilde{\Omega}'$. Hence the direct limit of equivariant extensions h_n is a homeomeorhism $h : \Omega' \to \tilde{\Omega}'$ conjugating g to $\tilde{g}$.

It shows existence of a conjugacy h for any given H. Uniqueness is obvious: $h|A$ consecutively determines the lifts $h|A^n$ by requirements of equivarience and continuous matching.

Finally, assume that H is quasi-conformal with dilatation K. Since g and $\tilde{g}$ are conformal, all the consecutive lifts of H to the annuli A^n are qc maps with the same dilatation K. By Proposition 2.10, their gluings (maps h_n) are K-qc maps as well. The direct limit h of K-qc extensions h_n is obviously K-qc as well.

The last statement is obvious due to the natural behavior of the Beltrami differentials under conformal liftings: $\mu_{H_n} = (g^{\circ n})^* \mu_H$ since $\tilde{g}^{\circ n} \circ H_n = H \circ g^{\circ n}$ where $g^{\circ n}$ and $\tilde{g}^{\circ n}$ are conformal. $\qquad\square$

Remark. We do not need to assume that the annuli Ω and Ω' are embedded to $\mathbb{C}$.

PROBLEM 4.35. *Is the assumption that all points in Ω escape automatically satisfied if* $\mathrm{mod}(\Omega) < \infty$?

18.4. Complex extensions of circle maps. In this section we will take a closer look at the holomorphic extensions of expanding cicle maps of class $\mathcal{E}$.

EXERCISE 4.36. *(i) For any $g \in \mathcal{E}$, there exist two $\mathbb{T}$-symmetric topological annuli $V \Subset V'$ (bounded by smooth Jordan curves) such that g admits a holomorphic extension to V and maps it onto V' as a double covering.*

Hint: Extend the Lyapunov metric from Exercise 4.32 to a neighborhood of $\mathbb{T}$.

(ii) Show that vice versa, property (i) imlies that $g \in \mathcal{E}$. Hint: Use the hyperbolic metric in V'.

(iii) Show that all points $z \in V \smallsetminus \mathbb{T}$ escape, i.e., $g^n z \in V' \smallsetminus V$ for some $n \in \mathbb{N}$.

Hints should go to an Appendix.

Thus property (i) can be used as a definition of an expanding circle map of class $\mathcal{E}$. In fact, only exterior part of the above extension is needed to reconstruct the circle map (it will be useful in what follows):

LEMMA 4.37. *Let $\Omega \subset \Omega' \subset \mathbb{C}$ be two open conformal annnuli whose inner boundaries coincide with the unit circle $\mathbb{T}$. Let $g : \Omega \to \Omega'$ be a holomorphic double covering. Then g admits an extension to*

a holomorphic double covering $G : V \to V'$, where $V \Subset V'$ are $\mathbb{T}$-symmetric annuli such that $\Omega = V \smallsetminus \bar{D}$ and $\Omega' = V' \smallsetminus \bar{D}$. If the outer boundary of Ω is contained in Ω', then $V \Subset V'$ and the restriction $G|\mathbb{T}$ is an expanding cicle map of class $\mathcal{E}$.

PROOF. First show that g continuously extends to $\mathbb{T}$ (apply boundary properties of confomal maps to inverse branches of g **??**). Then use the Schwarz Reflection Principle. □

Consider a holomorphic extension $g : V \to V'$ of a map $g \in \mathcal{E}$ given by Exercise **??**. Thus $V \Subset V'$ are two $\mathbb{T}$-symmetric annuli neighborhoods of the circle. Let $A = (\bar{V}' \smallsetminus V) \smallsetminus \mathbb{D}$ be the "outer" fundamental annulus for g.

Given another map $\tilde{g} : \tilde{V} \to \tilde{V}'$ as above, we will mark the corresponding objects with "tilde".

PROPOSITION 4.38. *Any two expanding circle maps $g : V \to V'$ and $\tilde{g} : \tilde{V} \to \tilde{V}'$ are conjugate by a qc map $h : (V', V, \mathbb{T}) \to (\tilde{V}', \tilde{V}, \mathbb{T})$ commuting with the reflection τ about the circle. In fact, any equivariant qc map $H : A \to \tilde{A}$ between the fundamental annuli admits a unique extension to a qc conjugacy h as above. Moreover $\mathrm{Dil}(h) = \mathrm{Dil}(H)$.*

PROOF. Consider an equivariant qc map H as above with dilatation K. By Lemma 4.36 it can be uniquely lifted to an equivariant K-qc homeomorphism $h : V' \smallsetminus \bar{\mathbb{D}} \to \tilde{V}' \smallsetminus \bar{D}$. By **??**, h admits a continuous extension to the unit circle. Reflecting it to the interior of the circle (and then exploiting Proposition 2.10) we obtain a desired K-qc conjugacy $h : V' \to \tilde{V}'$. □

Let us endow the exterior $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$ of the unit disk, with the hyperbolic metric $\rho \equiv \rho_{\mathbb{C} \smallsetminus \bar{D}}$. The hyperbolic length of a curve γ will be denoted by $l_\rho(\gamma)$, while it Euclidean length will be denoted by $|\gamma|$.

LEMMA 4.39. *Let $g : V \to V'$ be an expanding circle map of class $\mathcal{E}$. Let Ω and Ω' be two (open) annuli whose inner boundary is the circle $\mathbb{T}$. Let $h : \Omega \to \Omega'$ be a homeomorphism commuting with g. Then h admits a continuous extension to a map $\Omega \cup \mathbb{T} \to \tilde{\Omega} \cup \mathbb{T}$ identical on the circle.*

************************************* unedited

PROOF. Given a set $X \subset A$, let $\tilde{X}$ denote its image by ω. Let us take a configuration consisting of a round annulus $L^0 = \mathbb{A}[r, r^2]$ contained in A, and an interval $I_0 = [r, r^2]$. Let $L^n = P_0^{-n} L^0$, and I_k^n denote the components of $P_0^{-n} I^0$, $k = 0, 1, \ldots, 2^n - 1$. The intervals I_k^n subdivide the annulus L^n into 2^n "Carleson boxes" Q_k^n.

Since the (multi-valued) square root map P_0^{-1} is infinitesimally contracting in the hyperbolic metric, the hyperbolic diameters of the boxes $\tilde{Q}_k^n$ are uniformly bounded by a constant C.

Let us now show that ω is a hyperbolic quasi-isometry near the circle, that is, there exist $\epsilon > 0$ and $A, B > 0$ such that

$$A^{-1}\rho(z, \zeta) - B \leq \rho(\tilde{z}, \tilde{\zeta}) \leq A\rho(z, \zeta) + B, \qquad (18.2)$$

provided $z, \zeta \in \mathbb{A}(1, 1 + \epsilon)$, $|z - \zeta| < \epsilon$.

Let γ be the arc of the hyperbolic geodesic joining z and ζ. Clearly it is contained in the annulus $\mathbb{A}(1, r)$, provided ϵ is sufficiently small. Let $t > 1$ be the radius of the circle $\mathbb{T}_t$ centered at 0 and tangent to γ. Let us replace γ with a *combinatorial geodesic* Γ going radially up from z to the intersection with $\mathbb{T}_t$, then going along this circle, and then radially down to ζ. Let N be the number of the Carleson boxes intersected by Γ. Then one can easily see that

$$\rho(z, \zeta) = l_\rho(\gamma) \asymp l_\rho(\Gamma) \asymp N,$$

provided $\rho(z, \zeta) \geq 10 \log(1/r)$ (here $\log(1/r)$ is the hyperbolic size of the boxes Q_k^n).

On the other hand

$$\rho(\tilde{z}, \tilde{\zeta}) \leq l_\rho(\tilde{\Gamma}) \leq CN,$$

so that $\rho(\tilde{z}, \tilde{\zeta}) \leq C_1 \rho(z, \zeta)$, and (18.2) follows.

But quasi-isometries of the hyperbolic plane admit continuous extensions to $\mathbb{T}$ (see, e.g., [**Th**]). Finally, it is an easy exercise to show that the only homeomorphism of the circle commuting with P_0 is identical. $\qquad\square$

**8

We will show next that "outer automorphisms" of circle maps move points bounded hyperbolic distance:

LEMMA 4.40. *Let* $g : V \to V'$ *be a map of class* $\mathcal{E}$. *Let* Ω *and* Ω' *be two open annuli in* $V \smallsetminus \bar{\mathbb{D}}$ *with inner boundary* $\mathbb{T}$, *and let* $h : \Omega \to \Omega'$ *be an automorphism of* g. *Then for any* $\delta > 0$ *there exists an* $R = R(\delta) > 0$ *such that* $\rho(z, hz) \leq R$ *for all points* $z \in \Omega$ *whose distance from the outer boundary of* Ω *is at least* δ.

PROOF. By Proposition 4.38, g is qc conjugate to the quadratic circle map $f_0 : z \mapsto z^2$. Of course, this conjugacy can be extended to a global qc homeomorphism of $\bar{C}$ (e.g., by **??**). Since qc homeomorphisms of $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$ are hyperbolic quasi-isometries (**??**), it is enough to prove the assertion for f_0. So, let us assume from now on that $g = f_0$.

Of course, the assertion is true for any compact subset of Ω. Hence we need to check it only near to the unit circle.

By 4.39, h admits a continuous extension to the unit circle. Of course, it still commutes with g on the circle. By Proposition 4.35, $h|\mathbb{T} = \mathrm{id}$. Hence for any $\epsilon > 0$ there exists an $r > 1$ such that $\mathbb{A}(1, r] \Subset \Omega$ and

$$|z - hz| < \epsilon \quad \text{for} \quad z \in \mathbb{A}(1, r].$$

Consider a fundamental annulus A of g compactly contained in $\mathbb{A}(1, r]$. By compactness, there exists an $R > 0$ such that

$$\rho(z, hz) \le R \quad \text{for} \quad z \in A.$$

Let $A^n = g^{-n}A$. Take some $z \in A^1$. Since $|z - hz| < \epsilon$, these points are obtained by applying the *same local branch* of the square root map g^{-1} to the points gz and $g(hz) = h(gz)$. Since the local branches of g^{-1} preserve the hyperbolic distance on $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$, we have: $\rho(z, hz) = \rho(gz, h(gz)) \le R$.

Replacing A by A^1, we obtain the same bound for any $z \in A^2$, etc. The conclusion follows.   □

18.5. External map (the connected case). To any quadratic-like map $f : U \to U'$ one can naturally associate an expanding circle map g of class $\mathcal{E}$ which captures dynamics outside the Julia set. For this reason g is called the *external map* of f.

The construction is very simple if the Julia set $J(f)$ is connected. In this case the basin of infinity $D_f(\infty) = \mathbb{C} \smallsetminus K(f)$ is simply connected and can be conformally mapped onto the complement of the unit disk:

$$R : \mathbb{C} \smallsetminus K(f) \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}.$$

Let $\Omega = R(V \smallsetminus K(f))$, $\Omega' = R(V' \smallsetminus K(f))$. These are two conformal annuli with smooth boundary. Moreover, the have a common inner boundary, the unit circle $\mathbb{T}$, while the outer boundary of Ω is contained in Ω'. Conjugating f by R we obtain a holomorphic double covering

$$g : \Omega \to \Omega', \quad g(z) = R \circ f \circ R^{-1}(z) \quad \text{for} \quad z \in \Omega.$$

By Lemma 4.37, g can be extended to an expanding circle map of class $\mathcal{E}$.

In fact, this map is not uniquely defined since the Riemann map R is defined up to post-composition with rotation $z \mapsto e^{2\pi i\theta}z$, $0 \le \theta < 2\pi$. A natural way to *normalize* g is to put its fixed point β to $1 \in \mathbb{T}$.

Note also that if f is replaced by an affinely conjugate map $A^{-1} \circ f \circ A$, where $A : z \mapsto \lambda z$, $\lambda \in \mathbb{C}^*$, then the Riemann map R is replaced by $R \circ A$, and the external map g remains the same. Thus, to any quadratic-like map f (with connected Julia set) prescribed *up to an*

affine conjugacy corresponds an expanding circle map g well-defined *up to rotation conjugacy.*

We will consider the case of disconnected Julia set in §**??**.

18.6. Uniqueness of the straightening. Let us first show that "external automorphisms" of quadratic-like maps admit a continuous extension to the Julia set by identity (compare with Lemma 4.39).

LEMMA 4.41. *Let $f : U \to U'$ be a quadratic-like map with connected Julia set. Let $W \subset U$ and $W' \subset U$ be two (open) annuli whose inner boundary is $J(f)$. Let $h : W \to W'$ be a homeomorphism commuting with f. Then h admits a continuous extension to a map $W \cup J(f) \to W' \cup J(f)$ identical on the Julia set.*

PROOF. Consider the Riemann mapping $R : \mathbb{C} \smallsetminus K(f) \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}$ and the external circle map $g : V \to V'$, $g|V \smallsetminus \bar{D} = R \circ f \circ R^{-1}$. Transfer the annuli W and W' to the g-plane. We obtain two annuli $\Omega = R(W)$ and $\Omega' = R(W')$ in $V \smallsetminus \bar{\mathbb{D}}$ attached to the unit circle $\mathbb{T}$. Of course, the homeomorphism $k : \Omega \to \Omega'$, $k = R \circ h \circ R^{-1}$, commutes with g.

By Lemma 4.40, k moves points near $\mathbb{T}$ bounded hyperbolic distance: $\rho_{\mathbb{C} \smallsetminus \bar{\mathbb{D}}}(k(z), z) \le R$. Since the Riemann mapping $R : \mathbb{C} \smallsetminus \bar{D} \to \mathbb{C} \smallsetminus K(f)$ is a hyperbolic isometry, the same is true for h:

$$\rho_{\mathbb{C} \smallsetminus K(f)}(z, h(z) \le R$$

for $z \in W$ near $J(f)$. By **??**, the Euclidean distance $|z - hz|$ goes to 0 as $z \to J(f)$, $z \in W$. It follows that the extension of h by the identity on the Julia set is continuous. $\qquad\square$

COROLLARY 4.42. *Let f and $\tilde{f}$ be two quadratic-like maps, and let a homeomorphism h conjugates f to $\tilde{f}$ in some neighborhoods of the filled Julia sets. Then h is uniquely determined on $J(f)$.*

PROBLEM 4.37. *Assume that quadratic polynomials f and $\tilde{f}$ are conjugate on the Julia sets only. Is the conjugacy unique?*

Let us now summarize the above results:

THEOREM 4.43. *Let us consider two quadratic-like maps $f : U \to U'$ and $\tilde{f} : \tilde{U} \to \tilde{U}'$ with connected Julia sets. Assume that they are topologically conjugate near their Julia sets by a homeomorphism $\psi : V \to \tilde{V}$. Assume also that we are given an equivariant homeomorphism $H : A \to \tilde{A}$ between the (closed) fundamental annuli of f and $\tilde{f}$.*

Then there exists a unique homeomorphism $h : U' \to \tilde{U}'$ conjugating f to $\tilde{f}$, coinciding with ψ on the Julia set $J(f)$, and coinciding with H on A.

If H is qc, then $h|\,U \smallsetminus K(f)$ is also qc with the same dilatation. If both H and ψ are qc, then h is qc, and

$$\mathrm{Dil}(h) = \max(\mathrm{Dil}\,H, \mathrm{Dil}(\psi|\,K(f)).$$

In particular, if f and g are hybrid equivalent by means of ψ, then $\mathrm{Dil}(h) = \mathrm{Dil}(H)$.

Proof. By the Lifting Construction of §18.3, H admits a unique equivariant extension to a homeomorphism $h : U \smallsetminus K(f) \to \tilde{U} \smallsetminus K(\tilde{f})$. This extension continuously matches with ψ on the filled Julia set. Indeed, $\psi^{-1} \circ h$ commutes with f on some exterior neighborhood of $K(f)$. By Lemma 4.41, this map continuously extends to the filled Julia set as identity. Hence h continuous extends to the filled Julia set as ψ.

If H is qc then $h|\,U \smallsetminus K(f)$ is qc with the same dilatation by Lemma 4.36. All the rest follows from the Bers Lemma 2.11. $\square$

Of course, we can always construct an equivariant qc map H between the fundamental annuli. Hence if two quadratic-like maps are topologically equivalent, then the conjugacy can be selected quasi-conformal outside the filled Julia set. If they are hybrid equivalent, then the dilatation of the conjugacy is completely controlled by the dilatation of H, which is in turn controlled by the geometry of the fundamental annuli (see **??**). In the case of global polynomials we can do even better:

Corollary 4.44. *Consider two quadratic polynomials $f : z \mapsto z^2 + c$ and $\tilde{f} : z \mapsto z^2 + \tilde{c}$ with connected Julia sets. If they are topologically conjugate near their filled Julia sets, then there is a global conjugacy $h : \mathbb{C} \to \mathbb{C}$ which is conformal on the basin of ∞. If f and $\tilde{f}$ are hybrid conjugate near their filled Julia sets, then $f = \tilde{f}$.*

Proof. By §**??**, the Riemann-Bötcher map $B_f : D_f(\infty) \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}$ conjugates f to $z \mapsto z^2$. Hence the composition

$$R : B_{\tilde{f}}^{-1} \circ B_f :\, D_f(\infty) \to D_{\tilde{f}}(\infty) \tag{18.3}$$

conformally conjugates f to $\tilde{f}$ on their basins of ∞. By the previous theorem, this conjugacy matches with the topological conjugacy on the filled Julia set giving us a desired global conjugacy h.

Moreover, If f and $\tilde{f}$ are hybrid equivalent, then $\mathrm{Dil}(h) = 0$ a.e. By Weil's Lemma **??**, h is conformal and hence affine. But if two quadratic polynomials in the normal form $z^2 + c$ are affinely equivalent, then they are equal. $\square$

The last statement of the above Corollary gives the uniqueness part of the Straightening Theorem.

18.6.1. *Picture.*

19. Harmonic measure on $J(f)$

CHAPTER 5

Parameter plane (the Mandelbrot set)

20. Definition and first properties

20.1. Notational convention. We will label the objects corresponding to a map f_c by c, e.g., $J_c = J(f_c)$, $\mathrm{Per}(f_c) = \mathrm{Per}_c$. We often use notation $c_\mathsf{o} \equiv base$ for a base parameter, so that $f_\mathsf{o} = f_{c_\mathsf{o}}$, $J_\mathsf{o} = J_{c_\mathsf{o}}$, etc.

20.2. Connectedness locus and polynomials $c \mapsto f_c^n(0)$. The Mandelbrot set presents at one glance the whole dynamical diversity of the complex quadratic family $f_c : z \mapsto z^2 + c$. Figure ... shows this set and its blow-ups in several places. It is remarkable that all this intricate structure is hidden behind the following one-line definition.

Recall the Basic Dichotomy for the quadratic maps: the Julia set $J(f_c)$ is either connected or Cantor (Theorem 4.3). By definition, the *Mandelbrot set M* consists of those parameter values $c \in \mathbb{C}$ for which the Julia set J_c is connected. It is equivalent to saying that the orbit of the critical point

$$0 \mapsto c \mapsto c^2 + c \mapsto (c^2 + c)^2 + c \mapsto \ldots \qquad (20.1)$$

is not escaping to ∞. Let us denote the nth polynomial in (20.1) by $v_n(c)$, so that $v_0(c) \equiv 0$, $v_1(c) \equiv c$, and recursively

$$v_{n+1}(c) = v_n(c)^2 + c. \qquad (20.2)$$

Note that $\deg \phi_n = 2^{n-1}$.

Though the polynomials v_n are not iterates of a single polynomial, they behave in many respects similarly to the iterated polynomials:

EXERCISE 5.1 (Simplest properties of M). *Prove the following properties:*

(i) *If $|v_n(c)| > 2$ for some $n \in \mathbb{N}$ then $v_n(c) \to \infty$ as $n \to \infty$. In particular, $M \subset \bar{\mathbb{D}}_2$.*

(ii) *$v_n(c) \to \infty$ locally uniformly on $\mathbb{C} \smallsetminus M$. Hence M is compact.*

(iii) *$\mathbb{C} \smallsetminus M$ is connected. Hence M is full and all components of int M are simply connected.*

(iv) *The set of normality of the sequence $\{v_n\}$ coincides with $\mathbb{C} \smallsetminus \partial M$.*

One can see a similarity between the Mandelbrot set (representing the whole quadratic family) and a fillied Julia set of a particular quadratic map. It is just the first indication of a deep relation between dynamical and parameter objects.

Note that Proposition 4.4 describes the real slice of the Mandelbrot set:

$$M \cap \mathbb{R} = [-2, 1/4].$$

20.3. Dependence of periodic points on c. What immediately catches the eye in the Mandelbrot set is the *main cardioid C* with a cusp at $c = 1/4$. The cardioid bounds a domain of parameter values c such that f_c has an attracting fixed point.

EXERCISE 5.2. *Show that the main cardioid is given by the equation*

$$c = \frac{1}{2}e^{2\pi i\theta} - \frac{1}{4}e^{4\pi i\theta}, \quad 0 \le \theta < 2\pi,$$

where $\sigma = e^{2\pi i\theta}$ is the multiplier of the neutral fixed point of f_c.

Let us now take a look at how periodic points move with parameter:

LEMMA 5.1. *Let f_* has a cycle $\{\alpha_k\}_{k=0}^{p-1}$ of period p with multiplier $\sigma_0 \ne 1$. Then for nearby c, the maps f_c have a cycle $\{\alpha_k(c)\}_{k=0}^{p-1}$ holomorphically depending on c. Its multiplier $\mathrm{mult}(c)$ holomorphically depends on c as well.*

PROOF. Consider an algebraic equation $f_c^p(z) = z$. For $c = c_o$ it has roots $z = \alpha_k$, $k = 0, \ldots, p - 1$ (and maybe others). Since

$$\left.\frac{\partial(f_c^p(z) - z)}{dz}\right|_{c=c_o,\, z=\alpha_k} = \sigma_0 - 1 \ne 0,$$

the Implicit Function Theorem yields the first assertion. The second assertion follows from the formula for the multiplier:

$$\sigma(c) = 2^p \prod_{k=0}^{p-1} \alpha_k(c).$$

$\square$

Thus periodic points of f_c as functions of the parameter are algebraic functions branched at parabolic points only.

20.4. Hyperbolic components. A parameter value $c \in \mathbb{C}$ is called *hyperbolic/parabolic/Siegel* etc. if the corresponding quadratic polynomial f_c is such.

PROPOSITION 5.2 (Hyperbolic components). *The set $\mathcal{H}$ of hyperbolic parameter values is contained in* int M. *If H is a component of* int M *intersecting $\mathcal{H}$ then $H \subset \mathcal{H}$.*

PROOF. Lemma 5.1 implies that the set of hyperbolic parameter values is open. Since parameters in $\mathbb{C} \smallsetminus M$ are not hyperbolic (according to our terminology: see §**??**), the boundary parameter values $c \in \partial M$!!! cannot be hyperbolic either. Thus $\mathcal{H} \subset$ int M.

Take some some hyperbolic parameter value $c_{\mathsf{o}} \in H_0$. The corresponding map f_{o} has an attracting cycle of some period p. By Theorem 4.7, this cycle contains a point α_0 such that

$$\upsilon_{pn}(c_0) \equiv f_{\mathsf{o}}^{pn}(0) \to \alpha_0 \text{ as } n \to \infty.$$

It is easy to see (Exercise!) that for nearby $c \in H$ we have:

$$\upsilon_{pn}(c) \equiv f_{\mathsf{o}}^{pn}(0) \to \alpha_0(c) \text{ as } n \to \infty,$$

where $\alpha_0(c)$ is the holomorphically moving attracting periodic point of f_c (Lemma **??**). But the sequence of polynomials $\upsilon_{pn}(c)$, $n = 0, 1, \ldots$, is normal in H (Exercise 5.1, (iv)). Hence it must converge in the *whole domain H* to some holomorphic function $\tilde{\alpha}(c)$ coinciding with $\alpha_0(c)$ near c_0. By analytic continuation, $\tilde{\alpha}(c)$ is a a periodic point of f_c with period dividing p.

Moreover, the cycle of this point attracts the critical orbit persistently in H. It is impossible if this cycle is repelling somewhere. Indeed, a repelling cycles can only attract an orbit which eventually lands at it. This property is not locally persistent since otherwise it would hold for *all* $c \in \mathbb{C}$ (while it is violated, say, for $c = 1$).

If $\tilde{\alpha}(c)$ were parabolic for some $c \in H$, then it could be made repelling for a nearby parameter value. Thus $\tilde{\alpha}(c)$ is attracting for all $c \in H$, so that $H \subset \mathcal{H}$.   □

COROLLARY 5.3. *Neutral parameters lie on the boundary of M.*

PROOF. Let c_0 be a neutral parameter, i.e., the map f_{o} has a neutral cycle. This parameter can be perturbed to make the cycle attracting. If c_0 belonged to int M then by Proposition 5.2 it would be hyperbolic itself – contradiction.   □

EXERCISE 5.3. (i) *Any parameter $c \in \partial M$ can be approximated by superattracting parameters;*
(ii) *Misiurewicz parameters form a countable dense subset of ∂M.*

A component Λ of int M is called *hyperbolic* if it consists of hyperbolic parameter values. Otherwise Λ is called *queer*. The reason for the last term is that it is generally believed that *there are no queer components*. In fact, it is a central conjecture in contemporary holomorphic dynamics:

CONJECTURE 5.4 (Density of hyperbolicity). *There are no queer components. Hyperbolic parameters are dense in $\mathbb{C}$.*

Because of Exersice 5.3 (i), the second part of the conjecture would follow from the first one. It is sometimes referred to as *Fatou's Conjecture.*

20.5. Primitive and satellite hyperbolic components.

PROPOSITION 5.5. *Let H be a hyperbolic component of period n of M, let $p/q \neq 0 \mod 1$, and let $r_{p/q} \in \partial H$ be a parabolic parameter with rotation number p/q. Then there is a hyperbolic component H' of period np attached to H at $r_{p/q}$.*

A hyperbolic component H' that was born from another hyperbolic component by the period increasing bifurcation described in Proposition is called *satellite*. All other hyperbolic components of M are called *primitive*. They appear as a result of a *saddle-node* bifurcation.

Parabolic points on ∂H with multiplier 1 are called the *roots* of H. (We will see below (Theorem 5.9) that any hyperbolic component has a single root.) In particular, the bifucation point $r_{p/q}$ is the root of the satellite component H'.

The type of the component can be easily recognized geometrically:

PROPOSITION 5.6. *Satellite components are bounded by smooth curves, while primitive components have cusps at their roots.*

21. Connectivity of M

21.1. Uniformization of $\mathbb{C} \smallsetminus M$. In this section we will prove the first non-trivial result about the Mandelbrot set established by Douady and Hubbard in early 1980's. The strategy of the proof is quite remarkable: it is based on the explicit uniformization of the complement $\mathbb{C} \smallsetminus M$ by $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$. Recall from §16.4.2 that for $c \in \mathbb{C} \smallsetminus M$, we have a well-defined function

$$a = \Phi_M(c) = \phi_c(c), \tag{21.1}$$

where ϕ_c is the Böttcher function for f_c extended to the domain Ω_c bounded by the critical figure-eight equipotential.

THEOREM 5.7. *The Mandelbrot set M is connected. The function Φ_M conformally maps $\mathbb{C} \smallsetminus M$ onto $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$. Moreover, it is tangent to the identity at ∞: $\Phi_M(c) \sim c$ as $c \to \infty$.*

We immediately obtain the parameter analogue of Corollary 4.25:

COROLLARY 5.8. *The Mandelbrot set has capacity 1.*

We will give two proofs of Theorem 5.7. The first proof is short and elementary (it was the original proof given by Douady and Hubbard). The second proof, though longer and more demanding, illuminates the meaning of formula (21.1) and the ideas of qc deformations.

21.2. An elementary proof. It is based upon the explicit formula (16.7) for the Böttcher coordinate near ∞,

$$\phi_c(z) = \lim_{n \to \infty} (f_c^n(z))^{1/2^n}, \tag{21.2}$$

where the root in the right-hand side is selected in such a way that it is tangent to the identity at ∞. The sratedy is to show that R_M is a holomorphic branched covering of degree 1.

Step 1: analyticity. Let us consider a set $\boldsymbol{\Omega} = \{(c, z) \in \mathbb{C}^2 : z \in \Omega_c\}$, where we the $\Omega_c \subset \mathbb{C} \smallsetminus K(f_c)$ is the maximal equipotential saturated domains of analyticity of the Böttcher function ϕ_c (see §16.4). It is easy to see that this set is open. Indeed, for any c_0, there exist an $R > 0$ and $\epsilon > 0$ such that $|f_c(z)| > 2|z|$ for all $c \in D(c_0, \epsilon)$ and $|z| > R$. Hence $\mathbb{C} \smallsetminus \bar{\mathbb{D}}_R \subset \mathbb{C} \smallsetminus K(f_c)$ for all (c, z) as above.

Now, if $\zeta_0 \in \mathbb{C} \smallsetminus K(f_\circ)$ then $f_\circ^n(\zeta_0) \in \mathbb{C} \smallsetminus \bar{\mathbb{D}}_R$ for some n. By continuity, $f_c^n(\zeta) \in \mathbb{C} \smallsetminus \bar{D}_0$ for all (c, ζ) sufficiently close to (c_0, ζ_0), and the openness follows.

We also see that the orbits of f_c, $c \in \boldsymbol{\Omega}$, escape to ∞ at a locally uniform rate, which implies that convergence in the Brolin formula (16.11),

$$G_c(z) = \lim_{n \to \infty} \frac{1}{2^n} \log |f_c^n z|,$$

is locally uniform on $\boldsymbol{\Omega}$. Hence $(c, z) \mapsto G_c(z)$ is a continuos function on $\boldsymbol{\Omega}$,[1] so that, the set $\boldsymbol{\Omega}' = \{(c, z) \in \boldsymbol{\Omega} : G_c(z) > G_c(0)\}$ is also open.

But for the same reason, convergence in the Böttcher formula (21.2) is locally uniformly on $\boldsymbol{\Omega}'$. Hence the Böttcher function $(c, z) \mapsto \phi_c(z)$ is holomorphic on $\boldsymbol{\Omega}'$. We conclude that the function $R_M : \mathbb{C} \smallsetminus M \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}$, $R_M(c) = \phi_c(c)$, is holomorphic on $\mathbb{C} \smallsetminus M$.

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[1]It also follows that this function is *pluriharmonic* on $\boldsymbol{\Omega}$, i.e., its slices to one-dimensional holomorphic curves in $\boldsymbol{\Omega}$ are harmonic.

Step 2: behavior at ∞. Let $c_n = f_c^n(c)$. Then $|c_{n+1}| = |c_n^2 + c_n| \geq \frac{1}{2}c_n^2$, provided $|c_n|$ is big enough. It follows that $|R_m(c)| \geq \frac{1}{2}|c| \to \infty$ as $c \to \infty$. Hence R_M exdends holomorphically to ∞ and $R_M(\infty) = \infty$. Moreover, since

$$\sqrt[2^n]{f_c^n(c)} = \sqrt[2^n]{c^{2^n}(1 + O(c))} \sim c \text{ as c} \to \infty,$$

$R_M(c) \sim c$ near ∞ as well.

Step 3: properness. Let us show that the map $\Phi_M : \mathbb{C} \setminus M \to \mathbb{C} \setminus \mathbb{D}$ is proper:

$$|\Phi_M(c)| \to 1 \text{ as } c \to \partial M.$$

Let us define $n(c) \in \mathbb{N} \cup \{0\} \cup \{\infty\}$ as the last moment n such that $f_c^n(c) \in \bar{\mathbb{D}}_2$. By Exercise 5.1(i), $n(c) = \infty$ iff $c \in M$. Moreover, $n(c) \to \infty$ as $c \to M$. Otherwise there would exist $N \in \mathbb{N}$ and a sequence $c_k \to c \in M$ such that $f_{c_k}^N(c) \in \mathbb{C} \setminus \bar{\mathbb{D}}_3$, implying that $f_c^n(c) \in \mathbb{C} \setminus \bar{\mathbb{D}}_3$, which is not the case.

Since the Green function is continuous on $\boldsymbol{\Omega}$,

$$K = \sup_{G_c(z): \ (c,z) \in \bar{\mathbb{D}}_3 \times \mathbb{T}_2} < \infty.$$

Since $z \mapsto G_c(z)$ is subharmonic on the whole plane $\mathbb{C}$ for any c, $G_c(z) \leq K$ for $(c, z) \in \bar{\mathbb{D}}_3 \times \bar{\mathbb{D}}_2$. Hence

$$G_c(c) = \frac{1}{n(c)} G_c(f_c^{n(c)}(c) \leq \frac{1}{n(c)} K \to 0 \text{ as c} \to \text{M}.$$

It follows that $|\phi_c(c)| = e^{G_c(c)} \to 1$ as $c \to M$, $c \in \mathbb{C} \setminus M$, as was asserted.

Conclusion. Thus, the map R_M is a branched covering, so that, it has a well-defined degree. But $R_M^{-1}(\infty) = \{\infty\}$, and by Step 2, ∞ is a simple preimage of itself. Hence $\deg R_M = 1$, and we are done.

21.3. Second proof.

21.3.1. *Step 1: Qc deformation.* The idea is to deform the map by moving around the Böttcher position of its critical value. To this end let us consider a two parameter family of diffeomorphisms $\psi_{\omega,q} : \mathbb{C} \setminus \mathbb{D} \to \mathbb{C} \setminus \mathbb{D}$ written in the polar coordinates as follows:

$$\psi = \psi_{\omega,q}(r, \theta) = (r^\omega, \theta + q \log r), \quad \omega > 0, \ q \in \mathbb{R}.$$

In terms of complex variabe $a = re^{i\theta} \in \mathbb{C} \setminus \mathbb{D}$ and complex parameter $\lambda = \omega + iq$, $\Re\lambda > 0$, this family can be expressed in the following

concise form:

$$\psi_\lambda(a) = |a|^{\lambda-1}a. \tag{21.3}$$

This family commutes with $f_0 : a \mapsto a^2$: $\psi(a^2) = \psi(a)^2$, and acts transitively on $\mathbb{C} \smallsetminus \mathbb{D}$, i.e., for any $a_\star$ and a in $\mathbb{C} \smallsetminus \mathbb{D}$, there exists a λ, such that $\psi_\lambda(a_\star) = a$. (Note also that ψ_λ are automorphisms of $\mathbb{C} \smallsetminus \mathbb{D}$ viewed as a multiplicative semigroup.)

Take now a quadratic polynomial $f_\star \equiv f_{c_\star}$ with $c_\star \in \mathbb{C} \smallsetminus M$. Let us consider its Böttcher function $\phi_\star : \Omega_\star \to \mathbb{C} \smallsetminus \mathbb{D}_\star$, where $\Omega_\star \equiv \Omega_{c_\star}$ is the complement of the figure eight equipotenial (see §16.4.2) and $\mathbb{D}_\star \equiv \mathbb{D}_{R_\star}$ is the corresponding round disk, $R_\star > 1$. Take the standard conformal structure σ on $\mathbb{C} \smallsetminus \mathbb{D}$ and pull it back by the composition $\psi_\lambda \circ \phi_\star$. We obtain a conformal structure $\mu = \mu_\lambda$ in $\Omega_\star$. Since ψ_λ commute with f_0 while the Böttcher function conjugates $f_\star$ to f_0, the structure μ is invariant under $f_\star$.

Let us pull this structure back to the preimages of $\Omega_\star$:

$$\mu^n \,|\, \Omega^n = (f_\star^n)^*(\mu),$$

where $\Omega_\star^n = f_\star^{-n}\Omega_\star$. Since μ is invariant on $\Omega_\star$, the structures μ^{n+1} and μ^n coincide on $\Omega_\star^n$, so that they are organized in a single conformal structure on $\cup \Omega_\star^n = \mathbb{C} \smallsetminus J(f_\star)$. Extend it to the Julia set $J(f_\star)$ as the standard conformal structure.

We will keep notation $\mu \equiv \mu_\lambda$ for the conformal structure on $\mathbb{C}$ we have just constructed. By construction, it is invariant under $f_\star$. Moreover, it has a bounded dilatation since holomorphic pullbacks preserve dilatation: $\|\mu_\lambda\|_\infty = \|(\psi_\lambda)^*(\sigma)\|_\infty < 1$.

By the Measurable Riemann Mapping Theorem, there is a qc map $h_\lambda : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ such that $(h_\lambda)_*(\mu_\lambda) = \sigma$. By Corollary **??**, $h_l a$ can be normalized so that it conjugates f_λ to a quadratic map $f_c \equiv f_{c(\lambda)} : z \mapsto z^2 + c(\lambda)$. Of course, the Julia set f_c is also Cantor, so that $c \in \mathbb{C} \smallsetminus M$.

This family of quadratic polynomials is the desired qc deformation of $f_\star$.

21.3.2. *Step 2: Analyticity.* We have to check three propertices of the map $\Phi_M : \mathbb{C} \smallsetminus M \to \mathbb{C} \smallsetminus \mathbb{D}$: analyticity, surjectivity, and injectivity. Let us take them one by one.

It is obvious from formula (21.3) that the Beltrami differential

$$\nu_\lambda = (\psi_\lambda)^*(\sigma) = \bar\partial\psi_\lambda/\partial\psi_\lambda$$

depends holomorphically on λ. Hence the Beltrami differential $(f_\star)^*(\nu_\lambda)$ on $\Omega_\star$ also depends holomorphically on λ (see Exercise 2.12). Pulling it back by the iterates of $f_\star$ and extending it in the standard way to

$J(f)$, we obtain by Lemma 2.20 a holomorphic family of Beltrami differentials μ_λ on $\mathbb{C}$. By Corollary 4.16, $c(\lambda)$ is holomorphic on λ as well.

21.3.3. *Step 3: Surjectivity.* Note that the map $\psi_\lambda \circ \phi_\star \circ h_\lambda^{-1}$ conformally conjugates the polynomial $f_c \equiv f_{c(\lambda)}$ near ∞ to $f_0 : z \mapsto z^2$. By Theorem 4.23, these properties determine uniquely the Böttcher map ϕ_c of f_c, so that $\phi_c = \psi_\lambda \circ \phi_\star \circ h_\lambda^{-1}$ with $c = c(\lambda)$. Since h_λ conjugates $f_\star$ to f_c, we have: $h_\lambda(c_*) = c$ and hence

$$\Phi_M(c) = \phi_c(c) = \psi_\lambda \circ \phi_\star(c_\star) = \psi_\lambda(a_\star),$$

where $a_\star$ is the Böttcher position of the critical value of $f_\star$. Since the family $\{\psi_\lambda\}$ acts transitively on $\mathbb{C} \smallsetminus \mathbb{D}$, any point $a \in \mathbb{C} \smallsetminus \mathbb{D}$ can be relasized as $\Phi_M(c)$ for some $c = c(\lambda)$.

21.3.4. *Step 4: Injectivity.* We have to check that if

$$\phi_c(c) = a = \phi_{\tilde{c}}(\tilde{c}) \tag{21.4}$$

for two parameter valus c and $\tilde{c}$ in $\mathbb{C} \smallsetminus M$, then $c = \tilde{c}$. We let $f \equiv f_c$, $\phi \equiv \phi_c$, $\tilde{f} \equiv f_{\tilde{c}}$, $\tilde{\phi} \equiv \phi_{\tilde{c}}$. Similarly, we will mark with "tilde" the dynamical objects associated with $\tilde{f}$ that naturally correspond to dynamical objects associated with f.

Let $R = \sqrt{|a|}$. Then the maps ϕ^{-1} and $\tilde{\phi}^{-1}$ map $\mathbb{C} \smallsetminus \bar{\mathbb{D}}_R$ onto the domains $\Omega \equiv \Omega_c$ and $\tilde{\Omega} \equiv \Omega_{\tilde{c}}$ respectively. Moreover, they extend continuously to the boundary circle mapping it onto the boundary figures eight $\Gamma = \partial\Omega$ and $\tilde{\Gamma} = \partial\tilde{\Omega}$, and this extension if one-to-one except that

$$\phi^{-1}(\pm\sqrt{a}) = 0 = \tilde{\phi}^{-1}(\pm\sqrt{a}).$$

Hence the conformal map $h = \tilde{\phi}^{-1} \circ \phi : \Omega \to \tilde{\Omega}$ admits a homeomorphic extension to the closure of its domain:

$$h : (\mathrm{cl}(\Omega), 0) \to (\mathrm{cl}(\tilde{\Omega}), 0).$$

Consider a domain $\Omega^0 = f(\Omega)$ (exterior of the equipotential passing through c) and the complementary Jordan disk $\Delta^0 = \mathbb{C} \smallsetminus \Omega^0$. We will describe a hierarchical decomposition of Δ^0 into topological annuli A_i^n, $n = 1, \ldots$, $i = 1, 2, \ldots, 2^n$. Let $\Omega^n = f^{-n}\Omega^0$ (so that $\Omega \equiv \Omega^1$). The boundary $\partial\Omega^n$ consists of 2^{n-1} disjoint figures eight. The loops of these figures eight bound 2^n (closed) Jordan disks Δ_i^n. The map f conformally maps Δ_i^n onto some Δ_j^{n-1}, $n \geq 1$. Let $A_i^n = \Delta_i^n \cap \mathrm{cl}(\Omega^{n+1})$. These are closed topological annuli each of which is bounded by a Jordan curve and a figure eight. They tile $\Delta^0 \smallsetminus J(f)$. The map f conformally maps A_i^n onto some A_j^{n-1}, $n \geq 1$.

Let us lift $h \equiv h_1$ to conformal maps $H_i : A_i^1 \to \tilde{A}_i^1$:

$$H_i \,|\, A_i^1 = (\tilde{f}\,|\,\tilde{A}_i^1)^{-1} \circ h \circ (f\,|\,A_i^1). \tag{21.5}$$

Since h is equivariant on the boundary of $\Omega^1 \smallsetminus \Omega^0$, it matches with the H_i on $\partial \Delta_i^1$. Putting these maps together, we obtain an equivariant homeomorphism $h_2 : \mathrm{cl}(\Omega^2) \to \mathrm{cl}(\tilde{\Omega}^2)$ conformal in the complement of the figure eight Γ:

$$h_2(z) = \begin{cases} h(z), & z \in \Omega^1, \\ H_i(z), & z \in A_i^1. \end{cases}$$

Since smooth curves are removable (recall §12), h_2 is conformal in $\Omega^2 \smallsetminus \{0\}$. Since isolated points are removable, h_2 is conformal in Ω^2. Thus h admits an equivariant conformal extension to Ω^2.

In the same way, h_2 can be lifted to four annuli A_i^2. This gives an equivariant conformal extension of h to Ω^3. Proceeding in this way, we will consecutively obtain an equivariant conformal extension of h to all the domains Ω^n and hence to their union $\cup \Omega^n = \mathbb{C} \smallsetminus J(f)$.

Since the Julia set $J(f)$ is removable (Theorem 2.31), this map admits a conformal extension through $J(f)$. Thus, f and $\tilde{f}$ are conformally equivalent, and hence $c = \tilde{c}$.

This completes the second proof of Theorem 5.7.

22. The Multiplier Theorem

22.1. Statement. Let us pick a favorite hyperbolic component H of the Mandelbrot set M. For $c \in H$, the polynomial f_c has a unique attracting cycle $\boldsymbol{\alpha}_c = \{\alpha_k(c)\}_{k=0}^{p-1}$ of period p. By Lemma 5.1, the multiplier $\lambda(c)$ of this cycle holomorphically depends on c, so that we obtain a holomorphic map $\lambda : H \to \mathbb{D}$. It is remarkable that this map gives an explicit uniformization of H by the unit disk:

THEOREM 5.9. *The multiplier map* $\lambda : H \to \mathbb{D}$ *is a conformal isomorphism.*

This theorem is in many respects analogous to Theorem 5.7 on connectivity of the Mandelbrot set. The latter gives an explicit dynamical uniformization of $\mathbb{C} \smallsetminus M$; the former gives the one for the hyperbolic component. The ideas of the proofs are also similar.

We already know that λ is holomorphic, so we need to verify that it is surjective and injective. The first statement is easy:

EXERCISE 5.4. *The multiplier map* $\lambda : H \to \mathbb{D}$ *is proper and hence surjective. In particular, H contains a superattracting parameter value.*

22.2. Qc deformation. Let $Z \subset H$ be the set of superattracting parameter values in H. Take some point $c_0 \in H \smallsetminus Z$, and let $\lambda_0 \in \mathbb{D}^*$ be the multiplier of the corresponding attracting cycle. We will produce a qc deformation of $f_* \equiv f_{c_0}$ by deforming the associated fundamental torus.

22.2.1. *Fundamental torus.* Take a little topological disk $D = \mathbb{D}(a_0, \epsilon)$ around the attracting periodic point a_0 of f_*. It is invariant under $g_0 \equiv f_*^p$ and the quotient of D under the action of f_* is a conformal torus T_0. Its fundamental group has one marked generator corresponding to a little Jordan curve around α_0.

By the Linearization Theorem (**??**), the action of g_0 on D is conformally equivalent to the linear action of $\zeta \mapsto \lambda_0 \zeta$ on $\mathbb{D}^*$. Hence the partially marked torus T_0 is conformally equivalent to $\mathbb{T}^2_{\lambda_0}$, so that λ_0 is the modulus of T_0 (see §1.4.2).

Let us select a family of deformations $\psi_\lambda : \mathbb{T}^2_{\lambda_0} \to \mathbb{T}^2_\lambda$ of T_{λ_0} to nearby tori. For instance, ψ_λ can be chosen to be linear in the logarithmic coordinates $(x, y) = \log \zeta$, $\tau = \log \lambda$:

$$x + y\tau_0 \mapsto x + y\tau; \quad x \in \mathbb{R}, \ y \geq 0.$$

This gives us a complex one-parameter family of Beltrami differentials $\nu_\lambda = \psi_\lambda^*(\sigma)$ on $T_0 \approx \mathbb{T}^2_{\lambda_0}$ (in what follows we identify T_0 with $\mathbb{T}^2_{\lambda_0}$).

EXERCISE 5.5. *Calculate ν_λ explicitly (for the linear deformation).*

22.2.2. *Qc deformation of f_*.* We can lift ν_λ to the disk D and then pull it back by iterates of f_*. This gives us a family of f_*-invariant Beltrami differentials μ_λ on the attracting basin of $\boldsymbol{\alpha}$. These Beltrami differentials have a bounded dilatation since the pull-backs under holomorphic maps preserve dilatation. Extend the μ_λ by 0 outside the attracting basin (keeping the notation). We obtain a family of measurable f_*-invariant conformal structures μ_λ on the Riemann sphere. Solving the Beltrami equation $(h_\lambda)_*(\mu_\lambda) = \sigma$ (with an appropriately normalization) we obtain a qc deformation of f_* (see Corollary 4.17):

$$f_{c(\lambda)} = h_\lambda \circ f_* \circ h_\lambda^{-1} : \ z \mapsto z^2 + c(\lambda). \tag{22.1}$$

Moreover, note that this deformation is conformal on the basin of ∞.

Let us show that the multiplier of the attracting fixed point of $f_{c(\lambda)}$ is equal to λ. Consider the torus T_λ associated with the attracting cycle of $f_{c(\lambda)}$. Then h_λ descends to a homeomorphism $H_\lambda : T_0 \to T_\lambda$ such that $(H_\lambda)_*(\nu_\lambda) = \sigma$. Since $(\psi_\lambda)_*(\nu) = \sigma$ as well, the map

$$\psi_\lambda \circ H_\lambda^{-1} : T_\lambda \to \mathbb{T}^2_\lambda$$

is conformal. Hence the partially marked torus T_λ has the same modulus as $\mathbb{T}^2_\lambda$, which is λ. But as we know, this modulus is equal to the multiplier of the corresponding attracting cycle.

This deformation immediately leads to the following important conclusion:

LEMMA 5.10. *All maps f_c, $c \in H \smallsetminus Z$, are qc equivalent (and the conjugacy is conformal on the basin of ∞). Moreover,* card $Z = 1$.

PROOF. Take some $c_0 \in H \smallsetminus Z$. By Proposition 2.16, the deformation parameter $c(\lambda)$ in (22.1) depends continuously on λ. Hence $c : \lambda \mapsto c(\lambda)$ is the local right inverse to the multiplier function. But holomorphic functions do not have continuous right inverses near their critical points. Consequently, c_0 is not a critical point of the multiplier function λ and, moreover, c is the local inverse to λ. It follows that any c near c_0 can be represented as $c(\lambda)$, and hence f_c is qc equivalent to f_{c_0}.

Let us decompose the domain $H \smallsetminus Z$ into the union of disjoint qc classes (with conformal conjugacy on the basin of ∞). We have just shown that each qc class in this decomposition is open. Since $H \smallsetminus Z$ is connected, it consists of a single qc class.

Furthermore, we have shown that λ does not have critical points in $H \smallsetminus Z$. Hence $\lambda : H \smallsetminus Z \to \mathbb{D}^*$ is an unbranched covering. By the Riemann-Hurwitz formula (for the trivial case of unbranched coverings), the Euler characteristic of $H \smallsetminus Z$ is equal to 0, i.e., $1 - \operatorname{card} Z = 0$. $\qquad\square$

Thus, every hyperbolic component H contains a unique superattracting parameter value c_H. It is called the *center* of H. We let $H^* = H \smallsetminus \{c_H\}$.

22.3. Injectivity. The following lemma will complete the proof of the Multiplier Theorem:

LEMMA 5.11. *Consider two parameter values c and $\tilde{c}$ in $H \smallsetminus Z$. If $\lambda(c) = \lambda(\tilde{c})$ then the quadratic maps f_c and $f_{\tilde{c}}$ are conformally equivalent on $\mathbb{C}$.*

The idea is to turn the qc conjugacy from Lemma 5.10 into a conformal conjugacy. To this end we need to modify the conjugacy on the basin of the attracting cycle. Let us start with the component D_0 of the basin containing 0.

22.4. Internal angles. For $c \in \bar{H}$, $\arg \lambda(c)$ is called the *internal angle* of c.

23. Structural stability

23.1. Statement of the result. A map $f_{\mathsf{o}} : z \mapsto z^2 + c_{\mathsf{o}}$ (and the corresponding parameter $c_{\mathsf{o}} \in \mathbb{C}$) is called *structurally stable* if for any $c \in \mathbb{C}$ sufficiently close to c_{o}, the map f_c is topologically conjugate to f_{o}, and moreover, the conjugacy $h_c : \mathbb{C} \to \mathbb{C}$ can be selected continuously in c (in the uniform topology). By definition, the set of structurally stable parameters is open. In this section we will prove that it is dense:

THEOREM 5.12. *The set of structurally unstable parameters is equal to the boundary of the Mandelbrot set together with the centers of hyperbolic components. Hence the set of structurally stable parameters is dense in $\mathbb{C}$. Moreover, any structurally stable map f_{o} is quasi-conformally conjugate to all nearby maps f_c.*

Notice that parameters $c_base \in \partial M$ are obviously unstable since the Julia set J_{o} is connected, while the Julia sets J_c for nearby $c \in \mathbb{C} \smallsetminus M$ are disconnected. The centers of hyperbolic components are also unstable since the topological dynamics near a superattracting cycle is different from the topological dynamics near an attracting cycle (the grand orbits on the basin of attraction are discrete in the latter case and are not in the former).

The proof of stability of other parameters will occupy §23.2 – §23.5. The desired conjugacies will be constructed as equivariant holomorphic motions.

A holomorphic motion $h_c : X_{\mathsf{o}} \to X_c$ of a set $X \subset \mathbb{C}$ is called *equivariant* if

$$h_c(f_{\mathsf{o}}(z)) = f_c(h_c(z)) \qquad (23.1)$$

whenever both points z and $f_{\mathsf{o}}(z)$ belong to X_{o}. If the X_c are f_c-invariant, this just means that the maps h_c conjugate $f_{\mathsf{o}}|X_{\mathsf{o}}$ to $f_c|X_c$. (Of course, we can apply this terminology not only to the quadratic family).

Notice that the equivariance property (23.1) means that the associated lamination (see §1.1) is invariant under the map

$$\mathbf{f} : (\lambda, z) \mapsto (\lambda, f_\lambda(z)). \qquad (23.2)$$

Since by the Second λ-lemma, holomorphic motions are automatically quasi-conformal in the dynamical variable, the last assertion of Theorem 5.12 will follow automatically.

23.2. J-stability. Let us first show that the Julia set J_c moves holomorpically outside the boundary of M. (Strictly speaking, this

step is not needed for the proof of Theorem 5.12 given below, but it gives a good illustration of the method.)

A map $f_\mathsf{o} : z \mapsto z^2 + c_\mathsf{o}$ (and the corresponding parameter $c_\mathsf{o} \in \mathbb{C}$) is called *J-stable* if for any $c \in \mathbb{C}$ sufficiently close to c_o, the map $f_c | J_c$ is topologically conjugate to $f_\mathsf{o} | J_\mathsf{o}$, and moreover the conjugacy $h_c : J_\mathsf{o} \to J_c$ depends continuously on c.

THEOREM 5.13. *The set of J-stable parameters is equal to $\mathbb{C} \smallsetminus \partial M$ and hence is dense in $\mathbb{C}$. Moreover, the corresponding conjugacies $h_c : J_\mathsf{o} \to J_c$ form a holomorphic motion of the Julia set over the component of $\mathbb{C} \smallsetminus \partial M$ containing o.*

PROOF. Let C be the component of $\mathrm{int}\, M$ containing c_o. By Corollary 5.3, C does not contain neutral parameters, and hence all periodic points are persistently hyperbolic over C, either repelling or attracting. Hence they depend holomorphically on $c \in C$. Since C is simply connected (Exercise 5.1 (iii)), these holomorphic functions $c \mapsto \alpha(c)$ are single valued. Moreover, they cannot collide since collisions could occur only at parabolic parameters. Thus, they provide us with a holomorphic motion $h_c : \mathrm{Per}_\mathsf{o} \to \mathrm{Per}_c$ of the set of periodic points.

This holomorphic motion is equivariant. Indeed, if

$$c \mapsto \alpha(c) = h_c(\alpha)$$

is a holomorphically moving periodic point then $c \mapsto f_c(\alpha(c))$ is also a holomorphically moving periodic point. Hence $f_c(\alpha(c)) = h_c(f_\mathsf{o}(\alpha))$ and we obtain:

$$f_c(h_c(\alpha)) = f_c(\alpha(c)) = h_c(f_\mathsf{o}(\alpha)).$$

By the First λ-lemma (3.1), this holomorphic motion extends to a continuous equivariant holomorphic motion of the closure of periodic points, which contains the Julia set. Moreover, this motion is automatically continuous in both variables (λ, z), and hence provides us with a family of topological conjugacies between J_o and J_c continuously depending on c. $\square$

EXERCISE 5.6. *An equivariant holomorphic motion of the Julia set is unique.*

23.3. Böttcher motion: connected case. In this section, we will show that the basin of infinity, $D_c(\infty)$, moves bi-holomorphically over any component of $\mathrm{int}\, M$.

PROPOSITION 5.14. *Let C be a component of $\mathrm{int}\, M$ with a base point o. Then there exists an equivariant bi-holomorphic motion $h_c : D_\mathsf{o}(\infty) \to D_c(\infty)$ of the basin of infinity over C.*

PROOF. Let $\phi_c : D_c(\infty) \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}$ be the Böttcher-Riemann uniformization of the basin of infinity (see Theorem 4.24). It is a holomorphic function in two variables on the domain $\{(c, z) : c \in C, z \in D_c(\infty)\}$ (see Step 1 of §21.2). It follows that $h_c = \phi_c^{-1} \circ \phi_{\mathsf{o}}$ is a bi-holomorphic motion of D_c over C. Since the ϕ_c conjugate f_c to $z \mapsto z^2$, this motion is equivariant. $\square$

EXERCISE 5.7. *Show that an equivariant bi-holomorphic motion of the basin of ∞ over C is unique.*

Now the first λ-lemma implies:

COROLLARY 5.15. *For any component C of* int M, *there is a unique equivariant holomorphic motion $h_c : \bar{D}_c(\infty) \to \bar{D}_c(\infty)$ over C which is bi-holomorphic on $D_c(\infty)$.*

If Q is a queer componenet then $\mathbb{C} = \bar{D}_c(\infty)$ for any $c \in Q$, and so, we obtain the Structural Stability Theorem in this case:

COROLLARY 5.16. *For a queer component Q of* int M, *there is a unique equivariant holomorphic motion $h_c : \mathbb{C} \to \mathbb{C}$ over Q which is bi-holomorphic on $D_c(\infty)$. Hence all parameters $c \in H$ are structurally stable.*

23.4. Motion of an attracting basin. For a hyperbolic parameter c, let $\boldsymbol{\alpha}_c$ stand for the corresponding attracting cycle, and let $D(\boldsymbol{\alpha}_c)$ be its basin.

PROPOSITION 5.17. *Let H be a hyperbolic component of* int M, *and let $c_{\mathsf{o}} \in H^*$. Then there is an equivariant smooth holomorphic motion of the attracting basin $D(\boldsymbol{\alpha}_c)$ over some neighborhood of c_{o}.*

To prove this assertion, we need three simple lemmas. The first one is concerned with local extension of smooth motions.

LEMMA 5.18. *Let us consider a compact set $Q \subset \mathbb{C}$ and a smooth holomorphic motion h_λ of a neighborhood U of Q over a parameter domain (Λ, o). Then there is a smooth holomorphic motion H_λ of the whole complex plane $\mathbb{C}$ over some neighborhood Λ' of o whose restriction to Q coincides with h_λ.*

PROOF. We can certainly assume that $\bar{U}$ is compact. Take a smooth cut-off function $\eta : \mathbb{C} \to \mathbb{R}$ supported in U such that $\eta|Q \equiv 1$, and let

$$H_\lambda = \eta\, h_\lambda + (1 - \eta)\, \mathrm{id}\,.$$

Clearly H is smooth in both variables, holomorphic in λ, coinsides with h on Q and with the identity outside U. As $H_0 = \mathrm{id}$, $H_\lambda : \mathbb{C} \to \mathbb{C}$ is a diffeomorphism for λ sufficiently close to o, and we are done. $\square$

The second lemma is concerned with lifts of holomorphic motions.

LEMMA 5.19. *Let $h_\lambda : V_{\mathsf{o}} \to V_\lambda$ be a holomorphic motion of a domain $V_{\mathsf{o}} \subset \mathbb{C}$ over a simply connected parameter domain Λ. Let $f_\lambda : U_\lambda \to V_\lambda$ be a holomorphic family of proper maps with critical points c_λ^k such that the critical values $v_\lambda^k = f_\lambda(c_\lambda^k)$ form orbits of h_λ.[2] Then h_λ uniquely lifts to a holomorphic motion $H_\lambda : U_{\mathsf{o}} \to U_\lambda$ such that*

$$f_\lambda \circ H_\lambda = h_\lambda \circ f_{\mathsf{o}}. \tag{23.3}$$

PROOF. Notice that (23.3) means that the lamination associated with the motion **H** is the pullback of the lamination associated with the motion **h** under the map **f** (23.2). Clearly, such a pullback unique if exists.

Let us take any regular value $\zeta_{\mathsf{o}} = f_{\mathsf{o}}(z_{\mathsf{o}}) \in V_{\mathsf{o}}$, and let $\phi(\lambda) = h_\lambda(\zeta_{\mathsf{o}})$ be its orbit. We would like to lift this orbit to a desired orbit of z_{o}, so we are looking for a holomorphic solution $z = \psi(\lambda)$ of an equation

$$f_\lambda(z) = \phi(\lambda) \tag{23.4}$$

with $\psi(z_{\mathsf{o}}) = \zeta_{\mathsf{o}}$. Since $\phi(\lambda)$ is a regular point of f_λ for any $\lambda \in \Lambda$, the Implicit Function Theorem implies that near any point (λ', z') satisfying (23.4), it admits a unique local analytic solution $z = \psi(\lambda)$. Since the maps f_λ are proper, this continuation along any path compactly contained in Λ cannot escape the domain U_λ. Since Λ is simply connected, $\psi(\lambda)$ extends to the whole domain Λ as a single valued holomorphic function.

Two different orbits $\lambda \mapsto \psi(\lambda)$ obtained in this way do not collide, for (23.4) would have two different solutions near the collision point. Hence they form a holomorphic motion of $V_{\mathsf{o}} \smallsetminus \{v_{\mathsf{o}}^k\}$ over Λ. By the First λ-lemma, this motion extends to the whole domain V_{o}.

Finally,

$$f_\lambda(H_\lambda(z_{\mathsf{o}})) = f_\lambda(\psi(\lambda)) = \phi(\lambda) = h_\lambda(\zeta_{\mathsf{o}}) = h_\lambda(f_{\mathsf{o}}(z_{\mathsf{o}}))$$

holds for any point $z_{\mathsf{o}} \in U_{\mathsf{o}}$ except perhaps finitely many exceptions (preimages of the critical values of f_{o}). By continuity, it holds for all $z_{\mathsf{o}} \in U_{\mathsf{o}}$. $\square$

The last lemma is concerned with dependence of the linearizing coordinate (the Königs function) on parameters

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[2]In particular, any holomorphic family of univalent maps $f_\lambda : U_\lambda \to V_\lambda$ is allowed.

LEMMA 5.20. *Let f_λ be a holomorphic family of germs near the origin over a parameter domain Λ such that 0 is a simply attracting point. Then the normalized linearizing coordinate ϕ_λ depends holomorphically on λ.*

PROOF. The linearizing coordinate ϕ_λ is given by an explicit Königs formula (16.3):

$$\phi_\lambda(z) = \lim_{n\to\infty} \sigma_\lambda^{-n} f_\lambda^n, \quad \text{where } \sigma_\lambda = f_\lambda'(0). \qquad (23.5)$$

Since analyticity is a local property, we need to verify the assertion near an arbitrary parameter $\lambda_\circ \in \Lambda$. There exist $\epsilon > 0$ and $\rho < 1$ such that $f_\circ(\mathbb{D}_\epsilon) \Subset \mathbb{D}_{\rho\epsilon}$. Then the same is true for λ in some neighborhood Λ' of $\lambda_\circ$. By the Schwarz Lemma, the orbits $\{f_\lambda^n(z)\}_{n=0}^\infty$ of points $z \in \mathbb{D}_\epsilon$ converge to 0 at a uniformly exponential rate: $|f_\lambda^n(z)| \le \rho^n$ for $\lambda \in \Lambda'$. This implies (by examining the proof of (16.3)) that convergence in (23.5) is uniform on $\Lambda' \times \mathbb{D}_\epsilon$. Hence $\phi_\lambda(z)$ is holomorphic on $\Lambda' \times \mathbb{D}_\epsilon$. $\square$

Proof of Proposition 5.17. Let $\boldsymbol{\alpha}_c = \{f_c^k(\alpha)\}_{k=0}^{p-1}$ be the attracting cycle of f_c, and let us consider the maps f_c^p near their fixed points α_c. Lemma 5.20 implies that there is a neighborhood $H' \subset H^*$ of $c_\circ$ and an $\epsilon > 0$ such that the inverse linearizing coordinate $\phi_c^{-1}(z)$ for f_c^p is holomorphic on $\Lambda' \times \mathbb{D}_\epsilon$. Let $V_c = \phi_c^{-1}(\mathbb{D}_\epsilon) \ni \alpha_c$, and let us consider a fundamental annulus $A_c = \mathrm{cl}(V_c \smallsetminus f_c(V_c))$.

By Theorem 4.7, the critical orbit $\mathrm{orb}_c(0)$ must cross A_c. By adjusting ϵ and shrinking H' if needed, we can ensure that it does not cross ∂A_c. Then it crosses A_c at a single point $v_n(c) = f_c^n(0) \in \mathrm{int}\, A_c$, where $n \in \mathbb{N}$ is independent of c. Its position in the linearizing coordinate, $a_c = \phi_c(v_n(c)) \in \mathbb{A}(\epsilon, \sigma_c \epsilon) \equiv \mathbb{A}_c$, depends holomorphically on c (here σ_c is the multiplier of $\boldsymbol{\alpha}_c$).

Let $Q_c = \partial A_c \cup \{a_c\}$. Let us define a smooth equivariant holomorphic motion $\mathbf{h}$ of a small neighborhood of Q_c over H' as follows: $h_c = \mathrm{id}$ near the outer boundary of A_c, $h_c : z \mapsto \sigma_c z/\sigma_\circ$ near the inner boundary of A_c, and $h_c : z \mapsto a_c z/a_\circ$ near a_c. By Lemma 5.18, this motion extends to a smooth motion of the whole plane over some neighborhood of $c_\circ$ (we will keep the same notation H' for this neighborhood). Let us restrict the motion to the fundamental annulus A_c (keeping the same notation h_c for it). By Lemma 5.19 (in the simple case when there are no critical points), this motion can be first extended to the forward orbit of A_c, (providing us with an equivariant holomorphic motion of $\mathbb{D}_\epsilon$). Then we can transfer it using the linearizing coordinates to a holomorphic motion of V_c, then extend it to an invariant neighborhood $\mathbf{V}_c = \bigcup_{k=0}^{p-1} f_c^k(V_c)$ of $\boldsymbol{\alpha}$, and finally we can use Lemma 5.19 to pull this motion back to all preimages of $\mathbf{V_c}$ (the assumption of Lemma 5.19

on the critical values is secured by the property that a_c is an orbit of the motion $\mathbf{h}$). It provides us with the desired equivariant holomorphic motion of the basin $D(\boldsymbol{\alpha}_c)$. $\square$

COROLLARY 5.21. *Let H be a hyperbolic component of* $\operatorname{int} M$, *and let $c_{\mathsf{o}} \in H^*$. Then there is an equivariant holomorphic motion of the whole plane $\mathbb{C}$ over some neighborhood of c_{o}. Hence all parameters $c \in H^*$ are structurally stable.*

PROOF. Since for $\mathbb{C} = \operatorname{cl}(D_c(\infty) \cup D(\boldsymbol{\alpha}_c))$ for $c \in H$, Propositions 5.14 and 5.17, together with the First λ-lemma yield the desired. $\square$

23.5. Böttcher motion: Cantor case. Let us finally deal with the complement of M.

PROPOSITION 5.22. *Let $c_{\mathsf{o}} \in \mathbb{C} \smallsetminus M$. Then there is an equivariant smooth holomomrphic motion of the basin of infinity, $D_c(\infty)$, over some neighborhood of c_{o}.*

The proof is similar to the one given in the attracting case, using the Böttcher coordinate in place of the linearizing coordinate. To implement it, we need a rotationally equivariant Extension Lemma:

LEMMA 5.23. *Let $R > r > 1$ and let $z \in \mathbb{A}(r, R)$. Let ϕ be a holomorphic function on a domain (Λ, o) with $\phi(\mathsf{o}) = z$. Then there is a smooth holomorphic motion H_λ of the whole complex plane $\mathbb{C}$ over some neighborhood Λ' of o such that*

(i) $H_\lambda(z) = \phi(\lambda)$;
(ii) $H_\lambda = \operatorname{id}$ *on* $\mathbb{C} \smallsetminus \mathbb{A}(r, R) = \operatorname{id}$;
(iii) *The H_λ commute with the rotation group $\zeta \mapsto e^{i\theta}\zeta$.*

PROOF. Let $\tau(\lambda) = \phi(\lambda)/z$, and let $h_\lambda(\zeta) = \tau(\lambda)\zeta$. This motion satisfies requirements (i) and (iii). To make it satisfy (ii) as well, we will use a smooth cut-off function $\phi : \mathbb{R} \to \mathbb{R}$ supported on a small neighborhood of $|z|$. Then the motion

$$H_\lambda(\zeta) = \phi(|\zeta|)\, h_\lambda(\zeta) + (1 - \phi(|\zeta|))\zeta$$

satisfies all the requirements. $\square$

Proof of Proposition 5.22. Let us consider the Böttcher coordinate ϕ_c of f_c near ∞. Since it depends holomorphically on c, there is a neighborhood $U \subset \mathbb{C} \smallsetminus M$ of c_{o} and an $R > 1$ such that the function $(c, z) \mapsto \phi_c^{-1}(z)$ is holomorphic on $U \times (\mathbb{C} \smallsetminus \bar{D}_R)$.

Let $V_c = \phi_c^{-1}(\mathbb{C} \smallsetminus \bar{\mathbb{D}}_R)$. By adjusting R and U if necessary, we can ensure that the $\operatorname{orb}_c(0)$ does not cross the boundary of the fundamental annulus $A_c = V_c \smallsetminus f_c(V_c)$. Then there is a unique n such that $v_n(c) =$

$f_c^n(0) \in \text{int } A_c$. Let us mark the corresponding point $a_c = \phi_c(v_n(c))$ in the annulus $\mathbb{A} = \mathbb{A}(R, R^2)$.

Applying lemma 5.23, we find a rotationally equivariant holomorphic motion $H_c : \mathbb{A} \to \mathbb{A}$ such that $H_c(a_\circ) = a_c$ and $H_c = \text{id}$ on $\partial\mathbb{A}$. Let us show that

EXERCISE 5.8. *Show that this holomorphic motion extends to a holomorphic motion* $H_c : \mathbb{C} \smallsetminus \mathbb{D}_R \to \mathbb{C} \smallsetminus \mathbb{D}_R$ *commuting with* $z \mapsto z^2$.

Let us now transfer H_c by means of the Böttcher coordinate to a holomorphic motion $h_c : V_c \to V_c$, $h_c = \phi_c^{-1} \circ H_c \circ \phi_\circ$. This motion is equivarinat, $\phi_c \circ f_\circ = f_c \circ \phi_c$, and has $v_n(c)$ as one of its orbits. By Lemma 5.19, it can be lifted to a holomorphic motion of $f_c^{-1}(V_c)$ that has $v_{n-1}(c)$ as its orbit. Moreover, by the uniqueness of the lift, it coincides on $V_\circ$ with the original motion h_c, which implies that it is equivariant. Then we can lift it further to $f^{-2}(V_c)$, and so on: in this way we will exhaust the whole basin of ∞. $\square$

Since $\mathbb{C} = \bar{D}_c(\infty)$ for $c \in \mathbb{C} \smallsetminus M$, Proposition 5.22 (together with the First λ-lemma) yields:

COROLLARY 5.24. *Let* $c_\circ \in \mathbb{C} \smallsetminus M$. *Then there is an equivariant holomorphic motion of the whole plane* $\mathbb{C}$ *over some neighborhood of* $c_\circ$. *Hence all parameters* $c \in H^*$ *are structurally stable.*

Corollaries 5.16, 5.21 and 5.24 cover all types of components of $\mathbb{C} \smallsetminus \partial M$, and together prove the Structural Stability Theorem (5.12).

23.6. Invariant line fields and queer components.

23.6.1. *Definition.* Informally speaking, a line field on $\mathbb{C}$ is a family of tangent lines $l(z) \in \mathrm{T}_z\mathbb{C}$ depending measurably on $z \in \mathbb{C}$.

Here is a precise definition. Any line $l \in \mathbb{C}$ passing through the origin is uniquely represented by a pair of centrally symmetric points $e^{\pm 2\pi i\theta} \in \mathbb{T}$ in the unit circle, or by a single number

$$\nu = e^{4\pi i\theta} \in \mathbb{T}, \quad \theta \in \mathbb{R}/(\mathbb{Z}/2). \tag{23.6}$$

The space of these lines form, by definition, the one-dimensional projective line $\mathbb{PR}^1$, and (23.6) provides us with its parametrization by the angular coordinate (and shows that $\mathbb{PR}^1 \approx \mathbb{T}$).

Let us now consider the projective tangent bundle over $\mathbb{C}$,

$$\mathrm{PT}(\mathbb{C}) = \mathbb{C} \times \mathbb{PR}^1$$

parametrized by $\mathbb{C} \times (\mathbb{R}/(\mathbb{Z}/2))$. A *line field* on $\mathbb{C}$ is a measurable section of $\mathrm{PT}(\mathbb{C})$ defined on some set $X \subset \mathbb{C}$ of positive area called its *(measurable) support*. In terms of the angular coordinate, we obtain

a measurable function $X \to \mathbb{R}/(\mathbb{Z}/2)$, $z \mapsto \theta(z)$.[3] In the circular coordinate ν, we obtain a measurable function $X \to \mathbb{T}$. *In what follows, we will always extend ν by 0 to the whole plane.*

EXERCISE 5.9. *Show that a line field on a Riemann surface S is given by a Beltrami differential $\nu(z)\dfrac{d\bar{z}}{dz}$ with $|\nu(z)| \in \{0, 1\}$.*

A line field on a set $J \subset \mathbb{C}$ is a line field on $\mathbb{C}$ whose support is contained in J. If such a line field exists (with a non-empty support) then area $J > 0$.

A line field is called *invariant* (under a holomorphic map f) if it is invariant under the natural action of f on the projective line bundle: $l(fz) = Df(z)l(z)$, or in the angular coordinate, $\theta(fz) = \theta(z) + \arg f'(z)$, or in the Beltrami coordinate, $f^*\nu = \nu$ (where the pullback is understood in the sense of Beltrami differentials).

If an invariant line field l is supported on a set X then we can pull it back by the dynamics to obtain an invariant line field supported on the set $\tilde{X} = \bigcup\limits_{n=0}^{\infty} f^{-n}(X)$.[4] Hence we can assume in the first place that *l is supported on a completely invariant set*: this will be our standing assumption.

23.6.2. *Existence criterion.*

PROPOSITION 5.25. *Let Q be a queer component of* int M. *Then any map f_c, $c \in Q$, has an invariant line field on its Julia set. In particular,* area $J(f_c) > 0$.

Vice versa, if f_c has an invariant line field on its Julia set then c belongs to a queer component of int M.

PROOF. Take some $c_{\mathsf{o}} \in Q$. By Corollary 5.16, there is an equivariant holomorphic motion h_c over (Q, c_{o}) which is bi-holomorphic on $D_c(\infty)$. Let us consider the corresponding Belrtami differentials $\mu_c = \bar{\partial} h_c / \partial h_c$, $c \in Q$. Each μ_c vanishes on $D_{\mathsf{o}}(\infty)$, however $\mu_c \neq 0$ for $c \neq c_{\mathsf{o}}$ (for otherwise, by Weyl's Lemma the map h_c would be affine, contrary to the fact the quadratic maps f_c and f_{o} are not affinely conjugate). Hence area(supp μ_c) > 0 for any $c \neq c_{\mathsf{o}}$, and all the more, area $J_{\mathsf{o}} > 0$. Moreover, since μ_c is f_{o}-invariant, the normalized Beltrami differential $\nu_c = \mu_c / |\mu_c|$ (where we let $\nu_c = 0$ outside supp μ_c) is

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[3]As always, a measurable function is considered up to an arbitrary change on null-sets.

[4]The pullback would fail at the critical points but we can always remove their grand orbits (as any other completely invariant null-set) from $\tilde{X}$.

also f_o-invariant, and hence determines an invariant line field on the Julia set J_o.

Vice versa, assume f_o has an invariant line field on J_o given by an invariant Beltrami differential ν_o. For any $\lambda \in \mathbb{D}$, the Beltrami differential $\lambda\nu_\mathsf{o}$ is also f-invariant. Let $h_\lambda : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ be the solution of the corresponding Beltrami equation tangent to the identity at infinity. Then the map $h_\lambda \circ f_\mathsf{o} \circ h_\lambda^{-1}$ is a quadratic polynomial $f_{\sigma(\lambda)} :$ $z \mapsto z^2 + \sigma(\lambda)$ (see §15.1.2). By Corollary 4.17, the map $\sigma : \mathbb{D} \to \mathbb{C}$ is holomorphic. Since the line field is non-trivial, it is not identically constant. Hence its image covers a neighborhood of c_o contained in $\operatorname{int} M$. So, it is contained in some component of $\operatorname{int} M$. By **??**, this component cannot be hyperbolic, so it must be queer. $\qquad\square$

Thus, Fatou's Conjecture (5.4) is equivalent to the following one:

CONJECTURE 5.26 (No Invariant Line Fields). *No quadratic polynomial has an invariant line field on its Julia set.*

23.6.3. *Uniqueness and ergodicity.* If a line field $l(z)$ is rotated by angle $2\pi\alpha$ with $\alpha \in \mathbb{R}/(\mathbb{Z}/2)$, the corresponding Beltrami differential is then multiplied by $\lambda = e^{4\pi i\alpha} \in \mathbb{T}$. Of course, if the original line field was f-invariant then so is the rotated one.

LEMMA 5.27. *A quadratic polynomial can have at most one, up to rotation, invariant line field on its Julia set.*

This will follow from the ergodicity of the action of f on the support of any invariant line field. Recall that a map $f : X \to X$ of a measure space is called *ergodic* if X cannot be decomposed into a disjoint unnion of two invariant (and hence completely invariant) subsets of positive measure. Equivalently, there are no non-constant measurable functions $\phi : X \to \mathbb{R}$ invariant under f, i.e., such that $\phi \circ f = \phi$.

LEMMA 5.28. *Let f be a quadratic polynomial, and let $l(z)$ be an invariant line field on $J(f)$. Then the action of f on $\operatorname{supp} l$ is ergodic.*

PROOF. Assume that $\operatorname{supp} l$ admits a disjoint decomposition $X_1 \sqcup X_2$ into two measurable invariant subsets of positive measure. Then the restriction of l to these sets gives us two invariant line fields l_i with disjoint supports. Let ν_i be the corresponding Beltrami differentials. Then we can consider a complex two-parameter family of Beltrami differentials $\nu_\lambda = \lambda_1\nu_1 + \lambda_2\nu_2$, where $\lambda = (\lambda_1, \lambda_2) \in \mathbb{D}^2$. Since $\|\nu_\lambda\|_\infty < 1$ for each λ, we can solve the corresponding Beltrami equations and obtain a two parameter family of qc maps $h_\lambda : (\mathbb{C}, 0) \to (\mathbb{C}, 0)$ tangent to the identity at infinity. Then the maps $h_\lambda \circ f_\mathsf{o} \circ h_\lambda^{-1}$ form a family of quadratic polynomials $f_{\sigma(\lambda)} : z \mapsto z^2 + \sigma(\lambda)$ (see §15.1.2).

By Proposition 2.16, the map $\sigma : \mathbb{D}^2 \to \mathbb{C}$ we have obtained this way is continuous (in fact, by Corollary 4.17, it is holomorphic). Hence it cannot be injective: there exist $\lambda \neq \kappa$ in $\mathbb{D}^2$ such that $\sigma(\lambda) = \sigma(\kappa)$. Then the map $\phi = h_\kappa^{-1} \circ h_\lambda$ commutes with f_o. But the only conformal automorphism of $D_\mathsf{o}(\infty)$ commuting with f_o is the identity (see Exercise 4.19). Hence $h_\lambda = h_\kappa$ implying that $\lambda = \kappa$ – contradiction.
□

Proof of Lemma 5.27. Assume we have two invariant line fields given by Beltrami differentials ν_i. Let $X_i = \operatorname{supp} \nu_i$. Notice that due to our convention, both differences, $X_1 \smallsetminus X_2$ and $X_2 \smallsetminus X_1$, are completely invariant sets. If $\operatorname{area}(X_2 \smallsetminus X_1) > 0$ then an invariant Beltrami differential ν which is equal to ν_1 on X_1 and is equal to ν_2 on $X_2 \smallsetminus X_1$ has a non-ergodic support, contradicting Lemma 5.28. Hence $\operatorname{area}(X_2 \smallsetminus X_1) = 0$, and for the same reason $\operatorname{area}(X_1 \smallsetminus X_2) = 0$, so that the set $Y = X_1 \cap X_2$ can be taken as a measurable support of both differentials.

By Lemma 5.28, f acts ergodically on Y. But the ratio ν_2/ν_1 is an invariant function on Y. By ergodicity, it is equal to const a.e. on Y, and we are done. □

23.6.4. *Uniformization of queer components.* We can now construct a dynamical uniformization of any queer component Q by a Beltrami disk. (Compare with the uniformizations of hyperbolic components of $\mathbb{C} \smallsetminus \partial M$ given by Theorems 5.7 and 5.9.)

For a base map f_o, let us select an invariant line field on J_o given by an f-invariant Beltrami differential ν_o. Then the Beltrami disk $\{\lambda \nu_\mathsf{o}\}_{\lambda \in \mathbb{D}}$ generates a holomorphic family of quadratic polynomials $f_{\sigma(\lambda)} : z \mapsto z^2 + \sigma(\lambda)$ (see the proof of the second part of Lemma 5.25). This is the desired uniformization:

PROPOSITION 5.29. *The map* $\sigma : (\mathbb{D}, 0) \to (Q, \mathsf{o})$ *is the Riemann mapping.*

PROOF. The map σ is a holomorphic embedding for the same reason as the map considered in the proof of Lemma 5.28. Let us show that it is surjective. Let $c \in Q$. By Corollary 5.16, the map f_c is conjugate to f_o by a qc homeomorphism h_c which is conformal outside J_o. Let $\mu_c = \bar{\partial}h_c/\partial h_c$ be the Beltrami differential of h_c, and let $\nu_c = \mu_c/|\mu_c|$. Since the latter differential determines an invariant line field on J_o, Lemma 5.27 yields:

$$\operatorname{supp} \mu_c = \operatorname{supp} \nu_c = \operatorname{supp} \nu_\mathsf{o}.$$

Since the differential μ_c is f_o-invariant, the ratio μ_c/ν_o (extended by 0 beyond $\operatorname{supp} \nu_\mathsf{o}$) is an f-invariant function. By ergodicity, it is const

a.e., so that $\mu_c = \lambda \nu_c$ for some $\lambda \in \mathbb{D}$. It follows that $c = \sigma(\lambda)$, and we are done. □

23.7. Quasi-conformal classification of the quadratic maps. We can now give a complete classification of the quadratic maps up to qc conjugacy:

PROPOSITION 5.30. *Any qc class in the parameter plane $\mathbb{C}$ of the quadratic family is one on the following list:*
- *the complement of the Mandelbrot set;*
- *a hyperbolic component of* int M *punctured at the center;*
- *a queer component of* int M*;*
- *the center of a hyperbolic component;*
- *a single point of the boundary of M.*

The first three types of maps are *deformable*, the last two are *qc rigid.*

PROOF. By the Structural Stability Theorem (5.12), each of the above listed sets is contained in some qc class. What we need to show that they belong to different qc classes.

Assume it is not the case: let c_{o} and c be two parameters in different sets but in the same qc class. Then the quadratic polynomials f_{o} and f_c are conjugate by a qc map h. Let $\mu = \bar{\partial}h/\partial h$ be the Beltrami differential of h, and let $r = 1/\|\mu\|_\infty$. Let us consider the Beltrami disk $\{\lambda\mu : |\lambda| < r\}$ and the corresponding qc deformation

$$f_{\sigma(\lambda)} : z \mapsto z^2 + \sigma(\lambda)$$

of f_{o} (see Corollary 4.17). Then $\sigma : \mathbb{D}_r \to \mathbb{C}$ is a hololmorphic map such that $\sigma(0) = c_{\mathsf{o}}$ and $\sigma(1) = c$. In particular, it is not identically constant and hence its image U is a domain in $\mathbb{C}$. But U is not contained in a single component of int M, so it must intersect ∂M, and hence it must intersect $\mathbb{C} \smallsetminus M$. Thus, U contains quadratic maps of both dichotomy types: with connected as well as Cantor Julia sets, which is impossible as all the maps in U are topologically conjugate. □

CHAPTER 6

Combinatorics of external rays

1. Dynamical ray portraits

1.1. Motivaing problems. Consider a quadratic polynomial $f = f_c$ with connected Julia set. As we know (§**??**), its basin of infinity is uniformized by the Böttcher map $\phi : D_f(\infty) \to \mathbb{C} \smallsetminus \mathbb{D}$, which conjugates f to $z \mapsto z^2$. If the Julia set was locally connected then by the Carathéodory theorem the inverse map would ϕ^{-1} extend continuously to the unit circle $\mathbb{T}$. This would give a representation of $f| J(f)$ as a quotient of the the doubling map $\theta \mapsto 2\theta \mod 1$ of the circle $\mathbb{R}/Z \approx \mathbb{T}$. This observation immeadiately leads to the followong problems:

1) Describe explicitly equivalence realtions on the circle corresponding to all possible Julia sets;

2) Study the problem of local conectivity of the Julia sets.

It turns out that the first problem can be addressed in a comprehensive way. The second problem is very delicate. However, even non-locally connected examples can be partially treated due to the fact that many external rays always land at some points of the Julia set. This is the main theme of the following discussion.

1.2. Landing of rational rays. We say that an external ray $\mathcal{R}^\theta$ *lands* at some point z of the Julia set if $\mathcal{R}^\theta(t) \to z$ as $t \to 0$. Two rays $\mathcal{R}^{\theta/2}$ and $\mathcal{R}^{\theta/2+1/2}$ will be called "preimages" of the ray $\mathcal{R}^\theta$. Obviously, if some ray lands, then its image and both its preimages land as well.

An external ray $\mathcal{R}^\theta$ is called *rational* if $\theta \in \mathbb{Q}$, and *irrational* otherwise. Dynamically the rational rays are characterized by the property of being either periodic or preperiodic:

EXERCISE 6.1. *Let $\mathcal{R} = \mathcal{R}^\theta$.*

a) *If θ is irraional then the rays $f^n(\mathcal{R})$, $n = 0, 1, \ldots$, are all distinct.*

Assume θ is rational: $\theta = q/p$, where q and p are mutually prime. Then

(i) *If p is odd then $\mathcal{R}$ is periodic: there exists an l such that $f^l(\mathcal{R}) = \mathcal{R}$.*

(ii) *If p is even then $\mathcal{R}$ is preperiodic: there are l and $r > 0$ such that $f^r(\mathcal{R})$ is a periodic ray of period l, while the rays $f^k(\mathcal{R})$, $k = 0, 1, \dots, r - 1$, are not periodic.*

How to calculate l and r?

THEOREM 6.1. *Let f be a polynomial with connected Julia set. Then any periodic ray $\mathcal{R} = \mathcal{R}_f^{p/q}$ lands at some repelling or parabolic point of f.*

PROOF. Without loss of generality we can assume that the ray $\mathcal{R}$ is periodic and hence invariant under some iterate $g = f^l$. Let $d = 2^l$. Consider a sequence of points $z_n = \mathcal{R}(1/d^n)$, and let γ_n be the sequence of arcs on $\mathcal{R}$ bounded by the points z_n and z_{n+1}. Then $g(\gamma_n) = \gamma_{n-1}$.

Endow the basin $D = D_f(\infty)$ with the hyperbolic metric ρ. Since $g : D \to D$ is a covering map, it locally preserves ρ. Hence the hyperbolic length of the arcs γ_n are all equal to some L.

But all the rays accumulate on the Julia set as $t \to 0$. By the relation between the hyperbolic and Euclidean metrics (Lemma 1.19), the Euclidean length of these arcs goes to 0 as $n \to \infty$. Hence the limit set of the sequence $\{z_n\}$ is a connected set consisting of the fixed points of g. Since g has only finitely many fixed points, this limit set consists of a single fixed point β. It follows that the ray $\mathcal{R}$ lands at $\beta \in J(f)$ (compare with the proof of Theorem 4.14).

Since $\beta \in J(f)$, it can be either repelling, or parabolic, or Cremer. But the latter case is excluded by the Necklace Lemma 4.15. $\square$

1.3. Inverse Theorem: periodic points are landing points. It is much harder to show that, vice versa, any repelling or parabolic point is a landing point of at least one ray:

1.3.1. *Repelling case.*

THEOREM 6.2. *Let f be a polynomial with connected Julia set. Then any repelling point a is the landing point of at least one periodic ray.*

PROOF. Replacing f with its iterate, we can assume without loss of generality that a is a fixed point. We will consider the linearizing coordinates ϕ and ψ near a based on the discussion and notation of §16.2. Let $\tilde{U}_i$ be the components of $\psi^{-1}(D(\infty))$. These components are permuted by the map $g : z \mapsto \lambda z$. The main step of the proof is to show that each component $\tilde{U}_i$ is periodic under this action. It will be done by studying the rate of escape of hyperbolic geodesics in $\tilde{U}$ to infinity.

Let us consider the Green function $G : \mathbb{C} \to \mathbb{R}_{\geq 0}$ of f (see §16.6). Recall that it is a continuous subharmonic function satisfying the functional equation $G(fz) = dG(z)$. Let us lift it to the dynamical plane of g. We obtain a continuous subharmonic function $\tilde{G} = G \circ \psi$ on $\mathbb{C}$ satisfying the functional equation $\tilde{G}(\lambda z) = d\tilde{G}(z)$. Letting $M_n = \max\limits_{|z|=|\lambda|^n} \tilde{G}(z)$, $M \equiv M_1$, we see that $M_n \leq d^n M$.

By Lemma 4.22, the domains $\tilde{U}_i$ are simply connected and the restrictins $\psi : \tilde{U}_i \to D(\infty)$ are the universal coverings. Let us fix one of these domains, $\tilde{U} = \tilde{U}_i$ and endow it with the hyperbolic metric ρ. Let γ be the hyperbolic geodesic in $\tilde{U}$ that begins at a point $u_0 \in \mathbb{T}_1$ and goes to ∞ (i.e., γ is the pullback of a straight ray in $\mathbb{C} \smallsetminus \bar{\mathbb{D}}$ by the $B \circ \psi : U \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}$). This geodesic must cross all the circles $\mathbb{T}_{|\lambda|^n}$; let u_n stand for the first crossing point, and let $\rho_n = \rho(u_0, u_n)$.

By Excersice 4.23 and the above estimate on M_n, we have:

$$\rho_n = \log \frac{G(u_n)}{G(u_0)} \leq \log \frac{M_n}{G(u_0)} = n \log d + O(1). \qquad (1.1)$$

Thus, the points u_n escape to infinity no faster than at linear rate.

The above discussion is generally applied, no matter whether the domain $\tilde{U}$ is periodic under g or not. Assuming now that it is *aperiodic*, we will argue that the points u_n must escape to infinity at a superlinear rate.

If $\tilde{U}$ is aperiodic then the action of the cyclic group $< g >$ on the orbit of $\tilde{U}$ is faithful, so that, $\tilde{U}$ is embeded into the quotient torus $\mathbb{T}^2 = \mathbb{C}^* / < g >$ under the natural projection $\mathbb{C}^* \to \mathbb{T}^2$. Let $W \subset \mathbb{T}^2$ be the image under this embedding.

It is now convenient to make the logarithmic change of variable on $\mathbb{C}^*$ that turns it to the cylinder $\mathbb{C}/\mathbb{Z}$. Then the complex scaling g becomes the translation $z \mapsto z + \tau$, where $\tau = \log i\lambda/2\pi \mod \mathbb{Z}$, the circles $\mathbb{T}_{|\lambda|^n}$ become the circles $\mathbf{T}_n = \{v : \operatorname{Im} v = n \operatorname{Im} \tau\}$, U becomes a domain $\mathbf{U} \subset \mathbb{C}/\mathbb{Z}$, the geodesic γ in U becomes a geodesic $\boldsymbol{\gamma}$ in $\mathbf{U}$, and points u_n turn into points $\mathbf{u}_n \in \boldsymbol{\gamma} \cap \mathbf{T}_n$. Let us parametrize $\boldsymbol{\gamma}$ by the length parameter $t \in \mathbb{R}_{\geq 0}$ so that $\boldsymbol{\gamma}(0) = \mathbf{u}_0$.

Let us endow the cylinder $\mathbb{C}/\mathbb{Z}$ and the torus $\mathbb{T}^2$ with the flat Euclidean metric so that the natural projection $\pi : \mathbb{C}/\mathbb{Z} \to \mathbb{T}^2$ is locally isometric. Then

$$\operatorname{dist}(\boldsymbol{\gamma}(t), \partial\mathbf{U}) \to \mathbf{0} \quad \text{as} \quad t \to \infty. \qquad (1.2)$$

Otherwise there would exist $\epsilon > 0$ and a sequence of points $\mathbf{x}_n \in \mathbb{C}/\mathbb{Z}$ such that $\operatorname{Im} \mathbf{x}_{n+1} > \operatorname{Im} \mathbf{x}_n + 2\epsilon$ and $D_n \equiv D(\mathbf{x}_n, \epsilon) \subset \mathbf{U}$. Since $\pi :$

$\mathbf{U} \to \mathbb{T}^2$ is a locally isometric embedding, the images $\pi(D_n)$ would be disjoint ϵ-disks in $\mathbb{T}^2$, which is impossible by compactness of $\mathbb{T}^2$.

Let $d\rho(\boldsymbol{\gamma}(t)) = \sigma(t)|dz|$. By Lemma 1.19 and (1.2),

$$\sigma(t) \asymp \frac{1}{\text{dist}(\boldsymbol{\gamma}(t), \partial \mathbf{U})} \to \infty \quad \text{as} \quad t \to \infty. \tag{1.3}$$

Let l_n stands for Euclidean length of the arc of $\boldsymbol{\gamma}$ bounded by $\mathbf{u}_{n+1}$ and $\mathbf{u}_n$, and let $\sigma_n = \inf \sigma(t)$ on that arc. Then

$$\rho(\mathbf{u}_{n+1}, \mathbf{u}_n) \geq \sigma_n l_n \geq \sigma_n \operatorname{Im} \tau.$$

Hence

$$\rho(\mathbf{u}_{n+1}, \mathbf{u}_0) \geq \operatorname{Im} \tau \sum_{k=0}^{n-1} \sigma_k,$$

and by (1.3)

$$\frac{1}{n} \rho(\mathbf{u}_{n+1}, \mathbf{u}_0) \to \infty \quad \text{as} \quad n \to \infty,$$

contradicting (1.1).

Thus, the domain $\tilde{U}$ is periodic under the action of g with some period q. Hence the image W of $\tilde{U}$ in $\mathbb{T}^2$ is the quotient of $\tilde{U}$ by the cyclic $< g^q >$. It follows that it is conformally equivalent to either an annulus $\mathbb{A}(1, r)$ or to the punctured disk $\mathbb{D}^*$ depending on whether g^q is hyperbolic or parabolic. In fact, the first option is realized. Indeed,

$$\Psi \equiv B \circ \psi : \cup \tilde{U}_i \to \mathbb{C} \smallsetminus \bar{\mathbb{D}}$$

is a covering map conjugating g to $z \mapsto z^d$. Hence it semi-conjugates $g^q : \tilde{U} \to \tilde{U}$ to $z \mapsto z^{d^q}$. Since $\tilde{U}$ is simply-connected, Ψ lifts to a conformal isomorphism $\hat{\Psi} : \tilde{U} \to \mathbb{H}$ conjugating g^q to $\tau : \zeta \to d^q \zeta$. But the latter is a hyperbolic map of $\mathbb{H}$, so that, $W \approx \mathbb{H}/ < \tau >$ is an annulus (with modulus $\pi/(q \log d)$).

To complete the proof, let us consider the simple closed hyperbolic geodesic Γ in W. It lifts to a hyperbolic geodesic $\tilde{\Gamma}$ in $\tilde{U}$ invariant under the action of $< g^q >$. Let δ be a fundamental arc on $\tilde{\Gamma}$ bounded by some point u and $g^{-q}u$. Then $g^{-qn}(\delta) \to 0$ as $n \to \infty$, so that, $\tilde{\Gamma}$ lands at 0 (in "negative" time).

Since $\psi : \tilde{U} \to D(\infty)$ is a covering map semi-conjugating g^q to f^q, $\psi(\tilde{\Gamma})$ is a hyperbolic geodesic in $D(\infty)$ invariant under f^q and hence escaping to infinity in positive time. But hyperbolic geodesics in $D(\infty)$ escaping to infinity are exactly the external rays of f.

Finally, $\psi(\tilde{\Gamma})$ lands at a in negative time since ψ is continuous at 0. $\square$

PROBLEM 6.2. *a) Modify the above proof to show that there are only finitely many domains $\tilde{U}_i$.*
b) Conclude that there are only finitely many rays landing at any repelling point, and all of these rays are periodic.

1.3.2. *Parabolic case.* A point $a \in K(f)$ is called *dividing* if $K(f) \smallsetminus \{a\}$ is disconnected.

EXERCISE 6.3. *Assume $K(f)$ is connected. Show that a repelling or parabolic periodic point $a \in K(f)$ is dividing if and only if there are more than one ray landing at a.*

1.3.3. *Cantor case.*

PROPOSITION 6.3. *Let $f_c : z \mapsto z^2 + c$ be a quadratic polynomial with Cantor Julia set, i.e., $c \in \mathbb{C} \smallsetminus M$. Then any external ray $\mathcal{R}^\theta$ that does not crash at a precritical point lands at some point of $J(f_c)$.*

1.4. Rotation sets on the circle. We will now briefly deviate from the complex dynamics to study "rotation cycles" on the circle.

The oriented cicle $\mathbb{T} \approx \mathbb{R}/\mathbb{Z}$ is certainly not ordered, but it rather *cyclically ordered.* Namely, any finite subset $\Theta \subset \mathbb{T}$ is ordered, $\Theta = (\theta_1 \ldots, \theta_n)$, up to cyclic permutation of its points and this order is compatible with the inclusions of the sets. We say that a tuple of points $(\theta_1, \ldots, \theta_n)$ of $\mathbb{T}$ is *correctly ordered* if their order is compatible with the cyclic order of $\mathbb{T}$.

Given two points $\theta_1, \theta_2 \in \mathbb{T}$, we let $\overline{(\theta_1, \theta_2)}$ be the (open) arc of $\mathbb{T}$ that begins at θ_1 and ends at θ_2 (which makes sense since $\mathbb{T}$ is oriented).

A tuple of two points (θ_1, θ_2) of Θ is called neighbors in Θ if the corresponding arc $\overline{(\theta_1, \theta_2)}$ does not contain other points of Θ. (Note that this relation is asymmetric.)

Given a subset Θ and an injection $g : \Theta \to \mathbb{T}$, we say that g is *monotone* if it preserves the cyclic order of finite subsets of Θ (i.e., if $(\theta_1, \ldots, \theta_n)$ is a correctly ordered tuple of points of Θ, then the tuple of points $(g(\theta_1), \ldots, g(\theta_n))$ is also correctly ordered).

EXERCISE 6.4. *Show that g is monotone on a finite set $\Theta \subset \mathbb{T}$ iff it maps any tuple of neighbors in Θ to a tuple of neighbors in $g(\Theta)$.*

Monotone bijections $g : \Theta \to \Theta$ are called *rotations of* Θ, and Θ is correspondingly called a *rotation set* for g.

Any finite rotation set $\Theta \subset \mathbb{T}$ has a well defined rational *rotation number $p/q \in \mathbb{Q}/\mathbb{Z}$.* Namely, take a point $\theta \in \Theta$ and let q be the period of θ, while p be the number of points in the orb(θ) contained in the semi-open arc $\overline{[\theta, g(\theta))}$.

EXERCISE 6.5. *Check that q and p are independent of the choice of θ.*

We will now analyze rotation cycles for the doubling map $g : \theta \mapsto 2\theta \mod 1$.

LEMMA 6.4. *Let Θ be a rotation cycle for g with rotation number p/q. Then complementary arcs to Θ (counted according to the action of g starting with the shortest one) have lengths $2^{k-1}/(2^q-1)$, $k = 1, \ldots q$.*

PROOF. Let $\omega_i = \overline{(\theta_i, \kappa_i)}$ be the complementary arcs to Θ, where $g(\theta_i) = \theta_{i+1}$, $g(\kappa_i) = \kappa_{i+1}$ ($i \in Z/qZ$). If some ω_i is shorter that half-circle then g maps it homeomorphically onto the arc ω_{i+1} of length $|\omega_{i+1}| = 2|\omega_i|$. So, if all the arcs ω_i were shorter than half-circle then we would arrive at the basic logical contradiction:

$$1 = \sum_{i \in Z/qZ} |\omega_{i+1}| = 2 \sum_{i \in Z/qZ} |\omega_i| = 2.$$

Thus, one of the arcs ω_i must be longer than half-circle. Let us call it ω_0, and let $|\omega_0| = (1 + \epsilon)/2$. This arc is the union of the half-circle $\xi = \overline{[\kappa_0', \theta_0)}$ and the arc $\eta = \overline{(\theta_0, \kappa_0')}$ of length $\epsilon/2$, where $\kappa_0' = \kappa_0 - 1/2$ is the point symmetric with κ_0. Moreover, under g, the arc ξ is bijectively mapped onto the whole circle $\mathbb{T}$, while η is homeomorphically mapped onto $\overline{(\theta_1, \kappa_1)} = \omega_1$. We see that $|\omega_1| = \epsilon$.

Since each arc ω_i, $i = 1, \ldots, q - 1$, is shorter than half-circle, it is mapped homeomorphically onto the arc ω_{i+1}, and $|\omega_{i+1}| = 2|\omega_i|$. Hence $|\omega_i| = 2^{i-1}\epsilon$, $i = 1, \ldots, q$. Since $q = 0$ in $\mathbb{Z}/q\mathbb{Z}$, we obtain the equation:

$$\frac{1 + \epsilon}{2} = |\omega_0| = |\omega_q| = 2^{q-1}\epsilon,$$

which gives us the desired value of ϵ. □

The arc ω_0 of $\mathbb{T} \setminus \Theta$ which is longer than half-circle is called *critical*. The shortest arc ω_1 is called *characteristic*.

PROPOSITION 6.5. *For the doubling map $g : \theta \mapsto 2\theta$ on $\mathbb{T}$ and any rational $p/q \in \mathbb{Q}/\mathbb{Z}$, there exists a unique rotaion cycle $\Theta_{p/q} \subset \mathbb{T}$ with rotation number p/q.*

PROOF. Let Θ be a rotation cycle with rotation number p/q. Let us consider its characteristic arc $\xi_1 = \overline{(\theta, \kappa)}$. Since κ is the neighbor of θ, we have: $\kappa = 2^l\theta \mod 1$, where $l = 1/p$ in $\mathbb{Z}/q\mathbb{Z}$. On the other hand, $\kappa = \theta + 1/(2^q - 1)$ by Lemma 6.4. Hence

$$(2^l - 1)\,\theta \equiv 1/(2^q - 1) \mod 1 \tag{1.4}$$

Since θ is g-periodic with period q, $2^q\theta = \theta \mod 1$, so that, $\theta = t/(2^q - 1)$. Plugging it into (1.4), we come up with the equation

$$(2^l - 1)\,t \equiv 1 \mod 2^q - 1. \tag{1.5}$$

Since l and q are mutually prime, so are $2^l - 1$ and $2^q - 1$, and hence (1.5) has a unique solution mod $2^q - 1$. This prove uniqueness of the rotation cycle.

Going backwards, we take the solution of (1.4), let $\kappa = g^l\theta$ and $\xi_1 = \overline{(\theta, \kappa)}$. Then $\xi_2 = g^l(\xi)$ is the arc of length $2/(2^q - 1)$ adjacent to ξ_1; $\xi_3 = g^{2l}(\xi)$ is the arc of length $2/(2^q - 1)$ adjacent to ξ_2, etc., up to the arc $\xi_q = g^{lq}(\xi)$ of length $2^{q-1}/(2^q - 1) > 1/2$. Since the total length of these arcs is equal to 1, their closures tile the whole circle $\mathbb{T}$, so that, $\Theta = \mathrm{orb}(\theta)$ is a rotation cycle of g^l with rotation number $1/q$. Since $pl = 1 \mod q$, we have: $g|\Theta = (g^l)^p|\Theta$, and hence $g|\Theta$ has rotation number p/q.

$\square$

EXERCISE 6.6. *Derive the uniqueness part of the last proposition directly from Lemma 6.4, without finding the rotation cycle explicitly.*

EXERCISE 6.7. *Analyse the structure of rotation sets on $\mathbb{T}$ with irrational rotation number. Prove that for any $\eta \in \mathbb{R}/\mathbb{Z}$, there exists a unique closed rotation set Θ_η on $\mathbb{T}$ with rotation number η.*

EXERCISE 6.8. *Analyze the structure of rotation cycles for the map $g_d : \theta \mapsto d\,\theta$. Prove that there are at most $d - 1$ cycles with a given rotation number.*

1.5. Fixed points and their combinatorial rotaion number.

1.5.1. *Combinatorial rotation number.* Let us now consider a polynomial f of degree d with connected Julia set. Let a be its repelling or parabolic fixed point, and let $\mathcal{R}_i \equiv \mathcal{R}^{\theta_i}$ be the rays landing at a. The set of angles $\Theta(a) = \{\theta_i\} \subset \mathbb{T}$ is called the *ray portrait* of a.

LEMMA 6.6. *The ray portrait $\Theta(a)$ is a rotation set for the map $g_d : \theta \mapsto d\,\theta$.*

PROOF. Let S_i be the complementary sectors to the rays, i.e., the connected components of $\mathbb{C} \smallsetminus \cup \mathcal{R}_i$. Each sector S is bounded by a pair of rays $(\mathcal{R}, \mathcal{R}')$, which can be ordered so that $\mathcal{R}$ is positively oriented rel S. Thus, we can order the rays, $(\mathcal{R}_0, \mathcal{R}_1, \ldots, \mathcal{R}_{n-1})$, so that, $\mathcal{R}_i$ and $\mathcal{R}_{i+1}$ are neighbors, and this ordering is well defined up to cyclic permutation of the rays. So, the rays are cyclically ordered.

The map f preserves this cyclic order. Indeed, it is a local homeomorphism near a, and hence it permutes the local sectors. It follows

that neigboring rays are mapped by f to neighboring ones, which is equivalent to preserving the cyclic order (compare Exercise 6.4).

But the cylic order of the rays $\mathcal{R}^{\theta_i}$ coincides with the cyclic order of their slopes θ_i at ∞. Since these slopes are permuted by the map g_d, the conclusion follows. $\square$

For a periodic point a of period p, the combinatorial rotation number is defined by considering it as a fixed point of f^p.

1.5.2. *The $\alpha-$ and $\beta-$ fixed points of a quadratic polynomial.* Let us now assume that $f = f_c$ is a quadratic polynomial $z \mapsto z^d + c$ with connected Julia set. It turns out that the two fixed points of f (which ure statically undistinguishable) play very different dynamical role.

The polynomail f has only one invariant ray, $\mathcal{R}^0$. By Theorem 6.1, this ray lands at some fixed point called β; moreover, this point is either repelling or parabolic with multiplier 1. (In the latter case, $c = 1/4$ is the cusp of the Mandelbrot set, and the two fixed points coincide.) The ray $\mathcal{R}^0$ is the only ray landing at β (for any other ray would be also invariant by Lemma 6.6). Thus β is the *non-dividing fixed point* (see Exercise 6.3).

Outside the cusp $c = 1/4$, f_c has the second fixed point called α. It is either attracting (for c in the main hyperbolic component $H_0 \subset M$ bounded by the main cardioid C – see §20) or neutral (for c on the main cardioid C), or repelling. If α is repelling or parabolic then by Theorem 6.2 it is a landing point of some periodic ray $\mathcal{R} = \mathcal{R}^\theta$. Since $\theta \neq 0 \mod 1$, the period q of this ray is greater than 1. Of course, all the rays $\mathcal{R}_i = f^i(\mathcal{R})$, $n = 0, 1, \ldots, q - 1$, also land at α, so that, α is the *dividing* fixed point.

By Lemma 6.6 the ray portrait $\Theta(\alpha) \subset \mathbb{T}$ is a rotation set for the doubling map $\theta \mapsto 2\theta$. By Proposition 6.5, it is in fact, a single rotation cycle. Hence the rays $\mathcal{R}_i$ are cyclically permuted by f with a combinatorial rotaion number p/q. This rotation number, $\rho(f_c) \equiv \rho(c)$, is also called the *combinatorial rotation number of f* (or of the corresponding parameter c).

The rays $\mathcal{R}_i$ divide the plane into q sectors S_i, $i = 0 \ldots, q - 1$, which cut off arcs ω_i at the circle at infinity. We studied these arcs in Lemma 6.4. Recall that the longest of these arcs, labeled $\omega_0 \equiv \omega_q$, is called critical, while the shortest, ω_1 is called characteristic. The corresponding sectors, $S_0 \equiv S_q$ and S_1, will be called in the same way.

LEMMA 6.7. *For $i = 1, \ldots, q - 1$, the map f univalently maps the sectors S_i onto S_{i+1}. The critical sector S_0 contains the critical point 0, while the characteristic sector S_1 contains the critical value c.*

PROOF. Let $\bar{S}_i$ be the compactification of the sector S_i at infinity obtained by adding the arc ω_1 to S_i. This is a topological triangle. For $i = 1, \ldots, q-1$, the boundary of S_i is homeomorphically mapped onto the boundary of S_{i+1}. By the Argument Principle, the whole triangle $\bar{S}_i$ is homeomorphically mapped onto $\bar{S}_{i+1}$. Hence there are no critical points in these S_i, so that, $0 \in S_0$.   □

Let $\alpha' = -\alpha$; this is the second preimage of the fixed point α. By symmetry, there are q rays $\mathcal{R}'_i$ landing at α' symmetric to the rays $\mathcal{R}_i$, so that, $f(\mathcal{R}'_i) = \mathcal{R}_{i+1}$, $i \in \mathbb{Z}/q\mathbb{Z}$. Altogether, the rays $\mathcal{R}_i$ and $\mathcal{R}'_i$ partition the plane into $q-1$ pairs of symmetric sectors S_i, S'_i, $i = 1, \ldots, q-1$ (bounded by two rays each) and a central domain $\Omega_0 \ni 0$ bounded by two pairs of symmetric rays.

LEMMA 6.8. *The central domain Ω_0 is mapped onto the characteristic sector S_1 as a double branched covering.*

PROOF. Each pair of symmetric rays that bound Ω_0 is mapped homeomorphically onto a characteristic ray that bound S_1, so we have a 2-to-1 map $\partial\Omega_0 \to \partial S_1$.

Let $\bar{\Omega}_0$ be the compactification of Ω_0 by two symmetric arcs η and η' at infinity (where the arc η appeared in the proof of Lemma 6.4). Each of these arcs is mapped homeomorphically onto the characteristic arc ω_1.

We see that the boundary of $\bar{\Omega}_0$ is mapped to the boundary of $\bar{S}_1$ as a double covering, and the conclusion follows.   □

In the following sections we will describe the set of parameters with a given combinatorial rotation number.

2. Geodesic laminations

3. Limbs and wakes of the Mandelbrot set

3.1. Stability of landing.

3.1.1. *Repelling case.* If a is a repelling periodic point of period p for a polynomial f then by the Implicit Function Theorem, a nearby polynomial $\tilde{f}$ has a unique repelling periodic point $\tilde{a}$ near a. We will refer to this point as the *perturbed a*.

LEMMA 6.9 (Stability Lemma). *Assume that a periodic ray $\mathcal{R} = \mathcal{R}^\theta(f)$ lands at a repelling periodic point a for a polynomial f. Then for $\tilde{f}$ sufficiently close to f, the corresponding ray $\tilde{\mathcal{R}}^\theta$ lands at the perturbed repelling periodic point $\tilde{a}$.*

REMARK 6.1. Let us emphasize that this lemma applies to both connected and disconnected cases.

PROOF. Without loss of generality we can assume that the point a is fixed. and the ray $\mathcal{R}$ is invariant.

Let us take a small disk $D = D(a, 2\epsilon)$ such that the local inverse branch g of f^{-1} is well defined in D and $g(D) \Subset D$. Then the same is true for $\tilde{f}$ sufficiently close to f.

Let us fix some equipotential level $t > 0$ such that $\mathcal{R}(\tau) \subset D(a, \epsilon)$ for $\tau \leq dt$. Let γ be the closed arc of $\mathcal{R}$ in between the potential levels t and dt.

Let us consider the inverse Böttcher function

$$\tilde{B}^{-1} \equiv B_{\tilde{f}}^{-1} : \mathbb{C} \smallsetminus \bar{\mathbb{D}}_{\rho(f)} \to \Omega_f$$

on its maximal domain fo definition. Let

$$\mathcal{R}_{\geq t} = B^{-1}\{e^{\tau}e^{2\pi i\theta} : \ t \leq \tau < \infty\}.$$

The notations $\mathcal{R}_{>t}$ and similarly notations for $\tilde{\mathcal{R}}$ (whenever they are well defined) are self-explanatory. The Böttcher formula (16.7) implies that $\tilde{B}^{-1}$ depends continuously on $\tilde{f}$ in the closed-open topology. Hence if $\tilde{f}$ is sufficiently close to f, then the ray $\tilde{\mathcal{R}}_{\geq t}$ (parametrized by the potential level) is well defined and ϵ-close to the ray $\mathcal{R}_{\geq t}$. Let $\tilde{\gamma} = [\tilde{a}, \tilde{b}]$ be the arc of $\tilde{\mathcal{R}}$ between the potential levels t and dt. It follows that $\tilde{\gamma} \subset D$, so that, the inverse branch $\tilde{g}$ is well defined on $\tilde{\gamma}$.

But $\tilde{b} = f(\tilde{a})$, so that, $\tilde{a} = g(\tilde{b})$. Thus the arc $\tilde{g}(\tilde{\gamma}) \subset D$ gives an extension of the ray $\tilde{\mathcal{R}}_{\geq t}$ to the ray $\tilde{\mathcal{R}}_{\geq t/d}$. Repeating this argument, we conclude that the arcs $\tilde{g}^{-n}(\tilde{\gamma})$ give an extension of $\tilde{\mathcal{R}}_{\geq t}$ to the full ray $\tilde{\mathcal{R}}_{t>0}$.   □

3.1.2. *Parabolic case.*

LEMMA 6.10. *Let f be a polynomial with connected Julia set. For a parabolic periodic point a with multimplier $\lambda = e^{2\pi i p/q}$, the combinatorial rotaion number coincides with its rotation number p/q.*

PROOF. Without loss of generality we can assume that a is a fixed point.

By Lemma **??**, the rays landing at a are tangent to the bisectors L_i of the repelling petals, which are permuted by the differential $Df(a)$ with rotation number p/q. Hence these rays are organized in q groups $\mathcal{G}_i = (\mathcal{R}_{ij})_j$, $i = 1, \ldots q$, so that the rays in $\mathcal{G}_i$ are tangent to L_i. The rays within one group are naturally ordered: $\mathcal{R}_k \succ \mathcal{R}_j$ if $\mathcal{R}_j$ is positively oriented relatively to the local sector S of zero angle bounded

by these rays (in other words, $\mathcal{R}_k$ is obtained from $\mathcal{R}_j$ by the anti-clockwise "rotation" in S). Since f is a local orientation preserving diffeomorphism near a, it permutes these groups preserving the order of the rays. It follows that under f^q each ray is mapped back onto itself, and hence it is permuted by f with rotation number p/q. $\square$

Putting this together with Proposition 6.5, we obtain:

LEMMA 6.11. *Assume that the α-fixed point of the quadratic polynomial $f_c : z \mapsto z^2 + c$ is parabolic with rotation number p/q. Then it is a landing point of q rays that are permuted with the same rotation number.*

Recall that $r_{p/q} \in \mathcal{C}$ is the parabolic parameter with rotation number p/q. Let $H_{p/q}$ stand for the satellite hyperbolic component attached to the main cardioid $\mathcal{C}$ at $r_{p/q}$ (see Proposition 20.5).

LEMMA 6.12. *For any rotation number $p/q \neq 0$, there exists a curve $c(t)$, $t \in [0, \epsilon)$ such that $c(0) = r_{p/q}$, $c(t) \in H_{p/q}$ for $t > 0$, and $\rho(c(t)) = p/q$.*

3.2. Limbs and wakes.

3.2.1. *Limbs.* Let $\mathcal{L}^*_{p/q}$ be the connected component of $M \smallsetminus \{r_{p/q}\}$ containing $H_{p/q}$, and let $\mathcal{L}_{p/q} = \mathcal{L}^*_{p/q} \cup \{r_{p/q}\}$. This set is called the p/q-*limb* of the Mandelbrot set, while $\mathcal{L}^*_{p/q}$ is called the "unrooted p/q-limb".

PROPOSITION 6.13. *For any $c \in \mathcal{L}_{p/q}$, the combinatorial rotation number $\rho(c)$ is equal to p/q.*

PROOF. By Lemma 6.11, it is true at the root $r_{p/q}$. By Lemma 6.12, it is also true on some curve $\gamma \in H_{p/q}$ landing at $r_{p/q}$. By stability of ray portraits at repelling points (Lemma 6.9), the combinatorial rotaion number $c \mapsto \rho(c)$ is a continuous function of $c \in \mathcal{L}^*_{p/q}$. Since the unrooted limb $\mathcal{L}_{p/q}$ is connected, while ρ can assume only rational values, it is constant on the whole limb $\mathcal{L}_{p/q}$. $\square$

3.2.2. *Characteristic parameter rays.* Since the Stability Lemma 6.9 applies to the disconnected case as well, the p/q-ray portrait at the α-fixed point persists at some open set containing the unrooted limb $\mathcal{L}^*_{p/q}$. Below we will give the precese description of this open set.

For a parameter c with a well defined finite ray portrait, let $\mathcal{R}^-_{\mathrm{dyn}}(c)$ and $\mathcal{R}^+_{\mathrm{dyn}}(c)$ be the the characteristic rays landing at the α-fixed point α_c of f_c, and let $S_{\mathrm{char}}(c)$ be the characteristic sector bounded by these rays. For $c \in \mathcal{L}_{p/q}$, let $\theta^-_{p/q} < \theta^+_{p/q}$ be the angles of the charcteristic rays (which are independent of c by by Proposition 6.13).

The corresponding objects in the parameter plane are the rays $\mathcal{R}^-_{\mathrm{par}}(p/q)$ and $\mathcal{R}^+_{\mathrm{par}}(p/q)$ with angles $\theta^-_{p/q}$ and $\theta^+_{p/q}$, and the p/q-*wake* $\mathcal{W}_{p/q}$, the component of $\mathbb{C} \smallsetminus \mathrm{cl}(\mathcal{C} \cup \mathcal{R}^-_{\mathrm{par}} \cup \mathcal{R}^+_{\mathrm{par}})$ containing the satellite hyperbolic component $H_{p/q}$.[1]

In what follows we sometimes suppress the label p/q and c, as long as this cannot lead to a confusion.

LEMMA 6.14 (Key Observation). *For $c \in \mathcal{R}^\pm_{\mathrm{par}}(p/q)$, the dynamical characteristic rays $\mathcal{R}^\pm_{\mathrm{dyn}}(c)$ do not land on $J(f_c)$.*

PROOF. Assume for definiteness that $c \in \mathcal{R}^-_{\mathrm{par}}(p/q)$. Then by the Basic Phase-Parameter relation, $c \in \mathcal{R}^-_{\mathrm{dyn}}$.

Let $\Theta = \{\theta_i\}_{i=0}^{q-1} \subset \mathbb{T}$ be the cycle of θ^- under the doubling map, where $\theta_1 = \theta^-$. Then $0 \in \mathcal{R}^{\theta_0}_{\mathrm{dyn}}$, so, the ray $\mathcal{R}^{\theta_0}_{\mathrm{dyn}}$ does not land on $J(f)$ but rather crashes at the critical point 0.

Going backwards along the cycle of rays $\mathcal{R}^{\theta_i}_{\mathrm{dyn}}$, we see that all the rays of this cycle crash at some precritical point. In particular, the characteristic rays do.   □

LEMMA 6.15. *The wake $\mathcal{W}_{p/q}$ contains the unrooted limb $\mathcal{L}^*_{p/q}$ and some component Ω of $(\mathbb{C} \smallsetminus M) \smallsetminus (\mathcal{R}^-_{\mathrm{par}}(p/q) \cup \mathcal{R}^+_{\mathrm{par}}(p/q))$. All the points in the wake have combinatorial rotaion number p/q.*

PROOF. By the Stablility Lemma 6.9 and the Key Observation, the parameter rays $\mathcal{R}^\pm_{\mathrm{par}}(p/q)$ cannot accumulate on a point $c \notin \mathcal{C}$ with rotation number p/q. In particular, they do not accumulate on the unrooted limb $\mathcal{L}^*_{p/q}$, which implies the first assertion.

It follows that the wake $\mathcal{W}_{p/q}$ intersects $\mathbb{C} \smallsetminus M$, and hence it contains the component Ω of $\mathbb{C} \smallsetminus M \smallsetminus (\mathcal{R}^-_{\mathrm{par}}(p/q) \cup \mathcal{R}^+_{\mathrm{par}}(p/q))$ such that $\mathcal{L}^*_{p/q} \subset \partial\Omega$. (Notice that $(\mathbb{C} \smallsetminus M) \smallsetminus (\mathcal{R}^-_{\mathrm{par}}(p/q) \cup \mathcal{R}^+_{\mathrm{par}}(p/q))$ consists of two components.)

Let us prove the last assertion. Assume there is a parameter $c_1 \in \mathcal{W}_{p/q}$ with $\rho(c_1) \neq p/q$. Let us fix a reference point $c_0 \in H_{p/q}$ and connect it to c_1 with a curve $c_t \subset \mathcal{W}_{p/q}$, $0 \leq t \leq 1$.

By the Stablility Lemma 6.9, there is a maximal interval $[0, \tau)$ such that $\rho(c_t) = p/q$ for $t \in [0, \tau)$. By Proposition 6.13, $c(\tau) \notin \mathcal{L}^*_{p/q}$, so $c(\tau) \in \Omega$. Then by Proposition 6.3 only two events can happen:

(i) The characateristc ray $\mathcal{R}^+_{\mathrm{dyn}}(c_\tau)$ lands at some periodic point $a \neq \alpha$ of $J(f_{c_\tau})$. But then by the Stability Lemma, this would also be

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[1]This definition is convenient to start with, but eventually it will be simplified (see Theorem 6.16).

the case for $c_{\tau-\epsilon}$ for $\epsilon > 0$ sufficiently small, contradicting definition of τ.

(ii) The characteristc ray $\mathcal{R}^-_{\mathrm{dyn}}(c_\tau)$ crashes at some precritical point. But then the critical value c_τ would belong to one of the ray $\mathcal{R}^{\theta_i}_{\mathrm{dyn}}(c_\tau)$ of the cycle of $\mathcal{R}^-_{\mathrm{dyn}}(c_\tau)$. Since for $c_{\tau-\epsilon}$, the critical value c belongs to the characteristic sector $S_{\mathrm{char}}(c)$, this can only be one of the characteristic rays $\mathcal{R}^\pm_{\mathrm{dyn}}(c_\tau)$. But then by the Basic Phase-Parameter relation, $c_\tau \in \mathcal{R}^\pm_{\mathrm{par}}$ contradicting the definition of Ω.

$\square$

THEOREM 6.16. *Both parameter rays $\mathcal{R}^\pm_{\mathrm{par}}(p/q)$ land at the root $r_{p/q}$. The wake $\mathcal{W}_{p/q}$ coincides with the domain bounded by the curve $\mathcal{R}^-_{\mathrm{par}}(p/q) \cup \mathcal{R}^+_{\mathrm{par}}(p/q) \cup r_{p/q}$ and containing $H_{p/q}$. The combinatorial rotation number is equal to p/q throughout the wake.*

PROOF. We know from the proof of Lemma 6.15 that the rays $\mathcal{R}^\pm_{\mathrm{par}}(p/q)$ cannot accumulate on a point $c \in M \smallsetminus \mathcal{C}$ with rotation number p/q. Let us show that they can neither accumulate on other points $c \in M \smallsetminus \mathcal{C}$.

Let $\rho(c) = r/s \neq p/q$. By the Stability Lemma, $\rho(\tilde{c}) = r/s$ for all $\tilde{c} \in D(c, \epsilon)$, provided $\epsilon > 0$ is sufficiently small. But if $\mathcal{R}^-_{\mathrm{par}}$ accumulates on c then all nearby parameter rays $\mathcal{R}^\theta_{\mathrm{par}}$ enter the disk $D(c, \epsilon)$. Take such a parameter ray in the domain Ω, and let $\tilde{c} \in D(c, \epsilon) \cap \mathcal{R}^\theta_{\mathrm{par}}$. Since $\tilde{c} \in W$, $\rho(\tilde{c}) = p/q$ by Lemma 6.15, and we have arrived at a contradiction.

Hence the rays $\mathcal{R}^\pm_{\mathrm{par}}$ can accumulate only the points of main cardioid $\mathcal{C}$. Let $\omega^\pm \subset \mathcal{C}$ be the limit sets of the rays $\mathcal{R}^\pm$. If one of then, say, ω^-, was not a single point, then we could find a rational point $p/q \in \mathrm{int}\,\omega^-$, and the ray $\mathcal{R}^-_{\mathrm{par}}$ would have to cross the satellite component $H_{p/q}$. Since it is certainly impossible, the limit sets $\omega^\pm$ are, in fact, single points, so that both rays land at some points $c_\pm$ of the main cardioid.

If $c_+ \neq c_-$ then the wake $\mathcal{W}_{p/q}$ would contain, besides $H_{p/q}$, some other satellite hyperbolic domain $H_{r/s}$. But the combinatorial rotation number in $H_{r/s}$ is equal to $r/s \neq p/q$ contradicting Lemma 6.15. This shows that the rays $\mathcal{R}^\pm_{\mathrm{par}}$ land at the root $r_{p/q}$, and the rest of the lemma easily follows. $\square$

The angles $\theta^\pm_{p/q}$ of the rays $\mathcal{R}^\pm_{\mathrm{par}}(p/q)$ landing at $r_{p/q}$ are also called the *external angels* of $r_{p/q}$.

3.3. Limbs and wakes attached to other hyperbolic components. One can generalize the above discussion to limbs attached to any hyperbolic component H in place of the main one, H_0. Let

$a_c = a_c^H$, $c \in H$, be the attracting periodic point of f_c, and let $\lambda_c = \lambda_c^H$ be its multiplier. On the boundary of H the point a_c becomes neutral. By the Multiplier Theorem 5.9, for $c \in \partial H$, the rotation number $\rho(a_c)$ assumes once every value $\theta \in \mathbb{R}/\mathbb{Z}$. Let $r_{p/q}(H) \in \partial H$ stand for the parabolic parameter with rotatin number p/q, i.e., $\rho(a_c) = p/q$.

THEOREM 6.17. *Let $p/q \neq 0$ mod 1. Then there are two parameter rays $\mathcal{R}_{\mathrm{par}}^{\pm}(p/q, H)$ landing at $r_{p/q}(H)$ such $\rho(a_c^H) = p/q$ in the wake $\mathcal{W}_{p/q}(H)$ bounded these rays, and moreover, this wake is a maximal region with this property.*

REMARK 6.2. Note that at this moment we do not claim that there are no other rays landing at $r_{p/q}(H)$ since we will use Theorem 6.17 to show this.

The *limb* $\mathcal{L}_{p/q}(H)$ of M attached to the parabolic point $r_{p/q}(H)$ is defined as in the case of the main component H_0.

In the case when H is itself a satellite hyperboplic component attached to H_0, we call $\mathcal{L}_{p/q}(H)$ and $\mathcal{W}_{p/q}(H)$ *secondary limbs* and *wakes* respectively.

In what follows, we will also need to know that the external angles $\theta_{p/q}^{\pm}(H)$ of a root point depend continuously on the internal angle p/q.

LEMMA 6.18.   • *Let $p/q \neq 0$. Then $\theta^{\pm}(r/s)(H) \to \theta_{p/q}^{-}(H)$ as $r/s \nearrow p/q$, and $\theta^{\pm}(r/s)(H) \to \theta_{p/q}^{+}(H)$ as $r/s \searrow p/q$.*
   • *Let $H = H_0$. Then $\theta_{p/q}^{\pm}(H) \to 0$ as $p/q \to 0$ mod 1.*
   • *Let H be a satellite hyperbolic component and $\theta_0^{\pm}(H)$ be the characteristic rays landing at the root of H. Then $\theta_{p/q}^{\pm}(H) \to \theta_0^{-}(H)$ a $p/q \searrow 0$ and $\theta_{p/q}^{\pm}(H) \to \theta_0^{+}(H)$ as $p/q \nearrow 1$.*

REMARK 6.3. The only reason why in the last statement we assume that H is satellite is that we do not know yet that there are rays landing at the root of a primitive hyperbolic component $H \neq H_0$.

3.4. No fake limbs. A *fake limb* of M is a component of $M \smallsetminus \bar{H}_0$ different from any limb $\mathcal{L}_{p/q}^*$.

LEMMA 6.19. *There are no fake limbs.*

PROOF. Let X be such a limb. Notice first that $\bar{X} \cap \mathcal{C} \neq \emptyset$, for otherwise M would be disconnected. Also, since X is connected, the combinatorial rotaion number $\rho(c)$ is independent of $c \in X$, so we have a well defined number $\rho(X) = p/q$.

Obviously $X \cap \partial \mathcal{W}_{p/q} = \emptyset$, so that, X is either contained in $\mathcal{W}_{p/q}$ or lies outside its closures. Let us first assume the latter. Then for $r/s \notin$

$\{p/q, 0\}$, any parameter $c_0 \in X$ can be connected to any parameter $c_1 \in H_{r/s}$ by a path $c_t \in \mathbb{C} \smallsetminus \bar{H}_0$, $t \in [0, 1]$, that does not cross $\partial \mathcal{W}_{p/q}$. But then by the Stability Lemma, $\rho(c_t) = p/q$ for all $t \in [0, 1]$, contradicting to $\rho(c_1) = r/s$.

Assume now that $X \subset \mathcal{W}_{p/q}$. Let us then consider the periodic cycle $\{f^n(a_c)\}_{n=0}^{q-1}$ of period q obtained by analytic continuation of the attracting cycle bifurcated from the α-fixed point at $r_{p/q}$. At this moment we do not know yet that the multiplier λ_c of this cycle is different from 1 throughout the wake $\mathcal{W}_{p/q}$, so that, the function $c \mapsto a_c$ can be multi-valued. Let $Z = \{c \in \mathcal{W}_{p/q} \cup \{r_{p/q}\} : |\lambda_c| \leq 1\}$. Since this is a finite union of disjoint Jordan disks, $\mathcal{W}_{p/q} \smallsetminus Z$ is connected. Notice also that X is not contained in Z since there are always satellite components attached to each compenent of Z. Let $c_0 \in X \smallsetminus Z$, and let k/l be the combinatorial rotation number of the periodic point a_{s_0}.

Let us consider the secondary wakes $\mathcal{W}^2_{r/s}$ attached to the satellite component $H_{p/q}$. Again, we have the anlternative: either $X \subset \mathcal{W}^2_{k/l}$ or $X \cap \bar{\mathcal{W}}^2_{k/l} = \emptyset$. But the former option is actually impossible since $\bar{\mathcal{W}}^2_{k/l}$ does not touch $\mathcal{C}$. The latter option is ruled out in the same way as above by taking a different rotation number $r/s \neq k/l$ and connecting c_0 to a seconday satellite component $H^2_{r/s}$ attached to $H_{p/q}$ with a path $c_t \in \mathcal{W}_{p/q} \smallsetminus Z$. $\qquad \square$

COROLLARY 6.20. *The Mandelbrot set admits the following partition:*

$$M = H_0 \cup \mathcal{C} \cup \bigcup_{p/q \in \mathbb{Q}/\mathbb{Z} \smallsetminus 0} \mathcal{L}^*_{p/q}.$$

COROLLARY 6.21. *The rays $\mathcal{R}^{\pm}_{\mathrm{par}}(p/q)$ are the only parameter rays landing at $r_{p/q}$.*

PROOF. If there was an extra ray $\mathcal{R}$ landing at $r_{p/q}$ then by Lemma 1.31 there would be an extra component of $M \smallsetminus \bar{H}_0$ attached to $\mathcal{C}$ at $r_{p/q}$. $\quad \square$

3.5. The α-rays and their holomorphic motion. Let us fix some combinatorial rotation number p/q. For $c \in \mathcal{W}_{p/q}$, let $\mathcal{R}^{\theta_i}_c$ be the dynamical rays landing at the α-fixed point α_c, and let

$$\mathcal{I}^{(0)}_c = \bigcup_i \mathcal{R}^{\theta_i}_{\mathrm{dyn}}(c) \cup \{\alpha_c\}.$$

This configuration of rays partition the plane into q sectors S_i described in Lemma 6.7.

Let $h_c : X_* \to X_c$ be a holomorphic motion of some dynamical set over a pointed parameter domain $(\Lambda, *)$ of the quadratic family

$z \mapsto z^2 + c$. We say that it *respects the Böttcher marking* if for any point $z \in X_* \smallsetminus J(f_*)$ we have:

$$B_c(h_c(z)) = B_*(z), \quad c \in \Lambda$$

(so that, the Böttcher coordinate B_c is the "first integral" of the motion).

PROPOSITION 6.22. *There is a holomorphic motion of the configuration $\mathcal{I}_c^{(0)}$ over the parabolic wake $\mathcal{W}_{p/q}$ that respects the Böttcher marking.*

PROOF. Let us select an arbitrary base point $* \in \mathcal{W}_{p/q}$.

By definition, $B_c(\mathcal{R}_c^\theta(t)) = e^{t+i\theta}$, where $t \in \mathbb{R}_+$ and $\theta \in \mathbb{R}/\mathbb{Z}$ are the Böttcher coordinates of the point $\mathcal{R}_c^\theta(t)$. Hence for $B_*(\mathcal{R}_*^{\theta_i}(t)) = B_c(\mathcal{R}_c^{\theta_i}(t))$, so that, $h_c(z) = B_c^{-1} \circ B_*(z)$ determines a motion of the external rays $\mathcal{R}_c^{\theta_i}$ over $\mathcal{W}_{p/q}$ respecting the Böttcher marking. This motion is holomorphic since the Böttcher function B_c depends holomorphically on c.

On the other hand, the point α_c obviously moves holomorphically over $\mathcal{W}_{p/q}$ as well, and we obtain the desired motion of the whole configuration $\mathcal{I}_c^{(0)}$. □

Let

$$\mathcal{I}_c^{(n)} = f^{-n}(\mathcal{I}_c^{(0)}). \tag{3.1}$$

3.6. MLC on the main cardioid.

THEOREM 6.23. *The Mandelbrot set is locally connected at any point of the main cardioid $\mathcal{C}$.*

PROPOSITION 6.24. *For any irrational $\theta \in \mathbb{R}/\mathbb{Z}$, there is a single parameter ray $\mathcal{R}^\eta$ landing at the point $c(\theta) \in \mathcal{C}$ with internal angle θ.*

4. Misiurewisz wakes and decorations

5. Topological model

Part 3

Puzzle and parapuzzle

Part 4

Little Mandelbrot copies

CHAPTER 7

Primitive copies

1. Quadratic-like families

1.1. Definitions. Let $\Lambda \subset \mathbb{C}$ be a domain in the complex plane. A *quadratic-like family* $\mathbf{g}$ *over* Λ is a family of quadratic-like maps $g_\lambda : U_\lambda \to U'_\lambda$ depending on $\lambda \in \Lambda$ such that:

- The *tube* $\mathbb{U} = \{(\lambda, z) : \lambda \in \Lambda, \ z \in U_\lambda\}$ is a domain in $\mathbb{C}^2$;
- $g_\lambda(z)$ is holomorphic in two variables on $\mathbb{U}$.

As usual, we assume that the critical value of each f_λ is located at the origin 0.

We will now formulate several additional assumptions which will make a quadratic-like family nice. First of them is minor. We say that $\mathbf{g}$ *extends beyond* $\mathbb{U}$ if there exists a domain $\Lambda' \ni \Lambda$ and a quadratic-like family $G_\lambda : V_\lambda \to V'_\lambda$ over Λ' such that for $\lambda \in \Lambda$, g_λ is an adjustment (see §17) of G_λ.

We call a quadratic-like family $\mathbf{g} : U_\lambda \to U'_\lambda$ over Λ *proper* if

- The domains Λ, U_λ and U'_λ are bounded by smooth Jordan curves;
- $\mathbf{g}$ admits an extension beyond $\mathbb{U}$;
- For $\lambda \in \partial \Lambda$, $g_\lambda(0) \in \partial U'_\lambda$.

(The first two assumptions are imposed only for the sake of convenience.) Obviously $g_\lambda(0) \neq 0$ for $\lambda \in \partial \Lambda$, so that we have a well defined winding number of the curve $\lambda \mapsto g_\lambda(0)$, $\lambda \in \partial \Lambda$, around 0. We call it the winding number of $\mathbf{g}$ and denote $w(\mathbf{g})$. A proper family $\mathbf{g}$ is called *unfolded* if $w(\mathbf{g}) = 1$. By the Argument Principle, any proper unfolded quadratic-like family has a unique parameter value $*$ such that f_* has a superattracting fixed point, i.e., $f_*(0) = 0$. We will select $*$ as the *base point* in Λ.

Finally, we want the fundamental annulus $A_\lambda = U'_\lambda \smallsetminus \bar{U}_\lambda$ of g_λ to move holomorphically with λ. So, assume that there is a holomorphic motion $h_\lambda : \bar{A}_* \to \bar{A}_\lambda$ respecting the boundary dynamical relation, i.e., such that

$$h_\lambda(g_* z) = g_\lambda(h_\lambda(z)) \quad \text{for} \quad z \in \partial U_*.$$

For a technical reason, we impose the following boundary assumption on this motion:

Boundary extension. Let $\lambda \in \partial\Lambda$. The homeomorphisms $h_\mu :$ $\bar{U}'_* \smallsetminus \bar{U}_* \to \bar{U}'_\mu \smallsetminus \bar{U}_\mu$, $\mu \in \Lambda$, uniformly converge as $\mu \to \lambda$ to a continuous map $h_\lambda : \bar{U}'_* \smallsetminus U_* \to \bar{U}'_\lambda \smallsetminus U_\lambda$, which is one-to-one everywhere, except that $h_\lambda^{-1}(0)$ consists of two points on ∂U_*. (Note that ∂U_λ is a "figure eight" curve for $\lambda \in \partial\Lambda$.)

Denote this holomorphic motion by $\mathbf{h}$. We say that the quadratic-like family $\mathbf{g}$ is *equipped* with the holomorphic motion $\mathbf{h}$. Sometimes we will use notation $(\mathbf{g}, \mathbf{h})$ for an equipped quadratic-like family.

For equipped families, there is a natural choice of tubing (see §17.4.1) holomorphically depending on λ. Namely, select any tubing $t_* : \bar{A}_* \to \bar{\mathbb{A}}[r, r^2]$ for the base point, and then let

$$t_\lambda = t_* \circ h_\lambda^{-1}. \tag{1.1}$$

These are tubings since the holomorphic motion h_λ respects the boundary dynamical relations.

The Mandelbrot set $M(\mathbf{g})$ of the quadratic-like family is defined as $\{\lambda \in \Lambda : \ J(g_\lambda) \text{ is connected}\}$. If $\mathbf{g}$ is proper, then $M(\mathbf{g})$ is compactly contained in Λ.

Let us finish with a few terminological and notational remarks. Let $\pi : \mathbb{C}^2 \to \mathbb{C}$ stand for the projection onto the first coordinate. We call a set $\mathbb{U} \subset \mathbb{C}^2$ a *tube over* $\Lambda = \pi(\mathbb{U}) \subset \mathbb{C}$ if it is a fiber bundle over Λ whose *fibers* $U_\lambda = \mathbb{U} \cap \pi^{-1}\lambda$ are Jordan disks (either open or closed). For $X \subset \Lambda$, we let $\mathbb{U}|X = \mathbb{U} \cap \pi^{-1}X$.

1.2. Restricted quadratic family. In this section we will show that the quadratic family $\{f_c\}_{c \in \mathbb{C}}$ can be naturally restricted to a proper unfolded equipped quadratic-like family.

Fix some $r > 1$. Restrict the parameter domain $\mathbb{C}$ to the topological disk $D \equiv D_{r^2}$ bounded by the parameter equipotential of radius r^2. According to formula (**??**), this parameter domain is specified by the property that $f_c(0) \in \Omega_c(r^2) \equiv \Omega'_c$, where $\Omega_c(\rho)$ is the domain bounded by the dynamical equipotential of level ρ. Hence for $c \in D$, f_c restricts to a quadratic-like map $f_c : \tilde{\Omega}_c \to \tilde{\Omega}'_c$, where $\tilde{\Omega}_c \equiv \Omega_c(r)$. These quadratic-like maps obviously form a quadratic-like family over D, which we will call a *restricted quadratic family.*

The restricted quadratic family is proper. The first two properties of the definition are obvious. The main property, $f_c(0) \in \partial\tilde{\Omega}'_c$ for $c \in \partial D$, follows from formula (**??**). The winding number of this family is equal to 1. Indeed, when the *parameter c* runs once along the boundary ∂D, the *critical value $c = f_c(0)$* runs once around $0 \in D$.

The restricted quadratic family is equipped with the holomorphic motion of the fundamental annulus given by the Böttcher maps. Select 0 is the base point in D and let

$$B_c^{-1} : \mathbb{A}[r, r^2] \to \bar{\Omega}_c' \smallsetminus \Omega_c \qquad (1.2)$$

(note that $\mathbb{A}[r, r^2] = \Omega_0' \smallsetminus \Omega_0$). Since the Böttcher function $B_c^{-1}(z)$ is holomorphic it two variables (**??**), $\{B_c^{-1}\}_{c \in D}$ is a holomorphic motion. This motion admits the boundary extension (see the previous section), since for $c \in \partial D$, B_c^{-1} homeomorphically maps $\mathbb{C} \smallsetminus \mathbb{D}_r$ onto $\mathbb{C} \smallsetminus \Omega_c(r)$ except that two points on $\mathbb{T}_r$ collapse to 0 (see §**??**).

Thus the restricted quadratic family satisfy all the properties formulated in the previous section.

1.3. Straightening of quadratic-like families. The Mandelbrot set $M(\mathbf{g})$ of any quadratic-like family $\mathbf{g}$ can be canonically mapped into the genuine Mandelbrot set M. Namely, by the Straightening Theorem, for any $\lambda \in M(\mathbf{g})$ there is a unique quadratic polynomial $f_{c(\lambda)} : z \mapsto z^2 + c(\lambda)$, $c(\lambda) \in M$, which is hybrid equivalent to g_λ. The map $\chi : \lambda \mapsto c(\lambda)$ is called the *straightening* of $M(\mathbf{g})$.

We know that the straightening is not canonically defined outsed the Mandelbrot set but rather depends on the choice of the tubing. But for equipped families there is a natural choice given by (1.1). With this choice, the straightening χ admits an extension to the whole parameter domain Λ, which well be still denoted by χ.

Recall that D_r stands for the parameter disk bounded by the parameter equipotential of radius r (in the quadratic family). We can now formulate a fundamental result of the theory of quadratic-like families:

THEOREM 7.1. *Let $\mathbf{g}$ be a proper unfolded equipped quadratic-like family over Λ. Endow it with a holomorphic tubing given by (1.1). Then the corresponding straightening χ is a homeomorphism from Λ onto D_{r^2}.*

The proof of this theorem will be split into several pieces which are important on their own right.

1.4. The critical value moves transversally to h. We say that a holomorphic curve $\Gamma \subset \mathbb{C}^2$ is a *global transversal* to a holomorphic motion $\mathbf{h}$ if it transversally intersects each leaf of $\mathbf{h}$ at a single point.

LEMMA 7.2. *Under the assumptions of Theorem 7.1, the graph of the function $\lambda \mapsto g_\lambda(0)$, $\lambda \in \Lambda$, is a global transversal to the holomorphic motion $\mathbf{h}$ on $\mathbb{U}' \smallsetminus \mathbb{U}$.*

We will also express it by saying that *the critical value moves transversally to* **h**. The moral of this lemma is that *in the complex setting the transversality can come for purely topological reasons.*

PROOF. Take a point $z \in U'_* \smallsetminus U_*$ and consider its leaf

$$L_z = \{(\lambda, \zeta) \in \mathbb{C}^2 : \ \lambda \in \Lambda, \ \zeta = h_\lambda(z)\}.$$

Since the motion **h** admits a continuous extension to the boundary $\partial \Lambda$, the function $\psi : \lambda \mapsto h_\lambda(z)$ is continuous up to the boundary and $\psi(\lambda) \in U'_\lambda \smallsetminus U_\lambda$, $\lambda \in \partial \Lambda$. Since the tube $\mathbb{V} \equiv \mathbb{U}|\partial \Lambda$ is homeomorphic to the solid torus $\partial \Lambda \times \mathbb{D}$ over $\partial \Lambda$, the curve $\lambda \mapsto \psi(\lambda)$, $\lambda \in \partial \Lambda$, is homotopic to the zero curve $\lambda \mapsto 0$ in $\mathbb{V}$, i.e., these two curves can be joined by a continuous family of curves $\psi_t : \partial \Lambda \to \mathbb{V}$, $0 \leq t \leq 1$.

Consider now the curve $\phi : \lambda \mapsto f_\lambda(0)$, $\lambda \in \partial \Lambda$. Since **f** is proper, $\phi(\lambda) \in \partial \mathbb{V}$. Hence $\phi(\lambda) - \psi_t(\lambda) \neq 0$ for $\lambda \in \partial \Lambda$. It follows that the curves $\lambda \mapsto \phi(\lambda) - \psi(\lambda)$ and $\lambda \mapsto \phi(\lambda)$, $\lambda \in \partial \Lambda$, have the same winding number around 0. But the latter number is equal to 1, since **f** is unfolded. Hence the former number is also equal to 1. By the classical Argument Principle, the graphs of the functions ϕ and ψ have a single transverse intersection, and that is what we need. □

1.5. Uniformization of the complement of $M(\mathbf{g})$. In this section we will construct a dynamical (non-conformal) uniformization of $\Lambda \smallsetminus M(\mathbf{g})$ which generalizes the uniformization of $\mathbb{C} \smallsetminus M$ constructed in §**??**. This construction will illustrate how to relate the parameter and dynamical planes by means of holomorphic moions.

Let us consider a set $P = \{\lambda \in \Lambda : \ g_\lambda(0) \in U'_\lambda \smallsetminus U_\lambda\}$ (i.e., the set of parameters for which the critical point escapes under the first iterate through the fundamental annulus $A_\lambda = U'_\lambda \smallsetminus U_\lambda$). Note that all points in Λ sufficiently close to $\partial \Lambda$ obviously belong to P. We will show that P is an annulus naturally homeomorphic to the dynamical annulus $A_* = U'_* \smallsetminus U_*$.

To this end consider the graph of the function $\phi : \lambda \mapsto g_\lambda(0)$,

$$\Gamma = \{(\lambda, z) \in \mathbb{C}^2 : \ \lambda \in \Lambda, \ z = g_\lambda(0)\}.$$

By Lemma 7.2, this graph is a global transversal to the holomorphic motion **h**. Hence there is a well defined holonomy $\gamma : A_* \to \Gamma$ along the leaves of **f** , and it maps A_* homeomorphically onto a topological annulus $B \subset \Gamma$. Obviously, $\pi(B) = P$. Altogether, we have a homeomorphism $\pi \circ \gamma$ from A_* onto P. It follows, in particular that P is a topological annulus, whose inner boundary is a Jordan curve in Λ and the outer boundary is $\partial \Lambda$.

Let us consider the domain $\Lambda' = \Lambda \smallsetminus P$. The restriction of our quadratic-like family to this parameter domain is not proper any more. To restore this property, we have to restrict the dynamical domains as well. Let $V_\lambda = g_\lambda^{-1} U_\lambda$. For any $\lambda \in \Lambda'$, $g_\lambda(0) \in U_\lambda$; hence V_λ is a topological disk and $g_\lambda : V_\lambda \to U_\lambda$ is a quadratic-like map. This gives us a quadratic-like family over Λ'.

It is proper since by construction $g_\lambda(0) \in U_\lambda$ for $\lambda \in \partial \Lambda'$ (other technical properties required in the definition are even more obvious). It has winding number one since the function $\phi : \lambda \mapsto g_\lambda(0)$ does not have zeros in the annulus $\bar{R}$. It follows that the boundary curves $\phi : \partial \Lambda \to \mathbb{C}^*$ and $\phi : \partial \Lambda' \to \mathbb{C}^*$ are homotopic and hence they have the same winding number around 0.

Let us now equip this family with a holomorphic motion $h'_\lambda : A'_* \to A'_\lambda$ of the fundamental annulus $A'_\lambda = U_\lambda \smallsetminus V_\lambda$. This motion is obtained by lifting the motion h_λ by means of the double coverings $g_\lambda : A'_\lambda \to A_\lambda$,

$$
\begin{array}{ccc}
A'_* & \xrightarrow{\ h'_\lambda\ } & A'_\lambda \\
g_* \downarrow & & \downarrow g_\lambda \\
A_* & \xrightarrow{\ h_\lambda\ } & A_\lambda
\end{array}
$$

We need to check that this lift can be chosen holomorphic in λ. To this end take a point $z \in A_*$ and consider its orbit $\psi : \lambda \mapsto h_\lambda(z)$, $\lambda \in \Lambda'$. Take some $\zeta \in A'_*$ such that $g_*(\zeta) = z$. We want to find a holomorphic function $\psi' : \lambda \to h'_\lambda(\zeta)$ which makes the above diagram commutative, i.e., it should satisfy the equation:

$$g_\lambda(\psi'(\lambda)) = \psi(\lambda).$$

By the Implicit Function Theorem, this equation has a local holomorphic solution if $g'_\lambda(\zeta) \neq 0$, i.e., if ζ is not a critical point of g_λ. This condition is certainly satisfied in our situation.

By the λ-lemma, the original holomorphic motion $\mathbf{h}$ mathches with $\mathbf{h}'$ on the common boundary $\partial^i A_\lambda = \partial^o A'_\lambda$, so that together they provide a single holomorphic motion of the union $A_\lambda \cup A'_\lambda$ over Λ'.

Let $P' = \{\lambda \in \Lambda' : g_\lambda(0) \in A'_\lambda\}$. Applying the above result to the restricted quadratic-like family, we obtain a homeomorphism $\pi \circ \gamma' : A'_* \to P'$, where γ' is the holonomy along $\mathbf{h}'$. Since γ' matches with γ on the common boundary of the annuli, they give us a homeomorphism of the union of the dynamical annuli onto the union of parameter annuli, $A \cup A' \to P \cup P'$.

Proceeding in the same way, we will construct:

- A nest of parameter annuli P^n attached one to the next and the corresponding parameter domains $\Lambda^n = \Lambda^{n-1} \smallsetminus P^{n-1}$ (where $\Lambda^0 \equiv \Lambda$, $P^0 \equiv P$, $\Lambda^1 \equiv \Lambda'$). Moreover, $\cup P^n = \Lambda \smallsetminus M(\mathbf{g})$.
- A sequence of proper unfolded quadratic-like families

$$g_{n,\lambda} \equiv g_\lambda : V_\lambda^{n+1} \to V_\lambda^n \text{ over } \Lambda^n,$$

  where $V_\lambda^n = g_\lambda^{-n} U_\lambda'$ (thus $V_\lambda^0 \equiv U_\lambda'$, $V_\lambda^1 \equiv U_\lambda$ and $V_\lambda^2 \equiv V_\lambda$).
- A sequence of holomorphic motions $h_{n,\lambda}$ of the fundamental annulus $A_\lambda^n \equiv \bar{V}_\lambda^n \smallsetminus V_\lambda^{n+1}$ over Λ^n which equip $g_{n,\lambda}$; moreover $h_{n+1,\lambda}$ is obtained by lifting $h_{n,\lambda}$ by means of the coverings $g_\lambda : A_\lambda^n \to A_\lambda^{n-1}$. These holomorphic motions match on the common boundaries of the fundamental annuli.

Let $\gamma_n : A_*^n \to \Gamma$ be the holonomy along $\mathbf{h}_n$ (recall that Γ is the graph of the function $\phi : \lambda \mapsto f_\lambda(0)$. Since the holomorphic motions match on the common boundaries, these holonomies also match, and determine a continuous injection $\gamma : U_* \smallsetminus K(f_*) \to \Gamma$. Composing it with the projection π, we obtain a homeomorphism

$$\pi \circ \gamma : U_* \smallsetminus K(f_*) \to \Lambda \smallsetminus M(\mathbf{g})$$

between the dynamical and parameter annuli. Note that the inverse map is equal to $\gamma^{-1} \circ \phi$.

Composing the above homeomorphism with the tubing (1.1), we obtain a "uniformization" of $\Lambda \smallsetminus M(\mathbf{g})$ by a round annulus:

$$S : t_* \circ (\pi \circ \gamma)^{-1} = t_\lambda \circ \phi : \ \Lambda \smallsetminus M(\mathbf{g}) \to \mathbb{A}(1, r^2), \quad S(\lambda) = t_\lambda(g_\lambda(0)). \tag{1.3}$$

We call $S(\lambda)$ "the tubing position of the critical value of g_λ".

Remark. The above uniformization of $\Lambda \smallsetminus M$ is generally not conformal. However, in the case of a restricted quadratic family, it is the restriction of the conformal uniformization of $\mathbb{C} \smallsetminus M$. Indeed, in this case, the tubing t_λ turns into the Böttcher maps B_c (see (1.2)), the critical value $g_\lambda(0)$ turns into c, and formula (1.3) turns into formula (**??**) for the Riemann map $R : \mathbb{C} \smallsetminus M \to \mathbb{C} \smallsetminus \mathbb{D}$, $R(c) = B_c(c)$.

COROLLARY 7.3. *The Mandelbrot set $M(\mathbf{g})$ is connected and full.*

1.6. Adjustments of quadratic-like families. Include the "maximal" extension of the leaves up to the critical value

1.7. Quasi-conformality of the uniformization. Given a holomorphic motion $\mathbf{h}$ over Λ, let

$$\mathrm{Dil}(\mathbf{h}) = \sup_{\lambda \in \Lambda} \mathrm{Dil}(h_\lambda)$$

(which can be infinite). We say that the holomorphic motion $\mathbf{h}$ is K-qc if $\mathrm{Dil}(\mathbf{h}) \leq K$. In the following statement we will use the notations of §1.5.

LEMMA 7.4. *Under the assumptions of Theorem 7.1, assume that the tubing $t_* : A_* \to \mathbb{A}(r, r^2)$ and the holomorphic motion $\mathbf{h}$ are K-qc. Then the uniformization $S : \Lambda \smallsetminus M(\mathbf{g}) \to \mathbb{A}(1, r^2)$ is K-qc as well.*

In fact, we can make the dilatation depend only on $\mathrm{mod}(A_*)$ and $\mathrm{mod}(\Lambda \smallsetminus \Lambda')$, after an appropriate adjustment of the family $\mathbf{g}$.

LEMMA 7.5. *Let us consider a quadratic-like family $\mathbf{g}$ over Λ satisfying the assumptions of Theorem 7.1. This family can be adjusted to a family $\tilde{\mathbf{g}}$ over $\tilde{\Lambda}$ in such a way that the dilatation of the straightening $\tilde{\chi} : \tilde{\Lambda} \smallsetminus M(\tilde{\mathbf{g}}) \to D \smallsetminus M$ will depend only on $\mathrm{mod}(A_*)$ and $\mathrm{mod}(\Lambda \smallsetminus \Lambda')$.*

1.8. Looking from the outside. We are now ready to prove that the straightening is a homeomorphism outside the Mandelbrot sets.

LEMMA 7.6. *Under the assumptions of Theorem 7.1, the straightening $\chi : \Lambda \smallsetminus M(\mathbf{g}) \to D_{r^2} \smallsetminus M$ is a homeomorphism.*

PROOF. Let us consider the uniformizations $S : \Lambda \smallsetminus M(\mathbf{g}) \to \mathbb{A}(1, r^2)$ and $R : D \smallsetminus M \to \mathbb{A}(1, r^2)$ constructed above. Then

$$\chi = R^{-1} \circ S. \tag{1.4}$$

Indeed, let $\lambda \in \Lambda \smallsetminus M(\mathbf{g})$ and $c = \chi(\lambda) \in D \smallsetminus M$. Putting together (**??**) and (1.3), we obtain:

$$S(\lambda) = t_\lambda(g_\lambda(0)) = B_c(c) = R(c),$$

which is exactly (1.4). Since S and R are both homeomorphisms, χ is a homeomorphism as well. $\square$

1.9. Miracle of continuity. We will now show that the straightening is continuous on the boundary of $M(\mathbf{g})$:

LEMMA 7.7. *Under the assumptions of Theorem 7.1, the straightening χ is continuous at any point $\lambda \in \partial M(\mathbf{g})$ and moreover $\chi(\lambda) \in \partial M$.*

PROOF. First we will show that $\chi | \partial M(\mathbf{g})$ is a continuous extension of $\chi | \Lambda \smallsetminus M(\mathbf{g})$. Let $\lambda_n \in \Lambda \smallsetminus M(\mathbf{g})$ be a sequence of parameter values converging to some $\lambda \in \partial M$. Let $c_n = \chi(\lambda_n)$ and $c = \chi(\lambda) \in M$. We shoud show that $c_n \to c$. Let $g_\lambda : U \to U'$, $f_c : \Omega \to \Omega'$.

By Lemma 7.6, the map $\chi : \Lambda \smallsetminus \mathrm{int}\, M(\mathbf{g}) \to D \smallsetminus \mathrm{int}\, M$ is proper, and hence any limit point d of $\{c_n\} \subset D \smallsetminus M$ belongs to ∂M. We assert that $g_\lambda : U \to U'$ is qc conjugate to $f_d : V \to V'$. Indeed, the $g_{\lambda_n} : U_n \to U'_n$ are hybrid equivalent to the $f_{c_n} : \Omega_n \to \Omega'_n$ by means

of some qc maps $\psi_n : U'_n \to \Omega'_n$. By the straightening construction (see the proof of Lemma 4.33), the dilatation of ψ_n is equal to the dilatation of the tubing $t_{\lambda_n} = t_* \circ h_\lambda^{-1}$, which is locally bounded by the λ-lemma. By **??**, the sequence ψ_n is pre-compact in the topology of uniform convergence on compact subsets of U'. Take a limit map $\psi : U' \to \Omega'$. Since $g_{\lambda_n} \to g_\lambda$ uniformly on compact subsets of U and $f_{c_n} \to f_d$ (along a subsequence) uniformly on compact subsets of Ω, the map ψ conjugates g_λ to f_d, as was asserted.

But g_λ is also hybrid equivalent to f_c. Thus f_c and f_d are qc conjugate in some neighborhoods of their filled Julia sets. By **??**, they are qc conjugate on the whole complex plane. Since $d \in \partial M$, the Rigidity Theorem **??**(...) implies the desired: $c = d$ (and, in particular, $c \in \partial M$).

The above argument implies that χ continuously maps $\Lambda \smallsetminus \operatorname{int} M(\mathbf{g})$ into $D \smallsetminus \operatorname{int} M$. We still need to show that χ is continuous at any point $\lambda \in \partial M$ even if it is approached from the interior of $M(\mathbf{g})$. The argument is similar to the above except one detail. So, let now $\{\lambda_n\}$ be any sequence in Λ converging to λ. Let c_n, c and d be as above. Then the above argument shows that f_c is qc equivalent to f_d. But now we already know that $c \in \partial M$ (though this time we do not know it for d). Hence by the Rigidity Theorem **??**(...), $c = d$. $\qquad\square$

"Only by miracle can one ensure the continuity of straightening in degree 2" said Adrien Douady [**D1**]. As we have seen, a reason behind this miracle is quasi-conformal rigidity of the quadratic maps f_c with $c \in \partial M$ (**??**). Another reason is the λ-lemma (see §**??**). All these reasons are valid only for one-parameter families. There are no miracles in the polynomial families with more parameters, see [**DH2**, §...].

1.10. Analyticity of $\chi : \operatorname{int} M(\mathbf{g}) \to \operatorname{int} M$. *The assumptions of Theorem 7.1 will be standing until the end of this section*

1.10.1. *Hyperbolic components.* As in the case of the genuine Mandelbrote set, a component H of $\operatorname{int} M(\mathbf{g})$ is called hyperbolic if it contains a hyperbolic parameter value.

EXERCISE 7.1. *Show that:*

(i) *All parameter values in a hyperbolic component of $M(\mathbf{g})$ are hyperbolic;*

(ii) *Neutral parameter values belong to $\partial M(\mathbf{g})$ (compare Lemma **??**);*

LEMMA 7.8. *If P is a hyperbolic component of* int $M(\mathbf{g})$ *then there exists a hyperbolic component Q of* int M *such that $\chi : P \to Q$ is a proper holomorphic map.*

PROOF. Obviously the straightening of a hyperbolic map is hyperbolic. Hence $\chi(P)$ belongs to some hyperbolic component Q of int M. Moreover, since the hybrid conjugacy is conformal on the interior of the filled Julia set, it preserves the multiplies of attracting cycles. Hence

$$\mu_P(\lambda) = \mu_Q(c) \text{ for } c = \chi(\lambda),$$

where μ_P and μ_Q are the multiplier functions on the domains P and Q respectively. By the Implicit Function Theorem, both these functions are holomorphic. Moreover, by Theorem **??**, μ_Q is a conformal isomorphism onto $\mathbb{D}$. Hence $\chi = \mu_Q^{-1} \circ \mu_P$ is holomorphic as well.

By Lemma 7.7, the map $\chi : P \to Q$ is continuous up to the boundary and $\chi(\partial P) \subset \partial Q$. Hence it is proper. $\qquad\square$

1.10.2. *Queer components.* As in the quadratic case, a non-hyperbolic component of int $M(\mathbf{g})$ is called queer. Let us first extend Lemma **??** to quadratic-like families:

LEMMA 7.9. *Let P be a queer component of $M(\mathbf{g})$. Take a base point $* \in P$. Then there is a holomorphic motion $H_\lambda : U'_* \to U'_\lambda$ conjugating g_* to g_λ.*

PROOF. Since $M(\mathbf{g})$ is equipped, there is a holomorphic motion $h_\lambda : A_* \to A_\lambda$. Let $A_\lambda^n = g_\lambda^{-n} A_\lambda$. Since the critical point is non-escaping under the iterates of g_λ, A_λ^n is an annulus and $g_\lambda^n : A_\lambda^n \to A_\lambda$ is a covering map. By **??**, $\mathbf{h}$ can be consequtively lifted to holomorphic motions $h_{n,\lambda} : A_*^n \to A_\lambda^n$. By the λ-lemma (**??**), they automatically match on the common boundaries of the annuli, so that we have a single holomorphic motion $H_\lambda : U'_* \smallsetminus K(g_*) \to U'_\lambda \smallsetminus K(g_\lambda)$ conjugating g_* to g_λ. Since the sets $K(g_\lambda)$ are nowhere dense (see Corollary 4.32), the λ-lemma extension of H_λ to the whole domain U'_* still conjugates g_* to g_λ. $\qquad\square$

LEMMA 7.10. *Let H_λ be the holomorphic motion constructed in the previous lemma. Then the Beltrami differential*

$$\mu_\lambda(z) = \begin{cases} \frac{\bar{\partial} H_\lambda(z)}{\partial H_\lambda(z)}, & z \in K(g_*), \\ 0, & z \in \mathbb{C} \smallsetminus K(g_*), \end{cases}$$

holomorphically depends on $\lambda \in P$.

We can now prove an analogue of Lemma 7.8 for queer components.

LEMMA 7.11. *The straightening χ is holomorphic on the queer components of* int $M(\mathbf{g})$.

PROOF. Consider a queer component $P \subset$ int $M(\mathbf{g})$ with a base point $*$. For $\lambda \in P$, let $h_\lambda : U'_\lambda \to \Omega'_\lambda$ denote the hybrid conjugacy between $g_\lambda : U_\lambda \to U'_\lambda$ and its straightening $f_\lambda \equiv f_{\chi(\lambda)} : \Omega_\lambda \to \Omega'_\lambda$, and let $h \equiv h_*$. Then $f_* : \Omega_* \to \Omega'_*$ is qc equivalent to $f_\lambda : \Omega_\lambda \to \Omega'_\lambda$ by means of the map $\psi_\lambda : h_\lambda \circ H_\lambda \circ h^{-1}$, where $\{H_\lambda\}$ is the holomorphic motion from the previous lemma. Let $\phi_\lambda : \mathbb{C} \smallsetminus K(f_*) \to \mathbb{C} \smallsetminus K(f_\lambda)$ be the conformal conjugacy between the quadratic polynomials f_* and f_λ on the complements of their Julia sets. By **??**, the map

$$\Psi_\lambda(z) = \left\{ \begin{array}{ll} \psi_\lambda(z), & z \in K(f_*), \\ \phi_\lambda(z), & z \in \mathbb{C} \smallsetminus K(f_*), \end{array} \right.$$

is a global qc conjugacy between f_* and f_λ conformal outside the Julia set.

Let $\nu_\lambda = (h_\lambda)_* \mu_\lambda$, where μ_λ is the conformal structure on $K(g_*)$ considered in the previous lemma. Since h_λ is confomal a.e. on the Julia set, we have:

$$\Psi_\lambda^*(\sigma | K(f_\lambda)) = h_* \circ (H_\lambda)^* \circ h_\lambda^* \left(\sigma | K(f_\lambda)\right) = h_* \circ (H_\lambda)^* (\sigma | K(g_\lambda)) = h_* \mu_\lambda = \nu_\lambda.$$

Since the push forward-map

$$h_* : \mu \mapsto \nu, \quad \nu = \left(\frac{h'}{\overline{h'}} \mu\right) \circ h^{-1}$$

is a complex isomorphism between the spaces of Beltrami differentials. the previous lemma implies that ν_λ holomorphically depends on $\lambda \in P$. By the holomorphic dependence of the solution of the Beltrami equation on parameters (ref) and **??**, $f_\lambda(0) = \chi(\lambda)$ holomorphically depends on λ as well. $\square$

1.11. Discreteness of the fibers.

LEMMA 7.12. *For any $c \in M$, the fiber $\chi^{-1}(c)$ is finite.*

PROOF. Since $M(\mathbf{g})$ is compact, it is enough to show that the fibers are discrete. Assume that there exists some $c \in M$ with an infinite fiber $\chi^{-1}(c)$. Since $M(\mathbf{g})$ is compact, this fiber contains a sequence of distinct parameter values $\lambda_n \in \chi^{-1}(c)$ converging to some point $\lambda_\star \in \chi^{-1}(c)$ We will skip the subscript in all notations affiliated with the map $g_{\lambda_\star}$, i.e., $g_{\lambda_\star} \equiv g$, $U_{\lambda_\star} \equiv U$ etc.

Since χ is holomorphic on int M, $\lambda_\star$ cannot belong to int M unless it belongs to a queer component U such that $\chi | U \equiv const$. But in the latter case, we can replace $\lambda_\star$ by any boundary point of U. Thus we can always assume that $\lambda_\star \in \partial M$.

Since the quadratic-like family $g_\lambda : U_\lambda \to U'_\lambda$ is equipped, there exists an equivariant holomorphic motion $h_\lambda : A \to A_\lambda$ of the closed fundamental annulus $A_\lambda = \bar{U}'_\lambda \smallsetminus U_\lambda$, i.e., $h_\lambda(gz) = g_\lambda(h_\lambda z)$ for $z \in \partial A$. Extend it by the λ-lemma **??** to a holomorphic motion $h_\lambda : \mathbb{C} \smallsetminus U \to \mathbb{C} \smallsetminus U_\lambda$ (keeping the same notation for the extension). We will construct a holomorphic family of hybrid deformations G_λ of g, $\lambda \in \Lambda$, naturally generated by this holomorphic motion.

To this end let us first pull back the standard conformal structure to $\mathbb{C} \smallsetminus U$, $\mu_\lambda = h_\lambda^*(\sigma)$. Then extend μ_λ to a g-invariant conformal structure on $\mathbb{C} \smallsetminus K(g)$ by pulling it back by iterates of g. Finaly extend it to $K(g)$ as a strandard structure. This gives us a holomorphic family of g-invariant conformal structures on $\mathbb{C}$. We will keep the same notation μ_λ for these structures. Solving the Beltrami equations, we obtain a holomorphic family of qc maps $H_\lambda : \mathbb{C} \to \mathbb{C}$ such that $\mu_\lambda = (H_\lambda)^*(\sigma)$ and $\bar{\partial} H(z) = 0$ a.e. on $K(g)$. Conjugating g by these maps, we obtain a desired hybrid deformation $G_\lambda = H_\lambda \circ g \circ H_\lambda^{-1}$, $\lambda \in \Lambda$.

On the other hand, for maps $g_{\lambda_n} \equiv g_n$, we can construct the Beltrami differentials $\mu_{\lambda_n} \equiv \mu_n$ in a different way. Indeed, since the map g_n is hybrid equivalent to g, the equivariant map $h_{\lambda_n} \equiv h_n$ uniquely extends to a hybrid conjugacy (Theorem **??**). Let us keep the same notation h_n for this conjugacy.

The above two constructions naturally agree: $(h_n)^*\sigma = \mu_n$. Indeed, it is true on $\mathbb{C} \smallsetminus U$ by definition. It is then true on $U \smallsetminus K(f)$, since the Beltrami differentials are pulled-back under conformal liftings (see Lemma 4.36). Finally, it is true on the filled Julia set $K(g)$ since h_n is conformal a.e. on it.

Thus the qc maps $H_n : \mathbb{C} \to \mathbb{C}$ and $h_n : \mathbb{C} \to \mathbb{C}$ satisfy the same Beltrami equation. They also coincide at two points, e.g., at the critical point and at the β-fixed point of g (in fact, by Corollary 4.42 they coincide on the whole Julia set of g). By uniqueness of the solution of the Beltrami equation, $H_n = h_n$. Hence $G_n = g_n$. Returning to the original notations, we have

$$G_{\lambda_n}(z) = g_{\lambda_n}(z). \tag{1.5}$$

Take an $\epsilon > 0$ such that both functions $G_\lambda(z)$ and $g_\lambda(z)$ are well-defined in the bidisk $\{(\lambda, z) \in \mathbb{C}^2 : |\lambda - \lambda_\star| < \epsilon, \ z \in V \equiv g^{-1}U\}$. For any $z \in V$, consider two holomorphic functions of λ:

$$\Phi_z : (\lambda) = G_\lambda(z) \quad \text{and} \quad \phi_z(\lambda) = g_\lambda(z), \quad |\lambda - \lambda_\star| < \epsilon.$$

By (1.5), they are equal at points λ_n converging to λ_*. Hence they are identically equal.

Thus for $|\lambda| < \epsilon$, two quadratic-like maps, G_λ and g_λ, coincide on V. But it is impossible since the Julia set of G_λ is always connected, while the Julia set of g_λ is disconnected for some λ arbitrary close to $\lambda_\star$ (recall that we assume that $\lambda_\star \in \partial M(\mathbf{g})$). $\square$

COROLLARY 7.13. $\chi(\operatorname{int} M(\mathbf{g})) \subset \operatorname{int} M$.

Remark. Of course, it is not obvious only for queer components.

PROOF. Take a component P of $\operatorname{int} M$. We have proven that $\chi | P$ is a *non-constant* holomorphic function. Hence the image $\chi(P)$ is open. Since it is obviously contained in M, it must be contained in $\operatorname{int} M$. $\square$

1.12. Bijectivity. What is left is to show that the map $\chi : M(\mathbf{g}) \to M$ is bijective. By §1.5, the winding number of the curve $\chi : \partial \Lambda \to \mathbb{C}$ around any point $c \in D_{r^2}$ is equal to 1. By the Topological Argument Principle (§6.1),

$$\sum_{a \in \chi^{-1}c} \operatorname{ind}_a(\chi) = w_c(\chi, \partial \Lambda) = 1, \quad c \in D_{r^2}. \tag{1.6}$$

It immediately follows that the map $\chi : \Lambda \to \mathbb{D}_{r^2}$ is *surjective* (for otherwise the sum in the left-hand side would vanish fo some $c \in D_{r^2}$).

Let us show that χ is injective on the interior of $M(\mathbf{g})$. Indeed, if $a_0 \in \operatorname{int} M$, then by Corollary 7.13 $c = \chi(a_0) \in \operatorname{int} M$, and by Lemma 7.7 $\chi^{-1}(c) \subset \operatorname{int} M$. But by §1.10, $\chi | \operatorname{int} M$ is holomorphic and hence $\operatorname{ind}_a(\chi) > 0$ for any $a \in \operatorname{int} M$. It follows that the sum in the left-hand side of (1.6) actually contains only one term, so that c has only one preimage, a_0.

Finaly, assume that there is a point $c \in \partial M$ with more than one preimage. By the Topological Argument Principle, χ has a non-zero index at one of those preimages, say, a_1. Take another preimage a_2. Both a_1 and a_2 belong to ∂M.

Take a point $a_2' \notin \partial M(\mathbf{g})$ near a_2, and let $c' = \chi(a_2')$. By Exercise 1.35, χ is locally surjective near a_1, so that c' has a preimage a_1' over there. This contradicts injectivity of χ on $\Lambda \setminus \partial M(\mathbf{g})$.

This completes the proof of Theorem 7.1.

CHAPTER 8

Satellite copies

Part 5

Hints and comments to the exersices

Chapter 1

1.3. The space of ϵ-separated triples of points is compact. The Möbius transformation ϕ depends continuously on the triple $(\alpha, \beta, \gamma) = \phi^{-1}(0, 1, \infty)$ as obvious from the explicit formula

$$\phi(z) = \frac{z - \alpha}{z - \gamma} \cdot \frac{\beta - \gamma}{\beta - \alpha}.$$

(This can also be used to verify equivalence of the two topologies.)

1.5. The curvature of a metric $\rho(z)|dz|$ can be calculated by the formula:

$$\kappa(z) = -\frac{\Delta \log \rho(z)}{\rho(z)^2}.$$

$\mathrm{PSL}(2, \mathbb{R})$-invariance of the hyperbolic metric in the $\mathbb{H}$-model amounts to the identity:

$$\mathrm{Im}\, \phi(z) = \frac{\mathrm{Im}\, z}{|cz + d|^2}, \quad \phi(z) = \frac{az + b}{cz + d}.$$

Smooth isometries preserve angles between tangent vectors, and so conformal. In fact, one does not need to impose smoothness *a priori*. Any isometry is quasi-conformal (e.g., by the Pesin criterion, Theorem **??**), and hence conformal by Weyl's Lemma (2.9).

1.6. (It is a generality about discrete groups of isometries of locally compact spaces.) If proper discontinuity (see the definition in §1) was violated, then there would exist as sequence of distinct motions $\gamma_n : \mathbb{D} \to \mathbb{D}$, and sequence of points $x_n \to x \in \mathbb{D}$ such that $\gamma(x_n) \to y \in \mathbb{D}$. Then, since the γ_n are isometries, for any neighborhood $U \Subset \mathbb{D}$, the family of maps $\gamma_n : U \to \mathbb{D}$ would be uniformly bounded and equicontinuous. Hence it would be pre-compact, contradicting discreteness.

1.22. Let $U_n \Subset U$ be an increasing sequence of domains exhausting U, and let

$$\mathrm{dist}(\phi, \psi) = \sum \frac{1}{2^n} \sup_{z \in U_n} d_s(\phi(z), \psi(z)).$$

1.23. Consider a sequence of holomorphic functions $1/\phi_n(z)$ (which are the original functions written in terms of the local chart $1/z$ near ∞ in the target Riemann sphere). Apply the Hurwitz Theorem on the stability of roots of holomorphic functions.

1.20. Push the hyperbolic metric on $\mathbb{H}$ forward to $\mathbb{D}^*$ by the universal covering map $\mathbb{H} \to \mathbb{D}^*$, $z \mapsto e^{iz}$.

1.19. (iii) An ideal quadrilateral consisting of two adjacent triangles of the tiling gives us a fundamental domain of λ. In the $\mathbb{H}$-model, we can normalize it so that it is bounded by two vertical lines $x = \pm 1$ and two half-circles $|z \pm 1/2| = 1/2$. Then the boundary identifications are given by two parabolic deck transformations $z \mapsto z + 2$ and $z \mapsto z/(2z + 1)$. They generate the group of deck transformations, on the one hand, and the group Γ_2, on the other.

1.27. Without loss of generality, we can assume that $U = \mathbb{D}$, the functions ψ do not collide in $\mathbb{D}^*$, $\psi_1 \equiv \infty$ and $\psi \equiv \psi_2$ has a pole at 0. Then the functions ϕ_n are holomorphic on $\mathbb{D}$ and form a normal family on $\mathbb{D}^*$. By Exercise 1.26, we can assume that the ϕ_n are either uniformly bounded on each $\mathbb{T}_r$, $r \in (0, 1)$, or

$$\phi_n \to \infty \quad \text{uniformly on } \mathbb{T}_r. \tag{0.7}$$

In the first case, the Maximal Principle completes the proof, so assume (0.7) occurs. If $\phi_{n(k)}(0) \neq 0$ for a subsequence $n(k)$, then by the Minimum Principle $\psi_{n(k)} \to \infty$ uniformly on $\mathbb{D}_r$, and we are done. So, we can assume that $\phi_n(0) = 0$ for all n. Then the winding number of the curve $\phi_n : \mathbb{T}_r \to \mathbb{C}^*$ around 0 is positive. But by (0.7), the curve $\phi_n - \psi : \mathbb{T}_r \to \mathbb{C}^*$ eventually has the same winding number around 0 (r should be selected so that ψ does have poles on $\mathbb{T}_r$) and hence the equation $\phi_n(z) = \psi(z)$ has a solution in $\mathbb{D}_r$.

Chapter 3

3.1. A quadratic differential $\phi \in \mathcal{Q}$ can be represented as $\phi(z)dz^2$ where $\phi(z)$ is a holomorphic function on $\bar{\mathbb{C}} \smallsetminus \mathcal{P}$. Since $\int |\phi| < \infty$, this function can have at most simple poles at finite points z_i, $i = 1, \ldots, n - 1$, and $\phi(z) = O(|z|^{-3})$ near ∞ (which is equivalent to sayng that the differential $\phi(z)dz^2$ has a simple pole at ∞). Hence

$$\phi(z) = \sum_{i=1}^{n-1} \frac{\lambda_i}{z - z_i}$$

with $\sum \lambda_i = 0$ and $\sum \lambda_i \sum_{k \neq i} z_k = 0$. These two linear conditions are independent, and in fact, $(\lambda_1, \ldots, \lambda_{n-3})$ can be selected as global coordinates on the correspondent subspace (as the the right-most minor of the corresponding $2 \times (n - 1)$ matrix is equal to $z_{n-1} - z_{n-2} \neq 1$).

3.2. Let $[S_n, \phi_n]$ converge to $[S, \phi]$ in $\mathcal{T}(S_0)$. Then one can select representatives ϕ_n and qc maps $h_n : S_n \to S$ with $\mathrm{Dil}(h_n) \to 0$ such that $h_n \circ \phi_n = \phi$. Lift these maps to $\mathbb{H}$ normalizing the Φ_n at three

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points. Then use Theorem 2.12 to show that the Φ_n converge to Φ uniformly on $\mathbb{H}$.

3.4. Let $\{g_\alpha\}$ be the projective atlas on V. Let us write f in the local parameter $z = g_\alpha(x)$ (i.e., consider the function $f_\alpha = f \circ g_\alpha^{-1}$), and let us take its Schwarzian $Sf_\alpha(z)\,dz^2$. Let $\zeta = g_\beta(x)$ be another local chart (with an overlapping domain), and let $\zeta = A_{\beta\alpha}(z)$ be the transit Möbius map. Then $f_\beta \circ A_{\beta\alpha} = f_\alpha$, and the Chain Rule (3.4) translates into the property that the quadratic differential $Sf_\alpha(z)\,dz^2$ is the pullback of $Sf_\beta(\zeta)\,d\zeta^2$ under $A_{\beta\alpha}$. This means by definition that these local expressions determine a global quadratic differential on V.

Chapter 4

4.2. (i) Consider fixed points of f and their preimages.

(ii) It is a generality about full sets: a non-trivial loop γ in int K would separate $\mathbb{C} \smallsetminus K$.

4.3. (i) A full compact set $K \subset \mathbb{C}$ is connected if and only if its boundary ∂K is connected. A less obvious part: if $J = J_1 \sqcup J_2$ with both J_i compact, then $K = K_1 \sqcup K_2$, where K_i is a *hull* of J_i (i.e., the smallest full set containing J_i; it is obtained by adding all bounded compenents of $\mathbb{C} \smallsetminus J_i$ to J_i).

(ii) It is so in both cases of the dichotomy: for any connected set containing more than one point, or for a Cantor set.

4.12. For $z \in D_f(\boldsymbol{\alpha})$, $f^n \to \boldsymbol{\alpha}$ uniformly on a neighborhood of z. Let D be the component of int $K(f)$ containing z. Then by normality of the family $\{f^n | D\}$, the $f^n \to \boldsymbol{\alpha}$ uniformly on compact subsets of D.

4.13 (i) $D^0(\alpha)$ is the component of $\{z : f^{pn}(z) \to \alpha$ as $n \to \infty\}$ containing α.

(ii) Let $P_\infty = \cup P_n$. Then $f^p(\partial P_\infty) = \partial P_\infty$ since $f^p(\partial P_n) = \partial P_{n-1}$.

4.19. Note that the foliation by round circles is defined dynamically as the closures of the equivalence classes

$$z \sim \zeta : \exists n : g^n z = g^n \zeta,$$

sometimes called "small orbits". Hence a germ ϕ commuting with g must respect this foliation. It follows that ϕ is linear (even if it mapped just one round circle onto a round circle).

Chapter 5

5.1. (iii) Recall the proof of Proposition 4.2.

(iv) It follows from the dichotomy: $\phi_n \to \infty$ locally uniformly on $\mathbb{C} \setminus M$, and $|\phi_n(z)| < 2$ on M (as in Proposition 4.5).

5.3. (Compare with Theorem **??** (i) Since the family of functions ϕ_n is not normal near $c_* \in \partial M$, one of the equations $\phi_n(c) = 0$ or $\phi_n(c) = \pm\sqrt{c}$ should have roots arbitrary close to $c_* \in \partial M$.

(ii) Consider, for instance, the β-fixed point as a function of c (it branches only at the main cusp $1/4$). Then one of the equations $\phi_n(c) = \beta(c)$ or $\phi_n(c) = \sqrt{\beta(c) - c}$ should have roots arbitrary close to $c_* \in \partial M$.

5.8. For a point $\zeta = z^2 \in \mathbb{A}' = \mathbb{A}[R^2, R^4]$, let $H_c(\zeta) = (H_c(z))^2$. This map is correctly defined (does not depend on the choice of $z = \sqrt{\zeta}$), and is a self-homeomorphism of the annulus $\mathbb{A}'$ identical on $\partial \mathbb{A}'$ and commuting with the group of rotations. Moreover, it commutes with $z \mapsto z^2$ (by definition) and depends holomorphically on c. Now extend it further to $\mathbb{A}[R^4, R^8]$, and so on.

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