Chaotic infinite-dimensional dynamics

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1. Introduction to topological dynamics

Wikipedia on **Chaos:**

"Chaos theory describes the behavior of certain nonlinear dynamical systems that may exhibit dynamics that are highly sensitive to initial conditions (popularly referred to as the butterfly effect)."

What is a dynamical system? What is chaos?

1.1. Dynamical systems

We want to study the time-evolution of the state of a system.

The states of the system are described by the elements x of a certain set X.

The evolution of the system is described by a mapping $T : X \to X$.

That is:

$x \in X$: state of the system at time 0,
Tx: state of the system at time 1,
T^2x: state of the system at time 2,...

$T^n x$ state of the system at time $n \in \mathbb{N}$

Since we want to study the behaviour of $T^n x$ as $n \to \infty$ we want a notion of 'nearness', that is, X should carry a topology.

Definition.
A *(discrete) dynamical system* is given by a topological space X and a continuous map $T : X \to X$.

As usual, we define the n-*fold iterate of* T by

$$T^n = T \circ \ldots \circ T \quad (n \text{ times})$$

with

$$T^0 = I,$$

the identity on X.

For $x \in X$ we call

$$\text{orb}(x, T) = \{x, Tx, T^2x, \ldots\} = \{T^n x : n \geq 0\}$$

the *orbit* of x under T. The point x is the called the *starting point* or *initial point*.

Example. $T : [0, \pi] \to [0, \pi], Tx = \sin x$. For any $x \in [0, \pi]$ one observes that

$$T^n x \to 0 \quad \text{as } n \to \infty.$$

Note that $x = 0$ is a fixed point of T, that is $T0 = 0$.

Example. $T : [0, 1] \to [0, 1], Tx = x^2$. Then

$$T^n x = x^{2^n}, n \geq 0.$$

Hence, for any $x \in [0, 1)$ one finds that

$$T^n x \to 0 \quad \text{as } n \to \infty,$$

while

$$T^n 1 = 1 \to 0 \quad \text{as } n \to \infty,$$

Note that $x = 0$ and $x = 1$ are fixed points of T.

Exercise. Let $T : X \to X$ be a dynamical system. Show that: if $T^n x \to y$ as $n \to \infty$ then y is a fixed point of T.

Definition. Let $T : X \to X$ be a dynamical system.
(a) A point $x \in X$ is called a *fixed point* if $Tx = x$.
(b) A point $x \in X$ is called a *periodic point* if there is some $n \in \mathbb{N}$ such that $T^n x = x$. Each such n is called a *period* of x.

Example. For the dynamical system

$$T : [-1, 1] \to [-1, 1], x \to -x,$$

every point is periodic of period 2.

1.2. A particular dynamical system

We study the dynamical system

$$T : \mathbb{C} \to \mathbb{C}, z \to z^2.$$

We have that

$$T^n z = z^{2^n}, n \geq 0.$$

Consequently:

- if $|z| < 1$ then $T^n z \to 0$,
- if $|z| > 1$ then $T^n z \to \infty$.

But what happens if $|z| = 1$? We consider the new dynamical system

$$T : \mathbb{T} \to \mathbb{T}, z \to z^2,$$

where $\mathbb{T} = \{z : |z| = 1\}$ is the unit circle.

We first search for fixed points on $\mathbb{T}$:

$$Tz = z \quad \Longleftrightarrow \quad z^2 = z \quad \Longleftrightarrow \quad z = 1.$$

In contrast, there are many periodic points. Let $n \geq 1$:

$$\begin{aligned} T^n z = z \quad &\iff \quad z^{2^n} = z \\ &\iff \quad z^{2^n - 1} = 1 \\ &\iff \quad z = \exp\left(2\pi i \frac{k}{2^n - 1}\right), k = 0, 1, \ldots, 2^n - 2. \end{aligned}$$

Proposition. *The periodic points of T form a dense set (in $\mathbb{T}$).*

What happens to the other points of $\mathbb{T}$?

In order to better understand the mapping properties of T we write

$$z = e^{2\pi i x} \quad \text{with} \quad 0 \leq x < 1.$$

Then x has a dyadic representation

$$x = \frac{x_1}{2^1} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \ldots = \sum_{k=1}^{\infty} \frac{x_k}{2^k}$$

with

$$x_k = 0 \text{ or } 1, \quad k \geq 1.$$

We then have

$$\begin{aligned} Tz &= (\exp(2\pi i x))^2 \\ &= \exp(2\pi i 2x) \\ &= \exp\left(2\pi i 2(\frac{x_1}{2^1} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \ldots)\right) \\ &= \exp\left(2\pi i (x_1 + \frac{x_2}{2^1} + \frac{x_3}{2^2} + \ldots)\right) \\ &= \exp(2\pi i x_1) \exp\left(2\pi i (\frac{x_2}{2^1} + \frac{x_3}{2^2} + \ldots)\right) \\ &= \exp\left(2\pi i (\frac{x_2}{2^1} + \frac{x_3}{2^2} + \ldots)\right). \end{aligned}$$

In other words: If

$$z = e^{2\pi i x} \quad \text{with} \quad x = \frac{x_1}{2^1} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \dots$$

then

$$Tz = e^{2\pi i y} \quad \text{with} \quad y = \frac{x_2}{2^1} + \frac{x_3}{2^2} + \frac{x_4}{2^3} + \dots :$$

T acts as shift in the dyadic representation of x:

$$(x_1, x_2, x_3, \dots) \to (x_2, x_3, x_4, \dots).$$

This gives us a new understanding of the periodic points for T.

Proposition. If

$$z = e^{2\pi i x}, \quad x = \frac{x_1}{2^1} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \dots,$$

and if the sequence $(x_1, x_2, x_3, \dots)$ is periodic (of period n) then z is periodic for T (of period n).

Proof. Applying T n-fold to z corresponds to an n-fold left-shift of the sequence $(x_1, x_2, x_3, \dots)$, which leaves this sequence unchanged. $\square$

For example, the periodic sequence

$$(1, 0, 1, 0, 1, \dots)$$

corresponds to the periodic point

$$z = e^{2\pi i \frac{2}{3}} \quad \text{of period } 2.$$

Again the question: what happens to the other points z of $\mathbb{T}$? We need only study the effect of left-shifting the representing sequence $(x_1, x_2, x_3, \ldots)$!

Proposition. If

$$z = e^{2\pi i x}, \quad x = \frac{x_1}{2^1} + \frac{x_2}{2^2} + \frac{x_3}{2^3} + \ldots,$$

and if the sequence $(x_1, x_2, x_3, \ldots)$ contains every finite 0-1-sequence, then the orbit of z under T is dense in $\mathbb{T}$.

Proof. Let $w = e^{2\pi i v} \in \mathbb{T}$,

$$v = \frac{v_1}{2^1} + \frac{v_2}{2^2} + \frac{v_3}{2^3} + \ldots$$

Let $\varepsilon > 0$. By continuity, there is some $\delta > 0$ such that

$$|y - v| < \delta \Longrightarrow \left| e^{2\pi i y} - w \right| = \left| e^{2\pi i y} - e^{2\pi i v} \right| < \varepsilon. \tag{1}$$

Fix $N \in \mathbb{N}$. By assumption, the sequence

$$(v_1, v_2, \ldots, v_N)$$

appears in $(x_1, x_2, x_3, \ldots)$; that is, there is some L (depending on N) such that our sequence has the form

$$(x_1, x_2, x_3, \ldots, x_L, v_1, v_2, \ldots, v_N, x_{L+N+1}, x_{L+N+2}, \ldots).$$

Thus, an L-fold left-shift of that sequence gives us

$$(v_1, v_2, \ldots, v_N, x_{L+N+1}, x_{L+N+2}, \ldots).$$

Consequently, we have that

$$T^L z = e^{2\pi i y}$$

$$\text{with} \quad y = \frac{v_1}{2^1} + \frac{v_2}{2^2} + \ldots + \frac{v_N}{2^N} + \frac{x_{L+N+1}}{2^{N+1}} + \frac{x_{L+N+2}}{2^{N+2}} + \ldots.$$

For this y we find that

$$\begin{aligned}
|y-v| &= \Big|\frac{v_1}{2^1}+\frac{v_2}{2^2}+\ldots+\frac{v_N}{2^N}+\frac{x_{L+N+1}}{2^{N+1}}+\frac{x_{L+N+2}}{2^{N+2}}+\ldots \\
&\qquad -\Big(\frac{v_1}{2^1}+\frac{v_2}{2^2}+\frac{v_3}{2^3}+\ldots\Big)\Big| \\
&= \Big|\frac{x_{L+N+1}}{2^{N+1}}+\frac{x_{L+N+2}}{2^{N+2}}+\ldots \\
&\qquad -\Big(\frac{v_{N+1}}{2^{N+1}}+\frac{v_{N+2}}{2^{N+2}}+\ldots\Big)\Big| \\
&= \Big|\sum_{k=N+1}^{\infty}\frac{x_{L+k}-v_k}{2^k}\Big| \\
&\le \sum_{k=N+1}^{\infty}\Big|\frac{x_{L+k}-v_k}{2^k}\Big| \\
&\le \sum_{k=N+1}^{\infty}\frac{2}{2^k} \qquad \text{(because } x_k, v_k \in \{0,1\}\text{)} \\
&= \frac{2}{2^N}.
\end{aligned}$$

For N sufficiently large, this becomes less than δ. By (1), this implies that

$$|T^L z - w| < \varepsilon$$

for N sufficiently large (recall that L depends on N).□

Clearly, there are 0-1-sequences that contain all finite 0-1-sequences, for example

$$(0,1,0,0,0,1,1,0,1,1,0,0,0,0,0,1,0,1,0,0,1,1,1,0,0,\ldots)$$

Corollary. There are points $z \in \mathbb{T}$ whose orbit under T is dense in $\mathbb{T}$. In fact, there is a dense set of such points.

Proof. Existence: clear by our previous discussion.

Now, let z have dense orbit $z, Tz, T^2z, T^3z, \dots$.
Then every T^nz also has dense orbit: the orbit of T^nz is

$$T^nz, T^{n+1}z, T^{n+2}z, \dots,$$

which is also dense in $\mathbb{T}$.$\square$

Our next aim is to show that T is chaotic. But what is chaos? As noted by Wikipedia, the mapping should exhibit the *butterfly effect*, that is, sensitive dependence on initial conditions. Here is one definition of this concept.

Definition. Let (X, d) be a metric space. A dynamical system $T : X \to X$ is said to have *sensitive dependence on initial conditions* if there exists some $\delta > 0$ such that, for every $x \in X$ and $\varepsilon > 0$, there exists some $y \in X$ with $d(x, y) < \varepsilon$ such that, for some $n \in \mathbb{N}_0$, $d(T^nx, T^ny) > \delta$.

Exercise. Show that $T : \mathbb{T} \to \mathbb{T}, z \to z^2$ has sensitive dependence on initial conditions. Show that one can take any $\delta < 2$.

Thus, $z \to z^2$ has

- a dense set of periodic points
- a point with dense orbit
- sensitive dependence on initial conditions

In short: It is *chaotic*!

1.3. Chaotic dynamical systems

The following definition is due to R. L. Devaney (1986).

Definition (chaos – preliminary version). Let (X, d) be a metric space without isolated points. A dynamical system $T : X \to X$ is said to be *chaotic (in the sense of Devaney)* if it satisfies the following conditions:

(i) T has a dense orbit;

(ii) T has a dense set of periodic points;

(iii) T has sensitive dependence on initial conditions.

However, there is a serious problem with the condition of sensitive dependence.

Example. Consider

$$T : (1, \infty) \to (1, \infty), \quad Tx = 2x.$$

Then

$$T^n x = 2^n x,$$

hence

$$|T^n x - T^n y| = 2^n |x - y| \to \infty, \quad x \neq y.$$

So, T has sensitive dependence w.r.t. the usual metric on $(1, \infty)$.
But

$$d(x, y) = |\ln x - \ln y|$$

is an equivalent metric on $(1, \infty)$.
Now

$$d(T^n x, T^n y) = |\ln(2^n) + \ln x - (\ln(2^n) + \ln y)| = d(x, y),$$

so T does not have sensitive dependence w.r.t. d.
Thus, for sensitive dependence one needs to specify the underlying metric.

NB. This example has no periodic points and no dense orbits. Thus, sensitive dependence on initial conditions alone does not suffice for chaos (cf. Wikipedia).

Fortunately, there is a way out: the other two conditions in Devaney's definition of chaos imply sensitive dependence on initial conditions!

Theorem (J. Banks, J. Brooks, G. Cairns, G. Davis, P. Stacey).
Let (X, d) be a metric space with infinitely many points and without isolated points.
If $T : X \to X$ has a dense orbit and has a dense set of periodic points then T has sensitive dependence on initial conditions.

Proof.
STEP 1. Claim: There exists $\eta > 0$ such that, for all $x \in X$ there is a periodic point p such that

$$d(x, T^n p) > \eta, \quad \text{for all } n \in \mathbb{N}_0.$$

Proof. Since X is not finite, there are two periodic points p_1, p_2 whose orbits are disjoint.
Let

$$0 < 2\eta < \inf_{m,n \in \mathbb{N}_0} d(T^m p_1, T^n p_2).$$

Let $x \in X$. Then either for $j = 1$ or for $j = 2$ we have that

$$d(x, T^n p_j) > \eta, \quad \text{for all } n \in \mathbb{N}_0.$$

□

STEP 2. Claim: T has sensitive dependence with $\delta := \eta/4 > 0$, that is

$$\forall x, \forall \varepsilon > 0 \; \exists \; y, n : d(x,y) < \varepsilon, d(T^n x, T^n y) > \eta/4.$$

Proof. Fix $x \in X$ and $\varepsilon > 0$.

By assumption there is a periodic point q:

$$d(x,q) < \min(\varepsilon, \eta/4). \tag{1}$$

Let q have period N.

By STEP 1 there is a periodic point p:

$$d(x, T^n p) > \eta, \quad \text{for all } n \in \mathbb{N}_0. \tag{2}$$

By continuity of T there is a neighbourhood V of p:

$$d(T^n p, T^n y) < \eta/4, \quad \text{for } n = 0, 1, \ldots, N \text{ and } y \in V. \tag{3}$$

T has a dense set of points with dense orbit (X has no isolated points). Hence there are $z \in X, k \in \mathbb{N}$:

$$d(x,z) < \varepsilon, \quad T^k z \in V.$$

Choose $j \in \mathbb{N}_0$ such that $k \leq jN < k + N$. By (2), (3) and (1) we have

$$\begin{aligned} d(T^{jN} q, T^{jN} z) &= d(T^{jN} q, T^{jN-k} T^k z) \\ &\geq d(x, T^{jN-k} p) - d(T^{jN-k} p, T^{jN-k} T^k z) - d(x,q) \\ &> \eta - \eta/4 - \eta/4 = \eta/2. \end{aligned}$$

Thus,

$$\text{either} \quad d(T^n x, T^{jN} q) > \eta/4 \quad \text{or} \quad d(T^n x, T^{jN} z) > \eta/4.$$

with

$$d(x, q) < \varepsilon \quad \text{and} \quad d(x, z) < \varepsilon.$$

□

Consequence:

• can drop sensitive dependence from definition of chaos
• extend the definition to arbitrary topological spaces (metric unnecessary)

Definition (chaos). Let X be a topological space without isolated points. A dynamical system $T : X \to X$ is said to be *chaotic (in the sense of Devaney)* if it satisfies the following conditions:

(i) T has a dense orbit;

(ii) T has a dense set of periodic points.

Question: how does one, in practice, determine if a given map is chaotic?

Problems:
• finding a specific point whose orbit is dense!
• finding periodic points: in general, one does not know T^n!

Two facts will help:
• easier condition implying a dense orbit $\longrightarrow$ topological transitivity
• inferring chaos of a map from chaos of another map $\longrightarrow$ topological conjugacy

1.4. Topological conjugacy

Every mathematical theory has a notion of isomorphism.

Here: when do we regard two dynamical systems

$$T : X \to X, \quad S : Y \to Y$$

as 'the same'?

There should be a homeomorphism

$$\phi : X \to Y$$

such that

$$\text{if} \quad x \in X \quad \text{corresponds to} \quad y \in Y$$
$$\text{then} \quad Tx \quad \text{corresponds to} \quad Sy,$$

i.e.

$$\phi(x) = y \Longrightarrow \phi(Tx) = Sy,$$

i.e.

$$\phi \circ T = S \circ \phi.$$

For many results it suffices to have ϕ continuous with dense range.

Definition (topological conjugacy). Let $T : X \to X$ and $S : Y \to Y$ be dynamical systems.
(a) S is *quasi-conjugate* to T if there exists a continuous map $\phi : X \to Y$ with dense range such that $S \circ \phi = \phi \circ T$, that is, the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle\phi}\downarrow & & \downarrow{\scriptstyle\phi} \\ Y & \xrightarrow{S} & Y \end{array}$$

commutes.
(b) If ϕ can be chosen to be a homeomorphism then T and S are *conjugate*.

Before we give examples, here is the reason why this notion will help us.

Proposition. Let $S : Y \to Y$ be quasi-conjugate to $T : X \to X$.
(a) If T has a point with dense orbit, the so does S.
(b) If T has a dense set of periodic points then so does S.
(c) If T is chaotic then so is S.

Proof.
(a) Let $x \in X$ have dense orbit under T. Consider $y = \phi(x) \in Y$.
From

$$\phi \circ T = S \circ \phi$$

it follows that

$$\phi \circ T^n = S^n \circ \phi.$$

Hence

$$\begin{aligned}\text{orb}(y, S) &= \{S^n y : n \geq 0\} = \{S^n \circ \phi(x) : n \geq 0\} \\ &= \{\phi \circ T^n(x) : n \geq 0\} = \phi\big(\{T^n(x) : n \geq 0\}\big) \\ &= \phi(\text{orb}((x, T)).\end{aligned}$$

Since orb(x, T) is dense and ϕ has dense range, y has dense orbit under S.
(b) similarly.
(c) from (a) and (b). $\square$

Exercise. Prove assertion (b).

In fact, *implicitly*, we have already used this result:

Let

$$\Sigma_2 = \left\{ \ (x_n)_{n\in\mathbb{N}} : x_n \in \{0, 1\} \ \right\} = \prod_{n=1}^{\infty}\{0, 1\}.$$

be the space of all 0-1-sequences, endowed with the product topology (the topology of coordinatewise convergence).

Let

$$\sigma : \Sigma_2 \to \Sigma_2, \quad \sigma(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots).$$

be the *backward shift* (in probability theory: *the Bernoulli shift*).

Then $\sigma : \Sigma_2 \to \Sigma_2$ is a dynamical system (why?).

$$\boxed{T : \mathbb{T} \to \mathbb{T}, z \to z^2 \quad \text{is quasi-conjugate to} \quad \sigma : \Sigma_2 \to \Sigma_2.}$$

In fact, let

$$\phi : \Sigma_2 \to \mathbb{T}, \quad \phi\big((x_n)_{n\geq 1}\big) = \exp\Big(2\pi i \sum_{n=1}^{\infty} \frac{x_n}{2^n}\Big).$$

Then ϕ is continuous (why?), surjective (why?) (but not injective - why?). Moreover, we have essentially already shown that

$$T \circ \phi = \phi \circ S$$

(where?).
Consequence: in order to show that T is chaotic it suffices to show that S has a dense orbit and a dense set of periodic points. This we essentially did...

Remark. An earlier example shows that sensitive dependence is **not** preserved under topological conjugacy!

Further examples.

(1) Consider

$$M : [0,1) \to [0,1), \quad x \to 2x(\text{mod}1),$$

where we identify 0 with 1. It is 'the same' as

$$T : \mathbb{T} \to \mathbb{T}, \quad z \to z^2.$$

In fact, a topological conjugacy is given by $\phi : [0,1) \to \mathbb{T}, t \to e^{2\pi i t}$.

Thus, M is chaotic.

(2) Two popular families of dynamical systems are given by

• the **logistic functions**

$$L_\mu : \mathbb{R} \to \mathbb{R}, \quad x \to \mu x(1-x).$$

• the **quadratic functions**

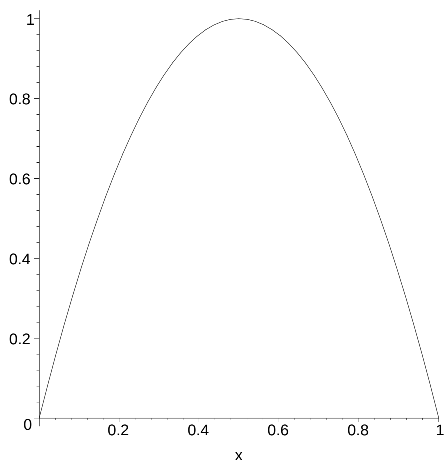
$$Q_c : \mathbb{R} \to \mathbb{R}, \quad x \to x^2 + c.$$

Exercise. Show that, any logistic system L_μ is conjugate to some quadratic system Q_c. (Hint: take ϕ of the form $ax + b$.)

(3) Of particular interest is the logistic function

$$L_4 : [0,1] \to [0,1], \quad x \to 4x(1-x),$$

restricted to $[0,1]$.

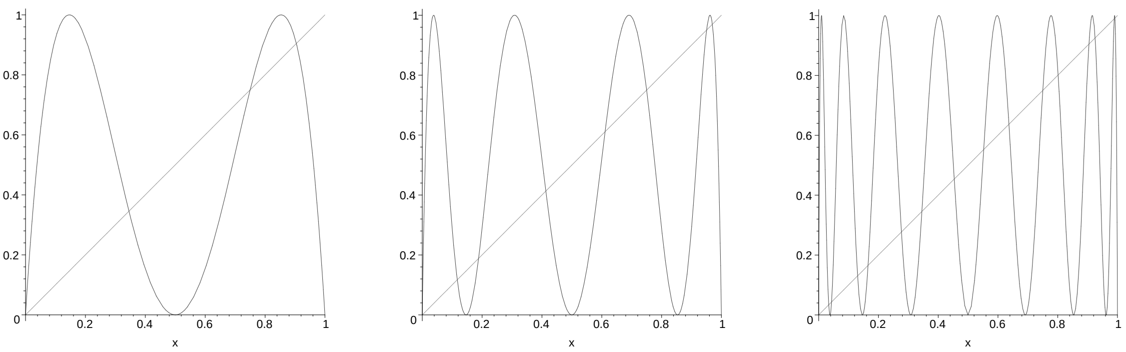


It is quasi-conjugate to $M : [0,1] \to [0,1], Mx = 2x(\text{mod}1)$ via

$$\phi(x) = \sin^2(\pi x).$$

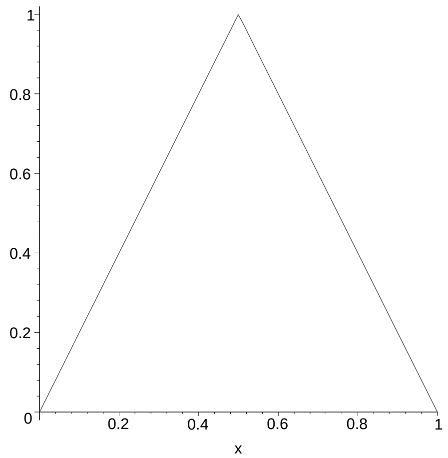
(do it!)
Thus, L_4 is chaotic.

This is not obvious from the graph of L_4.
But one gets an idea from looking at its iterates $L_4^n, n = 2, 3, 4$.
Recall that x is periodic of period n iff $L_4^n(x) = x$.



(3) Another popular dynamical system is given by the tent map

$$T_2 : [0,1] \to [0,1], \quad x \to \begin{cases} 2x & \text{if } 0 \le x \le \frac{1}{2}, \\ 2 - 2x & \text{if } \frac{1}{2} \le x \le 1. \end{cases}$$



Also the tent map is chaotic:

Exercise. Show that T_2 and L_4 are conjugate. (Hint: $\phi(x) = \sin^2(\frac{\pi}{2}x)$.)

1.5. The Birkhoff transitivity criterion

We turn to the other problem mentioned above:

Given a dynamical system, how do we see that it has a point with dense orbit? The existence of such a point is **not** obvious!

By a famous result of Birkhoff (1920), in many situations, existence of a dense orbit is equivalent to a formally much weaker condition.

Definition (topological transitivity). A dynamical system $T : X \to X$ is called *(topologically) transitive* if, for any pair U, V of non-empty open subsets of X, there exists some $n \geq 0$ such that $T^n(U) \cap V \neq \varnothing$.

Before continuing, let us note that transitivity is equivalent to a kind of *indecomposability* of the system.

Exercise. Show that transitivity is equivalent to the following:
- X cannot be written as $X = A \cup B$ with disjoint subsets A, B such that A is T-invariant and A and B have non-empty interior.

Now, if X has no isolated points, then transitivity is clearly weaker than the existence of a dense orbit:

In fact, let $x \in X$ have dense orbit under T. Let $U, V \neq \varnothing$ be open. Then

$$\exists n \geq 0 : T^n x \in U.$$

But $T^n x$ has itself a dense orbit (why? X has no isolated points!).
Thus

$$\exists k \geq 0 : T^k(T^n x) \in V,$$

hence

$$T^k(U) \cap V \neq \varnothing.$$

In many situations, the converse is also true:

Theorem (Birkhoff's transitivity criterion). Let X be a separable complete metric space X without isolated points and $T : X \to X$ a dynamical system. Then the following assertions are equivalent:

(i) T is topologically transitive;

(ii) there exists some $x \in X$ such that orb(x, T) is dense in X.

In that case the set of points in X with dense orbit is a dense G_δ-set.

Proof of (i)$\Rightarrow$(ii).

The proof is simple if we use the *Baire category theorem*.

Let (y_k) be a dense sequence in X. Then

$$\text{orb}(x,T) \quad \text{dense} \quad \Longleftrightarrow \forall k \forall m \exists n : d(T^n x, y_k) < \tfrac{1}{m},$$

i.e.

$$\begin{aligned}\mathcal{D} &:= \{x \in X : \text{orb}(x,T) \quad \text{dense}\} \\ &= \bigcap_k \bigcap_m \bigcup_n T^{-n}\big(\{y \in X : d(y, y_k) < \tfrac{1}{m}\}\big).\end{aligned}$$

By continuity of T, $\mathcal{D}$ is a countable intersection of open sets, i.e., a G_δ-set.

By the Baire category theorem, $\mathcal{D}$ is dense (in particular, non-empty) if, for all k, m,

$$\bigcup_n T^{-n}\big(\{y \in X : d(y, y_k) < \tfrac{1}{m}\}\big) \quad \text{is dense},$$

i.e.

$$\forall U \neq \varnothing \quad \text{open} \quad \exists n : T^n(U) \cap \{y \in X : d(y, y_k) < \tfrac{1}{m}\} \neq \varnothing.$$

But that is clear from transitivity. $\square$

Remark. There are mixed feelings on the Baire category theorem:

T. W. KÖRNER:
"The Baire category theorem is a profound triviality"
E. H. LIEB, M. LOSS (Preface to a book on 'Analysis'):
"Occasionally we have slick proofs, but we avoid unnecessary abstraction, such as the use of the Baire category theorem"

But the Baire category theorem has a 'constructive' proof, hence so has the Birkhoff transitivity theorem.

Example. A new proof that

$$T : \mathbb{T} \to \mathbb{T}, \quad z \to z^2$$

has a dense orbit:
Let $U \subset \mathbb{T}$ be open, non-empty.
Then U contains an arc γ of positive angle α.
The application of T to γ doubles this angle. Thus,

$$n\alpha > 2\pi \quad \Longrightarrow \mathbb{T} \subset T^n(\gamma) \subset T^n(U).$$

This is much stronger than transitivity.

We leave the following result as an exercise:

Proposition (continued...).
Let $S : Y \to Y$ be quasi-conjugate to $T : X \to X$.
(d) If T is transitive, them so is S.

Remark. It has by now been generally accepted in the literature that, in the definition of chaos for a general topological space, density of an orbit is replaced by the notion of transitivity. For this course we shall stick to our definition above.

2. Hypercyclic and chaotic operators

So far, all our dynamical systems were non-linear.

If we believe Wikipedia, this has to be so if we want to observe chaos.

But this is not so:

There are linear chaotic maps!

Setting:

Let X be a

• *Banach space*

or, more generally,

• a *topological vector space*, that is, a vector space X endowed with a topology such that the operations of addition,

$$X \times X \to X, \quad (x, y) \to x + y,$$

and scalar multiplication,

$$\mathbb{K} \times X \to X, \quad (\lambda, x) \to \lambda x,$$

($\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$) are continuous. We shall usually assume that, in fact,

• X is a *complete metric vector space*, that is, its vector space topology is induced by a complete metric d that is translation-invariant:

$$\forall x, y, z \in X \quad d(x, y) = d(x + z, y + z).$$

• Let $T : X \to X$ be a (continuous, linear) *operator*.

Then

$$T : X \to X$$

is a *linear dynamical system*.

Aim: find operators that have a dense orbit and a dense set of periodic points $\longrightarrow$ *linear chaos*.

2.1. A linear chaotic operator

Let f be an entire function, that is,

$$f : \mathbb{C} \to \mathbb{C} \quad \text{holomorphic.}$$

We consider the *operation of translation by 1*:

$$T : f(\cdot) \to f(\cdot + 1).$$

This is obviously a linear operation, defined on

$$H(\mathbb{C}) = \text{the space of entire functions.}$$

Thus we consider

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad Tf(z) = f(z+1).$$

It is a (continuous linear) operator, if we consider in $H(\mathbb{C})$ the topology of uniform convergence on compact subsets. To define a corresponding metric on $H(\mathbb{C})$, let

$$p_n(f) = \sup_{|z|\leq n} |f(z)|, \quad n \geq 1,$$

and

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(f-g)}{1+p_n(f-g)}, \quad f, g \in H(\mathbb{C}).$$

Then:

- d is a translation-invariant metric,
- $f_\nu \to f \iff \forall n \geq 1\ p_n(f - f_\nu) \to 0$
  $\iff f_\nu \to f$ uniformly on compact sets,
- d is a complete metric on $H(\mathbb{C})$.

The following simple fact will be useful:

- $p_n(f, g) < \frac{1}{2^n} \implies d(f, g) < \frac{2}{2^n}$.

Exercise. Prove these assertions.

We first show that the translation operator T has a dense orbit (Birkhoff, 1929).

For this we need a crucial result from complex approximation theory.

It is well known that every continuous function on $[0, 1]$ can be uniformly approximated by polynomials (Weierstrass' approximation theorem, 1885).

Less well known is the corresponding result for holomorphic functions.
The result is due to Runge (1885! PhD under Weierstrass 1880).

Theorem (Runge). Let K be a compact subset of $\mathbb{C}$ such that $\mathbb{C} \setminus K$ is a connected set ('K has no holes').
If f is defined and holomorphic on a neighbourhood of K, then, for any $\varepsilon > 0$, there exists a complex polynomial p such that

$$\sup_{z \in K} |f(z) - p(z)| < \varepsilon.$$

Corollary. $H(\mathbb{C})$ is separable.

Proof. By Runge, every entire function f can be uniformly approximated on any disk $|z| \leq n$ by a polynomial, hence also by a polynomial with 'rational' coefficients (from $\mathbb{Q} + i\mathbb{Q}$).

And the set is polynomial with 'rational' coefficients is countable. □

Theorem (Birkhoff). The translation operator

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad Tf(z) = f(z+1).$$

has a dense orbit.

What does this say?

Note that

$$T^n f(z) = f(z+n).$$

Thus, by Birkhoff, there is an entire function f such that

$$\{f(\cdot + n) : n \geq 0\}$$

is dense in $H(\mathbb{C})$.

In other words:

There is an entire function f such that, for any entire function g, there is a sequence (n_k) of positive integers such that

$$f(z + n_k) \to g(z) \quad \text{uniformly on compact sets.}$$

Proof. We have seen that $H(\mathbb{C})$ is a separable complete metric vector space. As such it does not have isolated points.

Thus, by the Birkhoff transitivity theorem, it suffices to show that T is transitive.

Let $U, V \neq \varnothing$ be open sets in $H(\mathbb{C})$. Let $f, g \in H(\mathbb{C})$ and $\varepsilon > 0$ be such that

$$\begin{aligned} &U_\varepsilon(f) := \{h \in H(\mathbb{C}) : d(f,h) < \varepsilon\} \subset U, \\ &U_\varepsilon(g) \subset V. \end{aligned}$$

Choose n such that

$$\frac{2}{2^n} < \varepsilon.$$

Now,

$$K := \{|z| \leq n\} \cup \{|z - 3n| \leq n\}$$

is a compact set without holes.

We define on K

$$h(z) = \begin{cases} f(z) & \text{if} \quad |z| \leq n, \\ g(z-3n) & \text{if} \quad |z-3n| \leq n. \end{cases}$$

Then h is even holomorphic on a neighbourhood of K.

By Runge, there is a polynomial P such that

$$\sup_{z \in K} |h(z) - P(z)| < \frac{1}{2^n}.$$

In particular,

$$\sup_{|z|\le n} |f(z) - P(z)| < \frac{1}{2^n},$$
$$\sup_{|z-3n|\le n} |g(z-3n) - P(z)| < \frac{1}{2^n}.$$

The second inequality implies that

$$\sup_{|z|\le n} |g(z) - P(z+3n)| < \frac{1}{2^n}.$$

In other words,

$$p_n(f, P) < \frac{1}{2^n},$$
$$p_n(g, T^{3n}P) < \frac{1}{2^n},$$

hence

$$d(f, P) < \frac{2}{2^n} < \varepsilon \quad \Longrightarrow \quad P \in U,$$
$$d(g, T^{3n}P) < \frac{2}{2^n} < \varepsilon \quad \Longrightarrow \quad T^{3n}P \in V.$$

Thus

$$T^{3n}(U) \cap V \neq \varnothing,$$

and T is transitive.$\square$

Remark. We have based the proof of Birkhoff's theorem (1929) on the Birkhoff transitivity theorem (1920).

This is not the original proof of Birkhoff.

He 'constructed' a function f with dense orbit in the form

$$f(z) = \sum_{k=0}^{\infty} P_k(z)$$

with certain polynomials P_k.

But recall that one can give a 'constructive' proof of the Birkhoff transitivity theorem. Applied this to Birkhoff's theorem essentially gives his original proof.

Remark. Nobody has yet seen a 'concrete' Birkhoff function.

But Voronin (1975) has shown that the Riemann ζ-function is almost a Birkhoff function in the right half of the critical strip:

Let g be holomorphic *and zero-free* in the disk $|z| < \frac{1}{4}$. Then there exists a sequence (n_k) of positive integers such that

$$\zeta(z + \tfrac{3}{4} + n_k i) \to g(z) \quad \text{uniformly on compact sets.}$$

What about **periodic points** for the translation operator T?

They are easy to find; for example, every 1-periodic entire function ($\sin(2\pi\cdot)$,...) is a fixed point for T.

In order to get a dense set of periodic points let's consider the functions

$$e_\lambda(z) := e^{\lambda z}.$$

Then

$$T^n e_\lambda(z) = e^{\lambda(z+n)} = e^{n\lambda} e_\lambda(z).$$

Thus, e_λ is a periodic point of T

$$\begin{aligned}
&\iff \quad \exists n \in \mathbb{N} : e^{n\lambda} = 1 \\
&\iff \quad \exists n \in \mathbb{N}, k \in \mathbb{Z} : n\lambda = 2\pi i k \\
&\iff \quad \lambda \in 2\pi i \mathbb{Q}.
\end{aligned}$$

But these are already sufficiently many to give us a dense set.

We shall need the following:

Lemma. Let $\Lambda \subset \mathbb{C}$ be a set with an accumulation point. Then the set

$$\text{span}\{e_\lambda : \lambda \in \Lambda\}$$

is dense in $H(\mathbb{C})$.

Proof. By assumption there are

$$\lambda_n \in \Lambda \quad \text{such that} \quad \lambda_n \to \lambda \quad (\lambda_n \neq \lambda, n \geq 1).$$

We write

$$e^{\lambda_n z} = e^{\lambda z} e^{(\lambda_n - \lambda) z} = e^{\lambda z} + e^{\lambda z}(\lambda_n - \lambda) z + e^{\lambda z} \frac{(\lambda_n - \lambda)^2 z^2}{2!} + \dots . \tag{1}$$

Thus,

$$e^{\lambda_n z} \to e^{\lambda z} \quad \text{uniformly on compact sets,}$$

(which is clear, anyway), hence

$$e^{\lambda z} \in \overline{\mathsf{span}}\{e^{\lambda_n z} : n \geq 1\}.$$

Next, (1) shows that

$$\frac{e^{\lambda_n z} - e^{\lambda z}}{\lambda_n - \lambda} \to z e^{\lambda z} \quad \text{uniformly on compact sets,}$$

hence

$$z e^{\lambda z} \in \overline{\mathsf{span}}\{e^{\lambda_n z} : n \geq 1\}.$$

We continue:

$$\frac{\frac{e^{\lambda_n z} - e^{\lambda z}}{\lambda_n - \lambda} - z e^{\lambda z}}{\lambda_n - \lambda} \to \frac{z^2}{2} e^{\lambda z} \quad \text{uniformly on compact sets,}$$

etc...

Thus,

$$z^k e^{\lambda z} \in \overline{\mathsf{span}}\{e^{\lambda_n z} : n \geq 1\}, \quad k \geq 0.$$

Now let $f \in H(\mathbb{C})$. Then

$$f(z) = e^{\lambda z}\big(e^{-\lambda z} f(z)\big) = e^{\lambda z}\Big(\sum_{k=0}^{\infty} a_k z^k\Big) = \sum_{k=0}^{\infty} a_k z^k e^{\lambda z} \in \overline{\mathsf{span}}\{e^{\lambda_n z} : n \geq 1\}.$$

$\square$

Corollary. The translation operator

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad Tf(z) = f(z+1).$$

has a dense set of periodic points.

Proof. We have seen that

$$e_\lambda, \quad \lambda \in 2\pi i\mathbb{Q},$$

are periodic for T.

But $2\pi i\mathbb{Q}$ has accumulation points in $\mathbb{C}$.

By the lemma, T has a dense set of periodic points. $\square$

Corollary. The translation operator

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad Tf(z) = f(z+1).$$

is a linear chaotic operator.

2.2. Hypercyclic and chaotic operators

We introduce the terminology that is used in linear dynamics.

Definition (Beauzamy 1986). Let X be a topological vector space. An operator $T : X \to X$ is said to be *hypercyclic* if it has a dense orbit, that is, if there is some $x \in X$ such that

$$\text{orb}(x,T) = \{T^n x : n \in \mathbb{N}_0\} \quad \text{is dense in } X.$$

Any such vector is called a *hypercyclic vector* for T. The set of hypercyclic vectors is denoted by

$$HC(T).$$

Five years later, Godefroy and Shapiro adopted Devaney's definition also for linear chaos.

Definition (Godefroy, Shapiro 1991). Let X be a complete metric vector space. An operator $T : X \to X$ is called *chaotic*, if

(i) T is hypercyclic;

(ii) T has a dense set of periodic points.

We have seen that chaos (in this sense) implies sensitive dependence on initial conditions. However, Godefroy and Shapiro have shown that, for linear operators, density of some orbit already suffices:

Proposition. Let $T : X \to X$ be a hypercyclic operator on a complete metric vector space.

Then T has sensitive dependence on initial conditions (with respect to any translation-invariant metric that induces the topology of X).

Proof. Let d be such a metric on X.

Let $x \in X, \varepsilon > 0$. We show that, for any $\delta > 0$, T satisfies the sensitivity condition
$$\exists\, y, n : d(x,y) < \varepsilon, d(T^n x, T^n y) > \delta.$$

Consider the open sets
$$U = \{z \in X\ ;\ d(0,z) < \varepsilon\}, \quad V = \{z \in X\ ;\ d(0,z) > \delta\}.$$

Since X has no isolated points, hypercyclicity implies transitivity.

Hence there is $n \in \mathbb{N}$ with $T^n(U) \cap V \neq \varnothing$, that is
$$\exists n \in \mathbb{N}, z \in X : \quad d(0,z) < \varepsilon,\ d(0, T^n z) > \delta.$$

Let
$$y = x + z.$$

Then
$$d(x,y) = d(x, x+z) = d(0,z) < \varepsilon \quad \text{and}$$
$$d(T^n x, T^n y) = d(T^n x, T^n(x+z))) = d(0, T^n z) > \delta.$$

$\square$.

(Linear) chaos has two ingredients: density of some orbit and density of the set of periodic points.

In the non-linear case, the second condition is difficult to verify (what is T^n?)

In the linear case we have a characterization that is often easily computable, provided the scalar field is $\mathbb{C}$.

Proposition. If X is a vector space over $\mathbb{C}$ and $T : X \to X$ is a linear mapping then the set of periodic points of T is given by

$$\begin{aligned} &\text{span}\{x \in X \ : \ Tx = \lambda x \text{ for some } \lambda \text{ such that } \lambda^n = 1 \text{ for some } n \in \mathbb{N}\} \\ &\qquad = \text{span} \bigcup_{q \in \mathbb{Q}} \text{Eig}(T, e^{2\pi i q}). \end{aligned}$$

Proof. If

$$Tx = \lambda x, \quad \lambda^n = 1,$$

then

$$T^n x = \lambda^n x = x.$$

And a linear combination of periodic points is again periodic (why?).

Conversely, suppose that

$$T^n x = x.$$

We decompose the complex polynomial $z^n - 1$ into a product of monomials

$$z^n - 1 = (z - \lambda_1)(z - \lambda_2) \dots (z - \lambda_n).$$

Observe that the roots λ_i, $i = 1, \dots, n$, are all different.

Substituting z by T gives

$$T^n - I = (T - \lambda_1 I)(T - \lambda_2 I) \dots (T - \lambda_n I).$$

Now, the complex polynomials

$$p_k(z) := \prod_{j \neq k} (z - \lambda_j), \quad k = 1, \dots, n$$

form a basis of the space of complex polynomials of degree strictly less than n (why?).

In particular there are $\alpha_k \in \mathbb{C}$, $k = 1, \ldots, n,$ such that

$$1 = \sum_{k=1}^{n} \alpha_k p_k(z),$$

hence

$$I = \sum_{k=1}^{n} \alpha_k p_k(T).$$

Define

$$y_k = p_k(T)x.$$

Then

$$(T - \lambda_k)y_k = (T - \lambda_k)p_k(T)x = \prod_{j=1}^{n}(T - \lambda_j I)x = (T^n - I)x = 0,$$

so that

$$x = Ix = \sum_{k=1}^{n} \alpha_k p_k(T)x = \sum_{k=1}^{n} \alpha_k y_k \in$$

span$\{x \in X \ : \ Tx = \lambda x$ for some λ such that $\lambda^n = 1$ for some $n \in \mathbb{N}\}$.

□

This shows: at least in the complex case, finding periodic points amounts to finding eigenvectors to suitable eigenvalues.

This is also a first hint that eigenvectors play an important role in linear dynamics...

Example. We consider the *backward shift*

$$B : \ell^2 \to \ell^2, \quad (x_1, x_2, x_3, \dots) \to (x_2, x_3, x_4, \dots)$$

on the complex Hilbert space

$$\ell^2 = \Big\{ (x_k)_{k\geq 1} : \sum_{k=1}^{\infty} |x_k|^2 < \infty \Big\},$$

endowed with the norm

$$\|x\|_2 := \Big(\sum_{k=1}^{\infty} |x_k|^2 \Big)^{1/2}.$$

Let T be a multiple of B:

$$T = \mu B, \mu \neq 0.$$

The eigenvectors are easy to determine: $Tx = \lambda x$ amounts to

$$\mu(x_2, x_3, x_4, \dots) = \lambda(x_1, x_2, x_3, \dots),$$

hence

$$x_{k+1} = \frac{\lambda}{\mu} x_k, \quad k \geq 1.$$

This recursion has the solution

$$x_k = \Big(\frac{\lambda}{\mu}\Big)^{k-1} x_1,$$

hence

$$x = x_1 \Big(1, \frac{\lambda}{\mu}, \Big(\frac{\lambda}{\mu}\Big)^2, \Big(\frac{\lambda}{\mu}\Big)^3, \dots \Big).$$

This belongs to ℓ^2 if and only if

$$|\lambda| < |\mu|.$$

This shows that there are **no periodic points** if $|\mu| \leq 1$, because in that case there are no eigenvectors of modulus 1.

Conversely, for $|\mu| > 1$, the **set of periodic points of T is dense** in ℓ^2.

By the previous proposition one has to show that

$$\text{span}\Big\{\Big(1, \frac{\lambda}{\mu}, \Big(\frac{\lambda}{\mu}\Big)^2, \dots\Big) : \lambda^n = 1 \quad \text{for some } n \in \mathbb{N}\Big\}$$

is dense in ℓ^2.

It suffices to show that its orthogonal complement is $\{0\}$.

Thus we have to show: if $y \in \ell^2$ such that

$$\langle x, y\rangle = \sum_{k=1}^{\infty} \Big(\frac{\lambda}{\mu}\Big)^{k-1} \overline{y_k} = 0 \quad \text{whenever} \quad \lambda^n = 1 \quad \text{for some } n \in \mathbb{N}$$

then $y = 0$.

But

$$\phi(z) = \sum_{k=1}^{\infty} \Big(\frac{z}{\mu}\Big)^{k-1} \overline{y_k}$$

defines a function that is holomorphic in $|z| < |\mu|$.

By our assumption it vanishes on the points $e^{2\pi i q}, q \in \mathbb{Q}$, of the unit circle. Since these points have an accumulation point, the identity theorem for holomorphic functions tells us that $\phi = 0$, hence

$$y_k = 0, \quad k \geq 1,$$

that is

$$y = 0.$$

We summarize:

Proposition. The operator

$$T = \mu B : \ell^2 \to \ell^2, (x_1, x_2, x_3, \ldots) \to \mu(x_2, x_3, x_4, \ldots)$$

has
- no periodic points if $|\mu| \leq 1$,
- a dense set of periodic points if $|\mu| > 1$.

Exercise. Show that the result remains true in the Banach sequence spaces $\ell^p, 1 \leq p < \infty$, of p-summable sequences and c_0 of convergent-to-0 sequences. (*Hint:* use a Hahn-Banach argument.)

Remark. We shall see later that, for $|\mu| > 1$, these maps are even chaotic.

2.3. Linear vs. non-linear and finite vs. infinite

We have seen that chaos can also occur for linear maps.

Why is this fact not generally known?

Possibly because it can only arise in infinite-dimensional systems.

This fact can be proved in various ways. A short proof is based on an eigenvalue investigation:

hypercyclicity prevents eigenvalues of the adjoint.

Let $T : X \to X$ be an operator on a metric vector space X.

The *dual* X^* of X is the space of continuous linear functionals on X.

The *adjoint* $T^* : X^* \to X^*$ of T is defined as follows:

for $y^* \in X^*$ let $T^*y^* \in X^*$ be such that $\quad (T^*y^*)(x) = y^*(Tx).$

Proposition. Let $T : X \to X$ be a hypercyclic operator. Then the adjoint T^* has no eigenvalue.

Proof. Suppose, on the contrary, that

$$T^*y^* = \lambda y^*, \quad y^* \neq 0,$$

and that T has a hypercyclic vector x.

Then

$$y^*(T^n x) = ((T^*)^n y^*)(x) = \lambda^n \, y^*(x), \quad n \geq 1.$$

But this is not possible:

• the left hand side is dense in $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ as n varies over $\mathbb{N}$, because $y^* \neq 0$ and x is hypercyclic;

• the right hand side is of the form $\lambda^n c$ with a constant $c \in \mathbb{K}$. But this is never dense in $\mathbb{K} = \mathbb{R}$.

Contradiction. □

Corollary. No linear operator on $\mathbb{C}^N$ is hypercyclic. A fortiori, no linear operator on $\mathbb{C}^N$ is chaotic.

Proof. A linear operator on $\mathbb{C}^N$ is given by an $N \times N$-matrix.

Its adjoint is given by the transposed matrix.

But every complex matrix possesses an eigenvalue. □

With a little more effort, the same result can be proved for $\mathbb{R}^N$.

Exercise. Let $T : \mathbb{K}^N \to \mathbb{K}^N$ be linear, where $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.
(a) Show that, if T has a dense set of periodic points then there is some $n \geq 1$ such that $T^n = I$, the identity operator. (Hint: First deduce that $\mathbb{K}^N$ has a basis consisting of periodic points. Then show that there is some $n \geq 1$ such that each basis vector has period n.)
(b) Deduce from (a) a new proof that no linear operator on $\mathbb{K}^N$ can be chaotic.

Remark. Since every metric vector space of finite dimension is isomorphic to some $\mathbb{R}^N$ or $\mathbb{C}^N$, no such space supports a hypercyclic operator (and even less so a chaotic operator).

As a corollary one obtains an interesting result on dense orbits in any space.

Corollary. Let X be a metric vector space, $T : X \to X$ an operator. If $x \in X$ is a hypercyclic vector then its orbit is linearly independent.

Exercise. Prove this result. (Hint: Show that, if the orbit is not linearly independent, then it lies in a finite-dimensional T-invariant subspace Y of X. But then T would be hypercyclic on Y.)

As to the comparison of non-linear with non-linear chaos we want to address one more question:

Does the linearity of a map make its dynamical structure much simpler?

The answer is: NO!

Theorem (Feldman 2001).
Let $f : K \to K$ be any (non-linear) dynamical system on a compact metric space K.

Then there exists an operator T on a separable Hilbert space H and a T-invariant compact subset $L \subset H$ such the systems $f : K \to K$ and $T|_L : L \to L$ are topologically conjugate.

In fact, one can take the *same* operator T (with, of course, different L) for any map $f : K \to K$.

The 'universal' operator T can be taken essentially as a multiple of a backward shift:

Let

$$H = \ell^2(\ell^2)$$

be the space of sequences

$$(x_n) \quad \text{such that each } x_n, n \geq 1, \text{ belongs to } \ell^2$$

such that

$$\|x\| := \Big(\sum_{n=1}^{\infty} \|x_n\|_{\ell^2}^2 \Big)^{1/2} < \infty.$$

Let B be the *backward shift operator*

$$B : \ell^2(\ell^2) \to \ell^2(\ell^2), \quad B(x_1, x_2, x_3, \ldots) = (x_2, x_3, x_4, \ldots),$$

and we define T by

$$T = 2B.$$

This operator works.

Proof.
STEP 1: Define a suitable embedding $\phi : K \to \ell^2(\ell^2)$.

First we can assume that the metric is bounded by 1 (else, replace d by the equivalent metric $d'(x,y) = \frac{d(x,y)}{1+d(x,y)}$).

As a compact metric space, K is separable: Let (y_k) be a dense sequence.

Then, define for $x \in K$

$$\phi(x) = \left(\left(\frac{1}{2^{k+n}} d(y_k, f^n(x)) \right)_k \right)_n \in \ell^2(\ell^2).$$

Then $\phi : K \to \ell^2(\ell^2)$ is continuous, for

$$\begin{aligned} \|\phi(x) - \phi(y)\|^2 &= \sum_{k,n} \frac{1}{2^{2(k+n)}} |d(y_k, f^n(x)) - d(y_k, f^n(y))|^2 \\ &\leq \sum_{k,n \leq N} \frac{1}{2^{2(k+n)}} |d(y_k, f^n(x)) - d(y_k, f^n(y))|^2 \\ &\qquad + \sum_{k>N \text{ or } n>N} \frac{4}{2^{2(k+n)}} \end{aligned}$$

can be made arbitrarily small by first choosing N large, then y close to x by continuity of f.

Also, ϕ is injective, for $\phi(x) = \phi(y)$ implies that

$$\frac{1}{2^k} d(y_k, x) = \frac{1}{2^k} d(y_k, y) \quad \text{for all } k \geq 1,$$

hence $x = y$ by density of the y_k.

As a continuous injection on a compact space, ϕ is a homeomorphism onto a compact subset, L say, of $\ell^2(\ell^2)$.

STEP 2: We show that $\phi \circ f = (2B) \circ \phi$. In fact,

$$\begin{aligned}\phi(f(x)) &= \left(\left(\frac{1}{2^{k+n}} d(y_k, f^{n+1}(x))\right)_k\right)_n \\ &= 2\left(\left(\frac{1}{2^{k+n+1}} d(y_k, f^{n+1}(x))\right)_k\right)_n = (2B)(\phi(x)),\end{aligned}$$

which, in addition, also shows that

$$(2B)(L) = (2B)(\phi(K)) = \phi(f(K)) \subset \phi(K) = L,$$

so that

$$2B : L \to L.$$

$\square$

In summary we have found that

> Linear chaos exists;
> Linear chaos is an infinite-dimensional phenomenon;
> Linear dynamics can be as complicated as non-linear dynamics.

2.4. The set of hypercyclic vectors

Let

$$T : X \to X$$

be a hypercyclic operator on a complete metric vector space.

By Birkhoff's transitivity theorem, the set

$$HC(T)$$

of hypercyclic vectors is always a dense G_δ-set.

This has a simple consequence:

Proposition.
Let $T : X \to X$ be a hypercyclic operator.

Then every vector in X is the sum of two hypercyclic vectors, that is,

$$X = HC(T) + HC(T).$$

Proof. Take $x \in X$. Then

$$HC(T) \quad \text{and} \quad x - HC(T) \quad \text{are dense } G_\delta\text{-sets.}$$

By the Baire category theorem, also

$$HC(T) \cap \big(x - HC(T)\big)$$

is a dense G_δ-set, in particular non-empty.

Thus there are

$$y, z \in HC(T)$$

such that

$$y = x - z,$$

hence

$$x = y + z.$$

□

As a consequence,

$$HC(T) \cup \{0\}$$

is **never a linear space, unless**

$$X = HC(T) \cup \{0\},$$

that is, unless every non-zero vector is hypercyclic.

The latter is extremely rare, but it can happen:

Theorem (Read, 1988).
There is an operator on ℓ^1 for which every non-zero vector is hypercyclic.

What makes this result difficult is that we want the operator to be defined on a complete space. Without this requirement there are plenty of examples, as we shall see below.

Thus,

$$HC(T) \cup \{0\}$$

is rarely a linear space.

But it **always** contains a dense vector subspace!

Theorem (Bourdon). Any hypercyclic operator on a complete metric vector space admits a dense linear subspace of vectors consisting of, except for zero, hypercyclic vectors.

Proof. Let x be a hypercyclic vector for $T : X \to X$.

We consider

$$Y = \text{span orb}(x, T).$$

Then any $y \in Y$ has the form

$$y = \sum_{k=0}^{n} a_k T^k x$$

with coefficients $a_k \in \mathbb{K}$, hence

$$y = P(T)x \quad \text{with } P(T) = \sum_{k=0}^{n} a_k T^k.$$

Let $y \neq 0$. Then $P(T) \neq 0$.

Suppose that we can show that

$$P(T) : X \to X$$

has dense range.

Then

$$T^n y = T^n P(T)x = P(T)(T^n x)$$

is dense in X (as n varies over $\mathbb{N}$) because x is hypercyclic.

Thus y is hypercyclic, so that

$$Y \setminus \{0\} \subset HC(T).$$

And Y is dense because it contains

$$\text{orb}(x, T).$$

□

To finish the proof it remains to show the following.

Lemma. If T is a hypercyclic operator then, for any non-zero polynomial P, $P(T)$ has dense range.

Proof. We give the proof only in the case of a complex normed space.

In the complex case, P decomposes into linear factors

$$P(z) = c(z - \lambda_1) \dots (z - \lambda_n),$$

hence

$$P(T) = c(T - \lambda_1 I) \dots (T - \lambda_n I).$$

Thus it suffices to show that each operator

$$T - \lambda I, \quad \lambda \in \mathbb{C},$$

has dense range.

This is an application of the Hahn-Banach theorem:

If

$$\text{ran}(T - \lambda I) = \{Tx - \lambda x : x \in X\}$$

was not dense, there would exist a continuous linear functional $y^* \neq 0$ on X that vanishes on Y, that is

$$y^*(Tx - \lambda x) = 0, \quad \text{for all } x \in X,$$

hence

$$(T^* y^*)x = y^*(Tx) = \lambda y^*(x) \quad \text{for all } x \in X,$$

hence

$$T^* y^* = \lambda y^*,$$

so that y^* would be an eigenvector for the adjoint T^*.

But we have already seen that this is impossible. $\square$

Remark. (a) Let $T : X \to X$ be a linear operator on a complex Banach space (say). Suppose that $x \in X$ is hypercyclic for T. Let

$$Y = \text{span orb}(x, T).$$

Since Y is T-invariant, we can define the operator

$$T|_Y : Y \to Y.$$

The proof of Bourdon's theorem implies that each non-zero vector in Y is hypercyclic for $T|_Y$.

This gives us plenty of examples of operators for which every non-zero vector is hypercyclic. What makes Read's theorem special is that his underlying space is complete.

(b) We remark that Bourdon's theorem, in fact, holds in full generality, that is, for all hypercyclic operators on all topological vector spaces.

3. The Hypercyclicity Criterion

Given an operator

$$T : X \to X,$$

how do we see that it is hypercyclic?

How do we see that it is chaotic?

As we have proved in the previous section, finding periodic points amounts to finding eigenvectors to certain eigenvalues.

Thus, showing that an operator has a dense set of periodic points is a well-defined and, often, easily feasible task.

We can therefore concentrate on the first question:

When is an operator hypercyclic?

By Birkhoff's transitivity theorem, it suffices to show that the operator is topologically transitive.

But even that can, at times, be difficult.

Thus, we are looking for new, possibly only sufficient, but easily applicable hypercyclicity criteria.

Here is one:

3.1. The Kitai-Gethner-Shapiro Criterion

The following was obtained in Carol Kitai's PhD thesis (1982) and, independently, by Gethner and Shapiro (1987).

Theorem (Kitai-Gethner-Shapiro). Let $T : X \to X$ be an operator on a complete metric vector space.
Suppose that there are dense subsets $Y_0, Y_1 \subset X$, and a map $S : Y_1 \to Y_1$, such that

(i) $T^n x \to 0$ for each $x \in Y_0$,

(ii) $S^n y \to 0$ for each $y \in Y_1$, and

(iii) $TSy = y$ for each $y \in Y_1$,

then T is hypercyclic.

Proof. By the Birkhoff transitivity theorem, we need only show that T is transitive.

Thus, let

$$U, V \neq \varnothing \quad \text{be open subsets of } X.$$

By density of Y_0 and Y_1 one can find points

$$y_0 \in Y_0 \cap U, \quad y_1 \in Y_1 \cap V.$$

Thus, using (iii)

$$T^n(y_0 + S^n y_1) = T^n y_0 + T^n S^n y_1 = T^n y_0 + y_1.$$

Now, by (i) and (ii),

$$y_0 + S^n y_1 \to y_0 \quad \text{and} \quad T^n y_0 + y_1 \to y_1.$$

Thus, if n is sufficiently large,

$$y_0 + S^n y_1 \in U \quad \text{and} \quad T^n y_0 + y_1 \in V,$$

hence

$$T^n(U) \cap V \neq \varnothing.$$

□

We stress that the mapping S need not have any structure: it need not be linear, nor continuous.

In spite of its slight appearance of technicality, this criterion tells us what to do:

- find a dense set on which the orbit of T tends to 0;
- find a dense set on which T has a right inverse S, and such that
- the orbit of S on that set tends to 0.

Let's consider some applications.

We have already seen that the multiple of the backward shift operator,

$$T : \ell^p \to \ell^p, (x_1, x_2, x_3, \ldots) \to \mu(x_2, x_3, x_4, \ldots), \quad 1 \leq p < \infty,$$

has

- a dense set of periodic points if $|\mu| > 1$;
- no periodic point if $|\mu| \leq 1$.

How about hypercyclicity?

If $|\mu| \leq 1$ then, for all $x \in \ell^p$,

$$\|Tx\| = |\mu| \|(x_2, x_3, x_4, \ldots)\| \leq \|x\|,$$

hence

$$\|T^n x\| \leq \|x\|, \quad n \geq 1.$$

Thus, every orbit is bounded, so that T **cannot be hypercyclic if** $|\mu| \leq 1$.

In all other cases, T is hypercyclic, as was shown by Rolewicz in 1969:

Theorem (Rolewicz). Let $1 \leq p < \infty$ and $|\mu| > 1$. Then the operator

$$T : \ell^p \to \ell^p, \quad (x_1, x_2, x_3, \ldots) \to \mu(x_2, x_3, x_4, \ldots),$$

is hypercyclic.

Proof. We follow the Kitai-Gethner-Shapiro program:

It is easy to find a dense set on which the orbit of T tends to 0:

The 'finite' sequences

$$x = (x_1, x_2, \ldots, x_N, 0, 0, \ldots)$$

form a dense set in ℓ^p, and we have

$$T^n x = 0 \quad \text{for } n \geq N,$$

hence

$$T^n x \to 0, \quad n \to \infty.$$

Next, T even has a right inverse on the whole space:

$$S : (x_1, x_2, x_3, \ldots) \to \frac{1}{\mu}(0, x_1, x_2, x_3, \ldots)$$

(note that T is not invertible because it is not injective).
Finally,

$$S^n x = \frac{1}{\mu^n}(0, 0, \ldots, 0, x_1, x_2, x_3, \ldots) \quad (n \text{ zeros}),$$

hence

$$\|S^n x\| = \frac{1}{|\mu|^n}\|x\| \to 0,$$

even for all $x \in \ell^p$.
Altogether, T is hypercyclic. $\square$

Corollary. The operators

$$T : \ell^p \to \ell^p, \quad (x_1, x_2, x_3, \ldots) \to \mu(x_2, x_3, x_4, \ldots), \quad |\mu| > 1,$$

are chaotic.

We turn to another classical operator, the differentiation operator

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad f \to f'$$

on the space of entire functions.

In 1952, MacLane showed that this operator is hypercyclic. In fact, he used a construction to find an entire function with dense orbit. We shall use here our criterion.

Theorem (MacLane). The operator

$$D : H(\mathbb{C}) \to H(\mathbb{C}), \quad f \to f'$$

is hypercyclic.

Proof. Again, a dense set of functions with orbits tending to zero is easily found.

By Runge's theorem, the polynomials form a dense set in $H(\mathbb{C})$.

But if P is a polynomial of degree N, then

$$D^n P = 0 \quad \text{for } n \geq N + 1,$$

hence

$$D^n P \to 0 \quad \text{as } n \to \infty.$$

A right inverse, even on all of $H(\mathbb{C})$, is also obvious:

$$S : H(\mathbb{C}) \to H(\mathbb{C}), \quad (Sf)(z) = \int_0^z f(w)dw.$$

And the orbits of any polynomial under S tend to zero. By linearity of S it suffices to show this for all monomials

$$z^N.$$

We have,

$$S^n(z^N) = N!\frac{z^{N+n}}{(N+n)!}.$$

Since, for all $R > 0$,

$$\sup_{|z|\leq R}\left|\frac{z^{N+n}}{(N+n)!}\right| = \frac{R^{N+n}}{(N+n)!} \to 0$$

the orbit of z^N under S tends to 0 in $H(\mathbb{C})$.$\square$

But differentiation is even chaotic:

Theorem. The operator

$$D : H(\mathbb{C}) \to H(\mathbb{C}), \quad f \to f'$$

is chaotic.

Proof. Eigenvectors of D are easy to come by. We had earlier defined

$$e_\lambda(z) = e^{\lambda z}.$$

Then

$$De_\lambda = \lambda e_\lambda.$$

By a previous result, the set of periodic points is exactly the span of the eigenvectors to eigenvalues of the form

$$\lambda = e^{2\pi i q}, \quad q \in \mathbb{Q}.$$

For D, this is therefore

$$\text{span}\{e_\lambda : \lambda = e^{2\pi i q}, \quad q \in \mathbb{Q}\}.$$

Since the points $e^{2\pi i q}, q \in \mathbb{Q}$, have an accumulation point, this span is dense (see Section 2.1). $\square$

One may wonder if Birkhoff's result on the hypercyclicity of the translation operator

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad (Tf)(z) = f(z+1)$$

can also be proved using the Kitai-Gethner-Shapiro criterion.

The answer is: yes, but the dense sets Y_0 and Y_1 are less obvious now.

While one clearly chooses as right inverse the translation to the right

$$S : H(\mathbb{C}) \to H(\mathbb{C}), \quad (Sf)(z) = f(z-1),$$

for the subspaces one can choose

$$Y_0 = Y_1 = \mathsf{span}\{f_{m,k} : k \geq 1, m \geq 0\},$$

where

$$f_{m,k}(z) = z^m \Big(\frac{\sin(z/k)}{z/k}\Big)^{m+1}.$$

Exercise. Show that, with this choice, the Kitai-Gethner-Shapiro criterion is satisfied. (Hint: for the density of $Y_0 = Y_1$ let $k \to \infty$, and use the fact that the polynomials are dense in $H(\mathbb{C})$.)

Thus it appears that, for Birkhoff's operator, our original proof using the Birkhoff transitivity theorem and Runge's theorem was simpler and more natural.

Yet another proof will be given in Section 3.3.

Remark. As is to be expected, not every hypercyclic operator satisfies the conditions of the Kitai-Gethner-Shapiro criterion. But it requires a little work to come up with a counter-example.

Thus it is of interest to weaken the conditions.

3.2. The Hypercyclicity Criterion

A weakening of the Kitai-Gethner-Shapiro criterion is easily available when we study its proof carefully.

The analysis shows that it suffices that

• the conditions hold only for a subsequence (n_k) instead of the full sequence,

• the iterates $S^n : Y_1 \to Y_1$ of a single map can be replaced by an arbitrary sequence $S_n : Y_1 \to X$,

• the right inverse condition '$TS = I$ on Y_1' can be replaced by an asymptotic condition '$T^n S_n \to I$ on Y_1'.

Altogether we obtain the famous Hypercyclicity Criterion due to Bès and Peris (1999):

Theorem (Hypercyclicity Criterion). Let $T : X \to X$ be an operator on a complete metric vector space.
Suppose that there are dense subsets $Y_0, Y_1 \subset X$, an increasing sequence (n_k) of positive integers and mappings $S_n : Y_1 \to X, n \geq 1$, such that

(i) $T^{n_k}x \to 0$ for each $x \in Y_0$,

(ii) $S_{n_k}y \to 0$ for each $y \in Y_1$, and

(iii) $T^{n_k}S_{n_k}y \to y$ for each $y \in Y_1$,

then T is hypercyclic.

Exercise. Prove the Hypercyclicity Criterion.

But now the conditions for hypercyclicity have become so weak that it is no longer clear if they are not also necessary for hypercyclicity. In fact, the following problem arose:

Problem (Bès/Peris, León/Montes, 1999.) Let X be a complete metric vector space (a Banach space, a Hilbert space). Does every hypercyclic operator on X satisfy the Hypercyclicity Criterion?

Over the last decade, a lot of effort has been put into answering that question.

At the outset, Bès and Peris (1999) realized a highly unexpected connection with another well-known but apparently unrelated problem in hypercyclicity.

To understand this question we need some preparation.

To any operator

$$T : X \to X$$

one can associate its direct sum

$$T \oplus T : X \times X \to X \times X, \quad (x, y) \to (Tx, Ty),$$

where $X \times X$ carries the product topology. For example, if X carries the metric d then the topology of $X \times X$ is induced by

$$\widetilde{d}((x_1, x_2), (\xi_1, \xi_2)) = d(x_1, \xi_1) + d(x_2, \xi_2).$$

Now, periodic points carry over immediately from T to $T \oplus T$, as is easy to see.

Proposition.

(a) If x and y are periodic for T then (x, y) is periodic for $T \oplus T$.

(b) If T has a dense set of periodic points then so does $T \oplus T$.

But what about hypercyclicity? This question was first posed by Herrero.

Problem (Herrero, 1992). Let X be a complete metric vector space (a Banach space, a Hilbert space). If $T : X \to X$ is hypercyclic, is then also $T \oplus T$ hypercyclic?

What does that question ask for exactly?

Suppose that x is a hypercyclic vector for T, that is,

> for every $y \in X, \varepsilon > 0$, there is some $n \in \mathbb{N}_0$ such that $d(y, T^n x) < \varepsilon$.

Do then exist two (necessarily T-hypercyclic) vectors x_1, x_2 such that

> for every $y_1, y_2 \in X, \varepsilon > 0$, there is some $n \in \mathbb{N}_0$ such that $d(y_1, T^n x_1) < \varepsilon$ and $d(y_2, T^n x_2) < \varepsilon$.

The answer would be 'yes' if we allowed different exponents n_1, n_2.

Note also that the choice $x_1 = x_2 = x$ does **not** work. In a way, the two T-hypercyclic vectors x_1 and x_2 need to be "very independent".

Looking at it from this angle there seems to be no reason why Herrero's question should have a positive answer.

On the other hand, if T satisfies the Hypercyclicity Criterion, then the answer is affirmative! This follows from the simple fact that also $T \oplus T$ satisfies the Hypercyclicity Criterion and that $X \times X$ is a also a complete metric vector space.

Exercise. Show that $T \oplus T$ satisfies the Hypercyclicity Criterion if T does.

Now, what Bès and Peris (1999) have found is that Herrero's question only has a positive answer if every hypercyclic operator satisfies the Hypercyclicity Criterion. More precisely:

Theorem (Bès, Peris). Let X be a complete metric vector space, and $T: X \to X$ an operator. Then the following assertions are equivalent.

(a) T satisfies the Hypercyclicity Criterion.

(b) $T \oplus T$ is hypercyclic.

Proof. It remains to show that (b)$\Longrightarrow$(a).

We introduce an intermediate condition:

(c) There exists a hypercyclic vector x_0 for T, a sequence $(x_k)_{k\geq 1}$ in X with $x_k \to 0$ and an increasing sequence (n_k) of positive integers such that

$$T^{n_k}x_0 \to 0 \quad \text{and} \quad T^{n_k}x_k \to x_0.$$

We shall show that (b)$\Longrightarrow$(c)$\Longrightarrow$(a).

(b)$\Longrightarrow$(c). Let (x_0, y_0) be hypercyclic for $T \oplus T$.

Then x_0 is necessarily hypercyclic for T.

We set $x_k = \frac{y_0}{k}, k \geq 1$. Then, $x_k \to 0$.

Also, for any $k \in \mathbb{N}$, (x_0, x_k) is hypercyclic for $T \oplus T$ (why?).

Hence there are positive integers n_k such that

$$d(T^{n_k}x_0, 0) < \frac{1}{k} \quad \text{and} \quad d(T^{n_k}x_k, x_0) < \frac{1}{k}.$$

And we can choose the n_k increasing.

(c)$\Longrightarrow$(a). We set

$$Y_0 = Y_1 := \{T^n x_0 : n \in \mathbb{N}\}.$$

By hypercyclicity of x_0, $Y_0 = Y_1$ is dense.

We define the mappings

$$S_{n_k} : Y_1 \to X, \quad S_{n_k}(T^n x_0) = T^n x_k.$$

(This is well-defined because the $T^n x_0$ are pairwise distinct.)

It remains to show that conditions (i), (ii) and (iii) of the Hypercyclicity Criterion are satisfied.

(i) For any $n \geq 1$,

$$T^{n_k}(T^n x_0) = T^n(T^{n_k} x_0) \to 0, \quad k \to \infty,$$

because $T^{n_k} x_0 \to 0$.

(ii) For any $n \geq 1$,

$$S_{n_k}(T^n x_0) = T^n x_k \to 0, \quad k \to \infty,$$

because $x_k \to 0$.

(iii) For any $n \geq 1$,

$$T^{n_k} S_{n_k}(T^n x_0) = T^{n_k}(T^n x_k) = T^n(T^{n_k} x_k) \to T^n x_0, \quad k \to \infty,$$

because $T^{n_k} x_k \to x_0$.$\square$

The Hypercyclicity Criterion is therefore equivalent to the fact that $T \oplus T$ is transitive, that is,

$$\forall U_1, V_1, U_2, V_2 \neq \varnothing \quad \text{open in } X \quad \exists n \in \mathbb{N} :$$

$$T^n(U_1) \cap V_1 \neq \varnothing, \quad T^n(U_2) \cap V_2 \neq \varnothing.$$

The latter condition can be considerably weakened.

Theorem (Bernal, GE). Let X be a complete metric vector space, and $T : X \to X$ an operator. Then T satisfies the Hypercyclicity Criterion if and only if the following condition (GS) is satisfied

$$\forall U, V, W \neq \varnothing \quad \text{open in } X \text{ with } 0 \in W \quad \exists n \in \mathbb{N} :$$

$$T^n(U) \cap W \neq \varnothing, \quad T^n(W) \cap V \neq \varnothing.$$

Note. This conditions appeared first in a paper by Godefroy and Shapiro, who showed that it implies hypercyclicity.

Proof. By the remark preceding the theorem it suffices to prove sufficiency of condition (GS).

STEP 1. The condition implies that T is transitive, and hence hypercyclic.

For, if $U', V' \neq \varnothing$ are open sets, they contain certain 2ε-neighbourhoods

$$U' \supset U_{2\varepsilon}(x'), \quad V' \supset U_{2\varepsilon}(y').$$

Applying (GS) to $U_\varepsilon(x'), U_\varepsilon(y'), U_\varepsilon(0)$ we find x_1, x_2 and some $n \in \mathbb{N}$ with

$$\begin{aligned} d(x_1, x') < \varepsilon, &\quad d(T^n x_1, 0) < \varepsilon, \\ d(x_2, 0) < \varepsilon, &\quad d(T^n x_2, y') < \varepsilon, \end{aligned}$$

hence

$$\begin{aligned} d(x_1 + x_2, x') &\leq d(x_1 + x_2, x_1) + d(x_1, x') = d(x_2, 0) + d(x_1, x') < 2\varepsilon, \\ d(T^n(x_1 + x_2), y') &\leq d(T^n(x_1 + x_2), T^n x_2) + d(T^n x_2, y') \\ &= d(T^n x_1, 0) + d(T^n x_2, y') < 2\varepsilon, \end{aligned}$$

which implies that

$$x_1 + x_2 \in U', \quad T^n(x_1 + x_2) \in V'.$$

STEP 2. We show that condition (c) of the previous proof holds.

With step 1, it follows from transitivity that $HC(T)$ is a dense G_δ-set.

On the other hand, we consider the set

$$M = \{x \in X : \exists (x_k) \to 0, \exists (n_k) : T^{n_k}x \to 0, T^{n_k}x_k \to x\}.$$

We have

$$M = \bigcap_k \bigcup_n \{x \in X : d(T^n x, 0) < \tfrac{1}{k}, \exists \xi : d(\xi, 0) < \tfrac{1}{k}, d(T^n \xi, x) < \tfrac{1}{k}\}.$$

By (GS) and the Baire category theorem, M is a dense G_δ-set (why?).

It follows, again with the Baire category theorem, that

$$M \cap HC(T) \neq \varnothing.$$

Thus there is a hypercyclic vector x_0 in M.

This shows that condition (c) in the proof of the Theorem of Bès and Peris is satisfied, which implies that T satisfies the Hypercyclicity Criterion.□

As an application of this result we have:

Theorem. Any chaotic operator on a complete metric vector space satisfies the Hypercyclicity Criterion.

In that case, by the Bès-Peris theorem, $T \oplus T$ is also hypercyclic. But we have already noted that $T \oplus T$ also has a dense set of periodic points. Hence:

Corollary. If T is a chaotic operator on a complete metric vector space then $T \oplus T$ is also chaotic.

Proof of the Theorem. We verify condition (GS) for T.

Thus let $U, V, W \neq \varnothing$ be open in X with $0 \in W$.

First, there exists a hypercyclic vector $y_0 \in V$. Then also $y_k = \frac{y_0}{k}, k \geq 1,$ is hypercyclic; so we can find an increasing sequence of positive integers with $T^{n_k} y_k \to y_0$. In addition, $y_k \to 0$.

Secondly, by chaoticity, there is a periodic point $x_0 \in U$. Since the sequence $(T^{n_k} x_0)$ takes only finitely many values, it has a constant subsequence. W.l.o.g. let

$$T^{n_k} x_0 = z \quad \text{for all } k \in \mathbb{N}.$$

Finally, by hypercyclicity of y_0, there is some $N \in \mathbb{N}$ with

$$z + T^N y_0 \in W.$$

We now collect: if k is sufficiently big we obtain, simultaneously,

- $x_0 + T^N y_k \in U$;
- $T^{n_k}(x_0 + T^N y_k) = T^{n_k} x_0 + T^N (T^{n_k} y_k) = z + T^N (T^{n_k} y_k) \in W$;
- $y_k \in W$;
- $T^{n_k} y_k \in V$.

This shows that (GS) holds. $\square$

The question remains: Does every hypercyclic operator satisfy the Hypercyclicity Criterion?

In 2007, the answer was finally shown to be: No!

Theorem.

(a) **(de la Rosa, Read).** There is a hypercyclic operator on a Banach space that does not satisfy the Hypercyclicity Criterion.

(b) **(Bayart, Matheron).** This is, in particular, true on any separable Hilbert space and on any of the spaces $\ell^p, 1 \leq p < \infty,$ and c_0.

The proofs are highly non-trivial.

We add that, as a consequence of a result of Herzog and Lemmert (1993), every hypercyclic operator on the space

$$\omega := \mathbb{C}^{\mathbb{N}},$$

endowed with the product topology, satisfies the Hypercyclicity Criterion.

But ω is well-known to be a pathological space in Hypercyclicity.

3.3. An eigenvalue criterion

We shall derive another criterion for hypercyclicity and chaos.

While it has stronger assumptions than our previous criteria, it is easily applicable when it works.

We have already noted the relevance of eigenvectors for periodic points. We now see that they can also lead to hypercyclicity, as was first observed by Godefroy and Shapiro (1991).

Theorem (Eigenvalue Criterion). Let $T : X \to X$ be an operator on a complete metric vector space.

(a) If the subspaces

$$\text{span}\{x \in X \ : \ Tx = \lambda x \text{ for some } \lambda \text{ with } |\lambda| < 1\},$$

$$\text{span}\{x \in X \ : \ Tx = \lambda x \text{ for some } \lambda \text{ with } |\lambda| > 1\}$$

are both dense then T is hypercyclic.

(b) If, in addition, the subspace

$$\text{span}\{x \in X \ : \ Tx = \lambda x \text{ for some } \lambda = e^{2\pi i q}, q \in \mathbb{Q}\}$$

is dense then T is chaotic.

The usefulness of this criterion is obvious:

- one often knows how to find eigenvectors,
- density of a span can often by tested by a Hahn-Banach argument.

Proof. (a) We show that the conditions of the Kitai-Gethner-Shapiro criterion are satisfied.

(i) Let Y_0 denote the first subspace. It is dense by assumption.

Let $x \in Y_0$. If

$$Tx = \lambda x, \quad |\lambda| < 1$$

then

$$T^n x = \lambda^n x \to 0.$$

In general, $x \in Y_0$ is a linear combination of such vectors. Hence also, by linearity,

$$T^n x \to 0.$$

(ii/iii) Let Y_1 denote the second subspace. It is dense by assumption.

There is a slight technical problem with the definition of S. To overcome this, we provide ourselves with an algebraic basis of Y_1 so that each basis vector is an eigenvector of modulus greater than 1.

On any basis vector $x \in Y_1$ we then define

$$Sx = \frac{1}{\lambda}x \quad \text{if } Tx = \lambda x\,.$$

And we extend S by linearity.

Clearly,

$$S : Y_1 \to Y_1 \quad \text{and} \quad TSx = x \quad \text{for all } x \in Y_1\,.$$

Also, for any basis vector

$$S^n x = \frac{1}{\lambda^n}x \to 0,$$

hence, by linearity, for all $x \in Y_1$,

$$S^n x \to 0.$$

(b) This is clear: we have already seen that the given subspace coincides with the space of periodic points for T. □

Before giving an application we note that there are hypercyclic operators without any eigenvectors. In that case, trivially, the eigenvalue criterion is not applicable.

Exercise. Consider the (bilateral) backward shift

$$B : \ell^1(v, \mathbb{Z}) \to \ell^1(v, \mathbb{Z}), \quad (x_k)_{k\in\mathbb{Z}} \to (x_{k+1})_{k\in\mathbb{Z}},$$

where $\ell^1(v, \mathbb{Z})$ is the weighted bilateral sequence space defined by

$$\ell^1(v, \mathbb{Z}) = \{(x_k)_{k\in\mathbb{Z}} : \|x\| := \sum_{n=-\infty}^{\infty} |x_n| v_n < \infty\},$$

with weights

$$v_n = \frac{1}{|n| + 1}.$$

(a) Show that B is a well-defined operator on X.
(b) Show that if

$$Bx = \lambda x, \quad x \neq 0,$$

then $\lambda \neq 0$ and

$$x = (\dots, (1/\lambda^2)x_0, (1/\lambda)x_0, x_0, \lambda x_0, \lambda^2 x_0, \dots), \quad \lambda \neq 0, \quad x_0 \neq 0$$

with some $x_0 \neq 0$. Deduce from this that B has no eigenvalues.
(c) Show that B satisfies the conditions of the Kitai-Gethner-Shapiro criterion and hence is hypercyclic (Hint: consider the subspace of finite sequences).

We shall now apply the eigenvalue criterion to give a new proof of the theorems of Birkhoff and MacLane, that is, the hypercyclicity of the operators

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad f(\cdot) \to f(\cdot + 1),$$
$$D : H(\mathbb{C}) \to H(\mathbb{C}), \quad f \to f'.$$

At first sight, these operators have little in common.

Also, the hypercyclicity proofs were very different: the one was based in an essential way on the Runge approximation theorem, the other on the Hypercyclicity criterion.

Our new proof not only allows us to give a unified approach to both results, it also extends it to a wide class of operators, namely all operators

$$T : H(\mathbb{C}) \to H(\mathbb{C})$$

that commute with D, that is,

$$TD = DT.$$

Unless $T = \lambda I$, this already implies that T is chaotic!

Theorem (Godefroy, Shapiro).
Suppose that $T : H(\mathbb{C}) \to H(\mathbb{C}), T \neq \lambda I$, is an operator that commutes with D, i.e.

$$TD = DT.$$

Then T is chaotic.

Proof. The proof requires several steps.

Step 1. We show that there exists an entire function $\Phi(z) = \sum_{k=0}^{\infty} a_k z^k$ such that T has the representation

$$Tf = \Phi(D)f = \sum_{k=0}^{\infty} a_k D^k f, \quad f \in H(\mathbb{C}).$$

Indeed, let

$$p_n(z) = z^n$$

denote the monomials. Then, using the assumption that T and D commute, we have that

$$D^k T p_n = T D^k p_n = 0 \quad \text{for } k > n.$$

Hence Tp_n is a polynomial of degree at most n.

Also, we find for $0 \le k \le n$ that

$$D^{n-k}Tp_n = TD^{n-k}p_n = n(n-1)\cdots(k+1)\,Tp_k.$$

Hence, for any $n \in \mathbb{N}_0$ and $z \in \mathbb{C}$,

$$\begin{aligned}
&\sum_{k=0}^{\infty}(Tp_k)(0)\frac{(D^k p_n)(z)}{k!} = \\
&\qquad = \sum_{k=0}^{n} \frac{(D^{n-k}Tp_n)(0)}{n(n-1)\cdots(k+1)} \ \frac{n(n-1)\cdots(n-k+1)}{k!} p_{n-k}(z) \\
&\qquad = \sum_{k=0}^{n} \frac{(D^{n-k}Tp_n)(0)}{(n-k)!} p_{n-k}(z) \\
&\qquad = \sum_{\nu=0}^{n} \frac{(D^{\nu}Tp_n)(0)}{\nu!} p_{\nu}(z) \qquad\qquad (\nu := n-k) \\
&\qquad = (Tp_n)(z).
\end{aligned}$$

By linearity we deduce for every polynomial f and $z \in \mathbb{C}$

$$\sum_{k=0}^{\infty}(Tp_k)(0)\frac{(D^k f)(z)}{k!} = (Tf)(z).$$

With

$$a_k = \frac{(Tp_k)(0)}{k!}$$

we thus have

$$Tf = \sum_{k=0}^{\infty} a_k D^k f = \Big(\sum_{k=0}^{\infty} a_k D^k\Big) f. \tag{1}$$

Using the continuity of T one can show that

$$\Phi(z) = \sum_{k=0}^{\infty} a_k z^k$$

defines an entire function and that

$$\Phi(D) = \sum_{k=0}^{\infty} a_k D^k$$

defines an operator on $H(\mathbb{C})$. (The proof requires some knowledge on Fréchet space operators).

Thus, by continuity, the identity (1) extends to all functions $f \in H(\mathbb{C})$.

Step 2. Now let, as usual,

$$e_\lambda(z) = e^{\lambda z}.$$

Then, by the representation of T,

$$Te_\lambda = \sum_{k=0}^{\infty} a_k D^k e_\lambda = \Big(\sum_{k=0}^{\infty} a_k \lambda^k\Big) e_\lambda = \Phi(\lambda) e_\lambda.$$

Hence e_λ is an eigenvector of T to the eigenvalue $\Phi(\lambda)$.

Step 3. We can now apply the eigenvalue criterion:

By Step 2, the space

$$\text{span}\{f \in H(\mathbb{C}) \ : \ Tf = \mu f \text{ for some } \mu \text{ with } |\mu| < 1\}$$

contains

$$\text{span}\{e_\lambda \ : |\Phi(\lambda)| < 1\}.$$

By the lemma in Section 2.1, this space is dense if the set

$$\{\lambda \in \mathbb{C} : |\Phi(\lambda)| < 1\}$$

has an accumulation point. But this is clear since it is a non-empty open set (note that Φ is not a constant since $T \neq \lambda I$).

For the same reason, the space

$$\text{span}\{f \in H(\mathbb{C}) \ : \ Tf = \mu f \text{ for some } \mu \text{ with } |\mu| > 1\},$$

is dense.

For the density of

$$\text{span}\{f \in H(\mathbb{C}) \ : \ Tf = \mu f \text{ for some } \mu = e^{2\pi i q}, q \in \mathbb{Q}\}$$

one has to use the open mapping theorem for holomorphic functions and Picard's theorem that any non-constant entire function omits at most one value. $\square$

An application of the eigenvalue criterion also allows us a finer study of the hypercyclicity of the operators

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad TD = DT, \quad T \neq \lambda I.$$

We know that each such operator T is hypercyclic, that is, there exists an entire function f with dense orbit under T.

Question: what are the possible rates of growth of hypercyclic functions f?

Already MacLane had considered this question.

He showed that there exists a D-hypercyclic entire function f of exponential type 1, that is, for all $\varepsilon > 0$ there is $M > 0$ with

$$|f(z)| \leq M e^{(1+\varepsilon)|z|}.$$

Bernal and Bonilla (2002) have attacked the same problem for general T, following an idea of Chan and Shapiro (1991).

The idea is to replace $H(\mathbb{C})$ by a space of entire functions of restricted growth:

For $\tau > 0$ let

$$E^2_\tau = \Big\{ f \in H(\mathbb{C}) : f(z) = \sum_{n=0}^{\infty} a_n z^n \quad \text{with} \quad \sum_{n=0}^{\infty} \Big(\frac{n!}{\tau^n}\Big)^2 |a_n|^2 < \infty \Big\}.$$

This is a Hilbert space under the natural norm

$$\|f\| = \Big(\sum_{n=0}^{\infty} \Big(\frac{n!}{\tau^n}\Big)^2 |a_n|^2 \Big)^{1/2}$$

that is continuously and densely embedded in $H(\mathbb{C})$. Moreover, every $f \in E^2_\tau$ satisfies the growth condition

$$|f(z)| \le M e^{\tau |z|} \quad \text{for } z \in \mathbb{C},$$

for some $M > 0$.

Exercise. Prove these statements.

Finally, one can show with the aid of the representation

$$T = \Phi(D)$$

that T also defines an operator on E^2_τ.

We can then follow the same procedure as before. First:

Lemma. The space E^2_τ contains all functions e_λ with $|\lambda| < \tau$. Moreover, if a set $\Lambda \subset \mathbb{D}_\tau := \{z : |z| < \tau\}$ has an accumulation point in $\mathbb{D}_\tau$ then

$$\text{span}\{e_\lambda : \lambda \in \Lambda\}$$

is dense in E^2_τ.

Exercise. Do the proof. (*Hint:* show that the orthogonal complement of the span is trivial. This will follow from the fact that a certain holomorphic function vanishes on Λ and hence on $\mathbb{D}_\tau$, cp. the argument on p. 36.)

As before we obtain the following.

Theorem. Suppose that $T : H(\mathbb{C}) \to H(\mathbb{C}), T \neq \lambda I,$ is an operator that commutes with D, i.e.

$$TD = DT.$$

Let

$$T = \Phi(D)$$

be its representation with an entire function Φ. Then T is hypercyclic on E^2_τ for any $\tau > \min\{|z| : |\Phi(z)| = 1\}$.

Since E^2_τ is densely embedded in $H(\mathbb{C})$ we obtain:

Corollary. Under the same assumptions as in the theorem we have:

For any

$$\tau > \min\{|z| : |\Phi(z)| = 1\}$$

there is a T-hypercyclic entire function with

$$|f(z)| \leq Me^{\tau|z|}, \quad z \in \mathbb{C}.$$

Proof of the theorem. We consider T on E^2_τ. As before we have that

$$Te_\lambda = \Phi(D)e_\lambda = \Phi(\lambda)e_\lambda.$$

Thus the functions

$$e_\lambda, \quad |\lambda| < \tau$$

are eigenvectors of T in E^2_τ (to the eigenvalue $\Phi(\lambda)$).

Hence, the set

$$\text{span}\{f \in E^2_\tau \ : \ Tf = \mu f \text{ for some } \mu \text{ with } |\mu| < 1\}$$

contains

$$\text{span}\{e_\lambda \ : |\lambda| < \tau, |\Phi(\lambda)| < 1\}.$$

By the lemma, the latter set is dense in E^2_τ provided that

$$\{z \ : |\Phi(z)| < 1\}$$

has an accumulation point in

$$\{z \ : |z| < \tau\}.$$

For this it suffices to find a point z with $|z| < \tau$ and $|\Phi(z)| = 1$.

But the latter is true since $\tau > \min\{|z| : |\Phi(z)| = 1\}$.

In the same way one shows that

$$\text{span}\{f \in E^2_\tau \ : \ Tf = \mu f \text{ for some } \mu \text{ with } |\mu| > 1\}$$

is dense in E^2_τ.

By the eigenvalue criterion, T is hypercyclic on E^2_τ. $\square$

The theorem seems to be the best growth result known for general operators T commuting with D.

For individual operators, much better results are available.

For $T = D$ we have that $\Phi(z) = z$, hence

$$\min\{|z| : |\Phi(z)| = 1\} = 1,$$

so that the theorem gives us: for any $\varepsilon > 0$ there is some D-hypercyclic function f with

$$|f(z)| \leq Me^{(1+\varepsilon)|z|}, \quad z \in \mathbb{C}.$$

MacLane had already shown the better result that type 1 is possible.

In fact, the following optimal result is known (GE, 1991): If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is any function with $\varphi(r) \to \infty$ as $r \to \infty$ then there is a D-hypercyclic function f with

$$|f(z)| \leq \varphi(|z|)\frac{e^{|z|}}{\sqrt{|z|}} \quad \text{for } |z| \text{ sufficiently large;}$$

but there is no D-hypercyclic function f that satisfies

$$|f(z)| \leq M\frac{e^{|z|}}{\sqrt{|z|}} \quad \text{for } z \neq 0.$$

For the translation operator

$$Tf(z) = f(z+1)$$

we have that

$$T = e^D$$

(why?), hence

$$\min\{|z| : |\Phi(z)| = 1\} = \min\{|z| : |e^z| = 1\} = 0,$$

so that the theorem gives us: for any $\varepsilon > 0$ there is some Birkhoff-hypercyclic function with

$$|f(z)| \leq Me^{\varepsilon|z|}, \quad z \in \mathbb{C}.$$

But Duyos-Ruiz (1983) had already shown that there are Birkhoff-hypercyclic functions of arbitrarily slow transcendental growth.

4. Classes of hypercyclic and chaotic operators

We have already met the first large class of chaotic operators, the operators

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad T \neq \lambda I, \quad \text{with } TD = DT.$$

This class covers the classical results of Birkhoff and MacLane.

The third classical result, the chaos for multiples of the backward shift on ℓ^p, leads to another class, weighted shift operators. This will also allow us to revisit MacLane's operator.

And, finally, we shall study the hypercyclicity of composition operators. This will, in particular, cover Birkhoff's result.

4.1. Weighted shifts

The basic model of all shifts is the backward shift

$$B : (x_1, x_2, x_3, \ldots) \to (x_2, x_3, x_4, \ldots).$$

Rolewicz has studied the hypercyclicity of multiples of this shift:

$$\mu B : (x_1, x_2, x_3, \ldots) \to (\mu x_2, \mu x_3, \mu x_4, \ldots).$$

In general, one might ask about hypercyclicity and chaos of shifts with arbitrary weights:

$$B_w : (x_1, x_2, x_3, \ldots) \to (w_2 x_2, w_3 x_3, w_4 x_4, \ldots);$$

here,

$$w = (w_k)$$

is an arbitrary weight sequence; we shall always assume that the weights are non-zero.

So far we have considered the sequence spaces ℓ^p and c_0 as underlying spaces.

More generally, we can consider these operators on any *sequence space* X, that is, any subspace of the space

$$\omega = \mathbb{K}^{\mathbb{N}} = \{(x_n)_{n\geq 1} : x_n \in \mathbb{K}\}$$

of all sequences.

We assume that X is endowed with a vector space topology in such a way that the canonical embedding

$$X \to \omega$$

is continuous; in other words, convergence in X implies coordinatewise convergence.

In that case, X is called a *topological sequence space* (in particular, a *Banach sequence space*, a *Fréchet sequence space*, a *complete metric sequence space*, etc.)

We start by characterizing when the (unweighted) shift B is hypercyclic.

For its proof we need the following general result.

Lemma. Let (M, d) be a metric space, $v_n, v \in M$. Suppose there is an increasing sequence (n_k) of positive integers such that

$$v_{n_k - j} \to v \quad \text{for every } j \in \mathbb{N}.$$

Then there exists an increasing sequence (m_k) of positive integers such that

$$v_{m_k + j} \to v \quad \text{for every } j \in \mathbb{N}.$$

Proof. By assumption,

$$\forall k \in \mathbb{N}\ \exists N_k > k\ \ \forall j = 0, \ldots, k : \quad d(v_{N_k - j}, v) < \frac{1}{k}.$$

Now let

$$m_k = N_k - k, \quad k \in \mathbb{N}.$$

Then

$$\forall k \in \mathbb{N}\ \exists m_k \in \mathbb{N}\ \forall j = 0, \ldots, k : \quad d(v_{m_k + k - j}, v) < \frac{1}{k},$$

that is,

$$\forall k \in \mathbb{N}\ \exists m_k \in \mathbb{N}\ \forall j = 0, \ldots, k : \quad d(v_{m_k + j}, v) < \frac{1}{k},$$

which implies the assertion (when we pass to an increasing subsequence of (m_k), if necessary).□

By e_n we denote the canonical unit sequences

$$e_n = (0, \ldots, 0, 1, 0, \ldots)$$

with the 1 in the nth position.

Theorem. Let X be a complete metric sequence space in which (e_n) forms a basis. Suppose that B defines an operator on X. Then B is hypercyclic if and only if there is an increasing sequence $(n_k)_k$ of positive integers such that

$$e_{n_k} \to 0 \quad \text{in } X.$$

We remark that, by the closed graph theorem, B is an operator on X as soon as B maps X into itself; that is, continuity is automatic.

Proof. For the **sufficiency** of the condition we use the Hypercyclicity Criterion.

We define $Y_0 = Y_1$ as the set of finite sequences, and $S : Y_1 \to X$ as the forward shift

$$S : (x_1, x_2, x_3, \ldots) \to (0, x_1, x_2, \ldots).$$

Clearly, $T^n x \to 0$ for all $x \in Y_0$. Thus, condition (i) holds.

For the proof of (ii) note that, by continuity of B,

$$B^j e_{n_k} = e_{n_k - j} \to 0$$

for all $j \geq 1$.

It follows from the lemma that there is a sequence (m_k) such that

$$S^{m_k} e_j = e_{m_k + j} \to 0$$

for all $j \geq 1$.

By linearity,

$$S^{m_k} x \to 0$$

for all $x \in Y_1$, hence condition (ii).

Condition (iii) is trivial because we even have $TS = I$.

For simplicity we shall do the proof of **necessity** in the case when X carries a norm $\|\cdot\|$.

We shall show that hypercyclicity of B implies that

for every $N \in \mathbb{N}, \varepsilon > 0$ there exists $n > N$ with $\|e_n\| < \varepsilon$.

Thus, fix $N \in \mathbb{N}$ and $\varepsilon > 0$.

By assumption, for every $x = (x_k) \in X$, the series

$$\sum_{k=1}^{\infty} x_k e_k$$

converges in X, hence each sequence $(x_k e_k)$ is bounded.

The uniform boundedness principle, applied to the operators

$$X \to X, \quad x \to x_k e_k, \quad k \geq 1,$$

implies that there is some $\delta > 0$ such that

$$\|x\| < \delta \quad \Longrightarrow \quad \forall k \in \mathbb{N}: \quad \|x_k e_k\| < \tfrac{\varepsilon}{2}. \tag{1}$$

Moreover, since convergence in X implies coordinatewise convergence, there is some $\eta > 0$ such that

$$\|x\| < \eta \quad \Longrightarrow \quad |x_1| < \tfrac{1}{2}. \tag{2}$$

Now let x be a hypercyclic vector for B with

$$\|x\| < \delta. \tag{3}$$

Then there exists some $n > N$ such that

$$\|B^{n-1}x - e_1\| < \eta. \tag{4}$$

In view of (1), it follows from (3) that

$$\|x_n e_n\| < \tfrac{\varepsilon}{2}.$$

In view of (2), it follows from (4) that

$$|x_n - 1| < \tfrac{1}{2},$$

hence

$$|x_n^{-1} - 1| < 1.$$

Altogether we have that

$$\begin{aligned}\|e_n\| &= \|x_n^{-1}x_ne_n\| \\ &\leq \|x_ne_n\| + \|(x_n^{-1}-1)x_ne_n\| \\ &\leq 2\|x_ne_n\| < \varepsilon,\end{aligned}$$

which had to be shown. □

We next want to characterize when B is chaotic.

For this we shall assume that (e_n) is an *unconditional basis*. In a Fréchet sequence space this can be characterized as saying that (e_n) is a basis such that, whenever (ε_n) is a 0-1-sequence then

$$(x_n) \in X \quad \Longrightarrow \quad (\varepsilon_n x_n) \in X.$$

Theorem. Let X be a Fréchet sequence space in which (e_n) forms an unconditional basis. Suppose that B defines an operator on X. Then the following assertions are equivalent:

(i) B is chaotic;

(ii) $\sum_{k=1}^{\infty} e_k$ converges in X;

(iii) B has a non-trivial periodic point.

Note that (ii) simply says that the constant sequences belong to X.

Condition (iii) is quite remarkable: a single non-trivial periodic point, without any hypercyclicity assumption, already makes B chaotic!

Proof. (i)$\Longrightarrow$(iii) is trivial.

(iii)$\Longrightarrow$(ii) Let

$$(x_1, x_2, x_3, \ldots) \neq 0$$

be periodic, of period N, say.

Then there is some $j \leq N$ such that $x_j \neq 0$, and we have, for $\nu \geq 1$,

$$x_{j+\nu N} = x_j.$$

Setting now all coordinates with indices $\neq j + \nu N$ zero, we see by unconditionality of the basis that

$$(0, \ldots, 0, x_j, 0, \ldots, 0, x_j, \ldots, 0, x_j, 0, \ldots) \in X$$

hence also

$$(0, \ldots, 0, 1, 0, \ldots, 0, 1, \ldots, 0, 1, 0, \ldots) \in X$$

where the non-zero entries have distance N.

Applying B $N-1$ times and adding the results we see that

$$(1, 1, 1, \ldots) \in X,$$

hence (ii).

(ii)$\Longrightarrow$(i). We shall do the proof in the case of a Banach sequence space X.

One can show that, by unconditionality of the basis, there is some $M > 0$ such that, for all $(x_n) \in X$ and all 0-1-sequences (ε_n),

$$\|(\varepsilon_n x_n)\| \leq M \|(x_n)\|. \tag{1}$$

Now, condition (ii) implies that $e_n \to 0$, hence, by the previous theorem, that B is hypercyclic.

Next, since

$$(1, 1, 1, \ldots) \in X,$$

unconditionality of the basis implies that every (periodic) 0-1-sequence belongs to X, hence, by linearity, every periodic sequence.

Now let $x = (x_n) \in X, \varepsilon > 0$. Since (e_n) is a basis, there is some $N \geq 1$ such that

$$\widetilde{x} := (x_1, x_2, x_3, \ldots, x_N, 0, \ldots)$$

has distance less than ε from x.

The periodic sequence

$$(x_1, \ldots, x_N, x_1, \ldots, x_N, x_1, \ldots, x_N, \ldots)$$

belongs to X.

Thus there is some $m \geq 1$ such that

$$(0, \ldots, 0, x_1, \ldots, x_N, x_1, \ldots, x_N, \ldots) \quad \text{with } mN \text{ leading zeros}$$

has norm less than $\frac{\varepsilon}{M}$.

By (1), the vector

$$\widetilde{y} := (0, \ldots, 0, x_1, \ldots, x_N, 0, \ldots, 0, x_1, \ldots, x_N, 0, \ldots, 0, x_1, \ldots, x_N, \ldots)$$
$$\text{with } mN \text{ leading zeros, and then repeated blocks of } (m-1)N \text{ zeros}$$

has norm less than ε.

But then

$$\begin{aligned} y &:= \widetilde{x} + \widetilde{y} \\ &= (x_1, \ldots, x_N, 0, \ldots, 0, x_1, \ldots, x_N, 0, \ldots, 0, x_1, \ldots, x_N, 0, \ldots, 0, x_1, \ldots) \\ &\quad \text{with repeated blocks of } (m-1)N \text{ zeros} \end{aligned}$$

is periodic.

And it has distance less than 2ε from x.□

It is now easy to transfer these results to arbitrary weighted shifts by means of a suitable conjugacy.

Let B_w be a weighted shift operator on some topological sequence space (Banach sequence space, etc.) X. We define new weights v_n by

$$v_n = \Big(\prod_{\nu=1}^{n} w_\nu\Big)^{-1}, \quad n \in \mathbb{N},$$

and consider the sequence space

$$X_v = \{(x_n)_{n\in\mathbb{N}} : (v_n x_n)_n \in X\}.$$

Then the diagonal transform

$$\phi_v : X_v \to X, \quad (x_n)_n \to (v_n x_n)_n,$$

is a vector space isomorphism, and we transfer the topology of X via ϕ_v to X_v. Then ϕ_v is a homeomorphism.

Then also X_v is a topological sequence space (Banach sequence space, etc.)

Moreover, for $(x_n) \in X_v$,

$$B_w(\phi_v(x_n)) = B_w(v_n x_n) = (w_{n+1} v_{n+1} x_{n+1}) = (v_n x_{n+1}) = \phi_v(B(x_n)),$$

so that the diagram

$$\begin{array}{ccc} X_v & \xrightarrow{B} & X_v \\ {\scriptstyle\phi_v}\big\downarrow & & \big\downarrow{\scriptstyle\phi_v} \\ X & \xrightarrow{B_w} & X \end{array}$$

commutes.

Hence,

$$B_w : X \to X \quad \text{and} \quad B : X_v \to X_v$$

are conjugate mappings.

Thus, B_w is hypercyclic on X if and only if B is hypercyclic on X_v.

Note that X_v inherits the property of (e_n) being a basis from X.

Thus we obtain the following:

Theorem. Let X be a complete metric sequence space in which (e_n) forms a basis. Suppose that the weighted shift B_w defines an operator on X. Then B_w is hypercyclic if and only if there is an increasing sequence $(n_k)_k$ of positive integers such that

$$\Big(\prod_{\nu=1}^{n_k} w_\nu\Big)^{-1} e_{n_k} \to 0 \quad \text{in } X.$$

In the special case of the spaces ℓ^p or c_0 we a obtain a result due to Salas (1995).

Corollary. Suppose that the weighted shift B_w defines an operator on $\ell^p, 1 \leq p < \infty$, or c_0. Then B_w is hypercyclic if and only if

$$\prod_{\nu=1}^{n} w_\nu, \quad n \geq 1,$$

is unbounded.

Note that this characterization does not depend on the particular spaces considered. In fact, it holds in any Banach sequence space with $\|e_n\| = 1$, $n \geq 1$.

As another example we consider the space $H(\mathbb{C})$ of entire functions.

Via the identification of an entire function

$$f(z) = \sum_{k=0}^{\infty} a_k z^k$$

with its sequence

$$(a_k)_{k\geq 0}$$

of Taylor coefficients it can be considered as a Fréchet sequence space.

Then,

$$\begin{aligned} v_{n_k} e_{n_k} \to 0 \quad &\Longleftrightarrow \quad v_{n_k} z^{n_k} \to 0 \quad \text{in } H(\mathbb{C}) \\ &\Longleftrightarrow \quad |v_{n_k}| R^{n_k} \to 0 \quad \text{for all } R > 0 \\ &\Longleftrightarrow \quad |v_{n_k}|^{1/n_k} \to 0. \end{aligned}$$

This gives us:

Corollary. Suppose that the weighted shift B_w defines an operator on $H(\mathbb{C})$. Then B_w is hypercyclic if and only if

$$\Big(\prod_{\nu=1}^{n} |w_\nu| \Big)^{1/n}, \quad n \geq 1,$$

is unbounded.

Since the differentiation operator

$$D : \sum_{k=0}^{\infty} a_k z^k \quad \to \quad \sum_{k=0}^{\infty} (k+1) a_{k+1} z^k$$

is a weighted shift with weights $w_k = k$, the corollary, in particular, includes MacLane's theorem on the hypercyclicity of D.

We turn to chaos.

Using the same argument as in the hypercyclic case we get:

Theorem. Let X be a Fréchet sequence space in which (e_n) forms an unconditional basis. Suppose that the weighted shift B_w defines an operator on X. Then the following assertions are equivalent:

(i) B_w is chaotic;

(ii) $\sum_{k=1}^{\infty}\Big(\prod_{\nu=1}^{k} w_\nu\Big)^{-1} e_k$ converges in X;

(iii) B_w has a non-trivial periodic point.

In the special case of the spaces ℓ^p we have:

Corollary. Suppose that the weighted shift B_w defines an operator on $\ell^p, 1 \leq p < \infty$. Then the following assertions are equivalent:

(i) B_w is chaotic;

(ii) $\displaystyle\sum_{k=1}^{\infty} \frac{1}{\Big|\prod_{\nu=1}^{k} w_\nu\Big|^p} < \infty$;

(iii) B_w has a non-trivial periodic point.

Note that, in contrast to hypercyclicity, the condition in terms of the weights depends on p.

A similar result holds for c_0.

On $H(\mathbb{C})$ we have:

Corollary. Suppose that the weighted shift B_w defines an operator on $H(\mathbb{C})$. Then the following assertions are equivalent:

(i) B_w is chaotic;

(ii) $\Big(\prod_{\nu=1}^{n} |w_\nu|\Big)^{1/n} \to \infty$;

(iii) B_w has a non-trivial periodic point.

Again, the condition in terms of the weights is satisfied for the differentiation operator D (with weights $w_k = k$), giving a new proof that differentiation is chaotic.

But one could just apply condition (iii): D has a non-trivial periodic point, the function e^z.

$\to$ the exponential function makes differentiation chaotic!

One more application:

Corollary. Every weighted shift B_w is chaotic on $\omega = \mathbb{K}^{\mathbb{N}}$.

The following exercise shows that the theorem characterizing chaotic weighted shift operators on general sequence spaces does not necessarily hold if the basis is not unconditional.

Exercise. Let X be the Banach sequence space defined by

$$X = \Big\{ x = (x_n) : \|x\| = \sum_{n=1}^{\infty} \Big| \frac{x_n}{n} - \frac{x_{n+1}}{n+1} \Big| < \infty \text{ and } \frac{x_n}{n} \to 0 \text{ as } n \to \infty \Big\};$$

Then (e_n) is a basis in X that is not unconditional. The (unweighted) shift B is an operator on X, has non-trivial periodic points (the constant sequences). It also satisfies the weights condition in our characterization of chaos. And yet, B is not chaotic because the constant sequences are the only periodic sequences.

Remark.
(a) *Weighted forward shifts*,

$$(x_1, x_2, x_3, x_4, \ldots) \to (0, w_1 x_1, w_2 x_2, w_3 x_3, \ldots)$$

are, of course, never hypercyclic: the only sequence that an orbit can approximate, is the zero sequence.

(b) There is a parallel theory for *bilateral weighted backward shifts*

$$B_w : (x_k)_{k \in \mathbb{Z}} \to (w_{k+1} x_{k+1})_{k \in \mathbb{Z}}$$

on sequence spaces over the index set $\mathbb{Z}$. The characterizing conditions there are considerably more difficult. For example, the bilateral (unweighted) backward shift B is – under the usual assumptions – hypercyclic iff there is an increasing sequence $(n_k)_k$ of positive integers such that, for any $j \in \mathbb{Z}$,

$$e_{j-n_k} \to 0 \quad \text{and} \quad e_{j+n_k} \to 0 \quad \text{in } X.$$

The advantage of bilateral shifts is that they allow for more freedom: unlike unilateral shifts, for example, they can have an inverse.

Or, as another example, Salas (1991) has found a hypercyclic weighted bilateral backward shift whose adjoint is also hypercyclic.

Theorem (Salas). There is a hypercyclic operator $T : X \to X$ (on a Hilbert space X) such that its adjoint $T^* : X^* \to X^*$ is also hypercyclic.

Note that the adjoint of a weighted bilateral backward shift is a weighted bilateral forward shift, and therefore essentially again a bilateral weighted backward shift (just reverse 'time': $n \to -n$).

Thus, one 'only' has to come up with weights so that both these shifts are simultaneously hypercyclic.

Note also that for such an operator T, neither T nor T^* may have eigenvectors, as we have seen. In particular, T cannot be chaotic.

Salas' result had another interesting consequence:

Corollary. There are two hypercyclic operators $S, T : X \to X$ (on a Hilbert space X) such that $T \oplus S$ is not hypercyclic (and not even cyclic).

Proof. We take Salas' example on a Hilbert space.

Then $T : X \to X$ and $T^* : X^* \to X^*$ are hypercyclic.

Suppose that

$$T \oplus T^* : X \times X^* \to X \times X^*, \quad (x, y^*) \to (Tx, T^* y^*)$$

is cyclic, that is, there are $x \in X, y^* \in X^*$ such that

$$Y := \text{span}\{(T^n x, (T^*)^n y^*) : n \in \mathbb{N}\} \quad \text{is dense in } X \times X^*.$$

We now consider the continuous linear functional

$$\varphi : X \times X^* \to \mathbb{K}, \quad \varphi(\xi, \eta^*) = -y^*(\xi) + \eta^*(x).$$

Then we have, for $n \in \mathbb{N}$,

$$\varphi(T^n x, (T^*)^n y^*) = -y^*(T^n x) + (T^*)^n y^*(x) = -y^*(T^n x) + y^*(T^n x) = 0,$$

hence also

$$\varphi(\xi, \eta^*) = 0 \quad \text{for all } (\xi, \eta^*) \in Y.$$

But since the functional φ is obviously not the null functional, Y cannot be dense, which is a contradiction.

Finally, to have two operators T, S on the same space, take for S the operator T^* considered as an operator on X (X is a Hilbert space!). $\square$

This result of Salas prompted Herrero's question if $T \oplus T$ can be hypercyclic if T is.

Finally, after the negative solution of Herrero's problem by de la Rosa and Read one may wonder if $T \oplus T$ must at least always be cyclic when T is hypercyclic.

The answer is NO!, also. Grivaux (2005) has shown that, for any hypercyclic operator T, if $T \oplus T$ is cyclic then it is already hypercyclic. (That's an application of the U-V-W- characterization of the Hypercyclicity Criterion.)

4.2. Composition operators

The translation operator

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad Tf(z) = f(z+1)$$

is a special composition operator

$$f \to f \circ \phi \quad \text{with } \phi(z) = z + 1.$$

Note that ϕ is an automorphism of $\mathbb{C}$.

More generally, one may study composition operators on arbitrary domains Ω in $\mathbb{C}$. The space $H(\Omega)$ is again endowed with the topology of uniform convergence on compact sets. The corresponding metric is defined by a sequence of seminorms

$$p_n(f) = \sup_{z \in K_n} |f(z)|$$

via the Fréchet combination

$$d(f,g) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{p_n(f-g)}{1 + p_n(f-g)}, \quad f, g \in H(\Omega);$$

here, (K_n) is an exhaustion of Ω of compact sets, that is, with

$$K_n \subset (K_{n+1})^\circ, \quad \bigcup_n K_n = \Omega.$$

Note that every compact set is contained in some K_n.

Thus we can ask about the hypercyclicity of an arbitrary composition operator

$$C_\phi : H(\Omega) \to H(\Omega), \quad f \to f \circ \phi,$$

where

$$\phi : \Omega \to \Omega$$

is an automorphism.

This problem was thoroughly studied by Bernal und Montes (1995).

It turns out that the hypercyclicity can be characterized in terms of the following condition.

Definition.
A holomorphic self-map $\phi : \Omega \to \Omega$ is called a *run-away sequence* if for every compact set $K \subset \Omega$ there exists some $n \in \mathbb{N}$ such that $\phi^n(K) \cap K = \varnothing$.

Clearly, the $\mathbb{C}$-automorphism

$$\phi(z) = z + 1$$

is run-away.

The following result was obtained:

Theorem (Bernal, Montes). Let Ω be a domain in $\mathbb{C}$ and

$$C_\phi : H(\Omega) \to H(\Omega), \quad f \to f \circ \phi$$

a composition operator with an automorphism ϕ of Ω.
(a) If Ω is finitely connected but not simply connected, then C_ϕ is never hypercyclic.
(b) If Ω is simply connected or infinitely connected, then C_ϕ is hypercyclic if and only if ϕ is run-away.

Example. For $\Omega = \mathbb{C}$ the automorphisms are

$$\phi(z) = az + b, \quad a \neq 0.$$

Then ϕ is run-away if and only if $a = 1$ and $b \neq 0$.

(Note that, for $a \neq 1$, ϕ has a fixed point.)

Example. For $\Omega = \mathbb{D}$ the automorphisms are

$$\phi(z) = e^{i\theta}\frac{z-a}{1-\overline{a}z}, \quad |a| < 1.$$

Then ϕ is run-away if and only if ϕ has no fixed point in $\mathbb{D}$.

We turn to the proof of the Theorem of Bernal and Montes.

We first show that the run-away property is always **necessary** for hypercyclicity.

Suppose that $f \in H(\Omega)$ is hypercyclic for C_ϕ.

If ϕ is not run-away then there is a compact set $K \subset \mathbb{C}$ such that

$$\phi^n(K) \cap K \neq \varnothing, \quad \text{for all } n \in \mathbb{N}.$$

Then we find $z_n \in K$ with

$$\phi^n(z_n) \in K.$$

But f is bounded on K, by a constant M, say.

Thus,

$$|C_\phi^n f(z_n)| = |f(\phi^n(z_n))| \leq M, \quad \text{for all } n \in \mathbb{N}.$$

On the other hand, since f is hypercyclic we can find some $n \in \mathbb{N}$ such that

$$\max_{z \in K} |C_\phi^n f(z) - (M+1)| < 1.$$

This is impossible since the z_n belong to K. $\square$

This consideration suffices to exclude the case of finite connectivity.

If the (finite) connectivity is at least 2 then it is known that there are only finitely many automorphisms. Thus ϕ cannot be run-away.

In the case of connectivity 1, Ω is conformally equivalent to either $\mathbb{C} \setminus \{0\}$, $\mathbb{D} \setminus \{0\}$ or an annulus. Bernal and Montes show that in the latter two cases there is no run-away ϕ, while in $\mathbb{C} \setminus \{0\}$ the run-away ϕ do not produce hypercyclic composition operators.

We turn to the case of simple or infinite connectivity:

The proof of **sufficiency** of the run-away property has two basic ingredients, one approximation theoretic, the other topological.

We first recall that a compact subset K of Ω is called Ω*-convex* if every hole of K (= connected component of $\mathbb{C} \setminus K$) contains a point of $\mathbb{C} \setminus \Omega$.

Then we have the following version of Runge's theorem for arbitrary domains.

Theorem (Runge). Let Ω be a domain in $\mathbb{C}$. Let K be an Ω-convex compact subset of Ω.
If f is defined and holomorphic on a neighbourhood of K, then, for any $\varepsilon > 0$, there exists a function $g \in H(\Omega)$ such that

$$\sup_{z \in K} |f(z) - g(z)| < \varepsilon.$$

The second ingredient gives us the right tool for being able to apply Runge.

Lemma. Let Ω be a domain of infinite connectivity and ϕ a run-away automorphism of Ω. If K be an Ω-convex subset of Ω then there is some $n \in \mathbb{N}$ such that $\phi^n(K) \cap K = \varnothing$ and $\phi^n(K) \cup K$ is Ω-convex.

The proof of this lemma was the most difficult part in the paper by Bernal and Montes.

Note that the lemma remains true for simply connected domains, but then the result is trivial.

With these preparations, the **proof of hypercyclicity** is no longer difficult: we show that $C_\phi : H(\Omega) \to H(\Omega)$ is topologically transitive.

Let $U, V \neq \varnothing$ be open subsets of $H(\Omega)$.

Choose $f \in U, g \in V$ and $\varepsilon > 0$ such that

$$U_\varepsilon(f) \subset U, \quad U_\varepsilon(g) \subset V.$$

As in the proof of Birkhoff's theorem there are $m \in \mathbb{N}$ and $\delta > 0$ such that

$$p_m(f_1, f_2) = \sup_{z \in K_m} |f_1(z) - f_2(z)| < \delta \quad \Longrightarrow \quad d(f_1, f_2) < \varepsilon.$$

Thus, in order to show that, for some $n \in N$

$$C_\phi^n(U) \cap V \neq \varnothing,$$

it suffices to find some $h \in H(\Omega)$ such that

$$\sup_{z \in K_m} |f(z) - h(z)| < \delta, \quad \text{and}$$
$$\sup_{z \in K_m} |g(z) - C_\phi^n h(z)| < \delta.$$

Thus, let us consider the compact set K_m and $\delta > 0$.

By making K_m larger, if necessary, we can assume that it is Ω-convex (fill unnecessary holes).

By the lemma, there is some $n \in \mathbb{N}$ such that $\phi^n(K_m) \cap K_m = \varnothing$ and $\phi^n(K_m) \cup K_m$ is Ω-convex.

We define on $\phi^n(K_m) \cup K_m$

$$F(z) = \begin{cases} f(z) & \text{if} \quad z \in K_m, \\ g((\phi^n)^{-1}(z)) & \text{if} \quad z \in \phi^n(K_m). \end{cases}$$

Then F is holomorphic on a neighbourhood of $\phi^n(K_m) \cup K_m$.

By Runge's theorem there exists a function $h \in H(\Omega)$ that approximates F on $\phi^n(K_m) \cup K_m$ up to an error δ, that is

$$\sup_{z \in K_m} |f(z) - h(z)| < \delta,$$
$$\sup_{z \in \phi^n(K_m)} |g((\phi^n)^{-1}(z)) - h(z)| < \delta,$$

hence also

$$\sup_{z \in K_m} |g(z) - h(\phi^n(z))| < \delta.$$

This had to be shown. $\square$

Remark. The proof has a curious consequence. We have seen that if ϕ is not run-away then one cannot approximate all constant functions. Thus, in the simply or infinitely connected case we can conclude: If C_ϕ has an orbit that has all constant functions in its closure then C_ϕ is hypercyclic.

We give an example of a hypercyclic composition operator on an infinitely connected domain (taken from a recent paper by Gorkin, León-Saavedra and Mortini).

Example. Consider the automorphism

$$\phi(z) = \frac{z - \frac{1}{2}}{1 - \frac{1}{2}z}$$

of the unit disk $\mathbb{D}$. Then take out the compact disk

$$K = \{z : |z| \leq \tfrac{1}{10}\}$$

and all its images and preimages:

$$\Omega := \mathbb{D} \setminus \bigcup_{n=-\infty}^{\infty} \phi^n(K).$$

Then Ω is a domain of infinite connectivity and

$$\phi : \Omega \to \Omega$$

is an automorphism.

Now, since ϕ has no fixed point in $\mathbb{D}$, it is run-away on $\mathbb{D}$ and hence also on Ω (alternatively, show that

$$\phi^n(z) = \frac{z - \frac{3^n-1}{3^n+1}}{1 - \frac{3^n-1}{3^n+1}z}.$$

Hence $\phi^n(z) \to -1$ uniformly on compact sets in $\mathbb{D}$; hence ϕ is run-away).

Therefore, C_ϕ is hypercyclic on Ω.

Recently, in joint work with R. Mortini, we considered the more general case when

$$\phi : \Omega \to \Omega$$

is only a holomorphic self-map.

The composition operator

$$C_\phi : H(\Omega) \to H(\Omega), \quad f \to f \circ \phi,$$

remains an operator on $H(\Omega)$, and we can ask when it is hypercyclic.

First we can note that ϕ has to be injective. For, if

$$\phi(z_1) = \phi(z_2)$$

then

$$f(\phi^n(z_1)) = f(\phi^n(z_2))$$

for all $f \in H(\Omega)$ and $n \geq 0$. Thus, $f \circ \phi^n$ can only approximate functions that take the same value at z_1 and z_2. If we want a hypercyclic function f then we must have $z_1 = z_2$.

For simply connected domains we found that this is the only additional restriction.

Theorem (Mortini, GE). Let ϕ be a holomorphic self-map of a simply connected domain.

Then C_ϕ is hypercyclic if and only if ϕ is injective and run-away.

The proof is essentially the same as in the automorphic case.

The finitely connected case extends also, but here one has to come up with new necessary conditions for hypercyclicity that lead to a contradiction.

Theorem (contd...). Let ϕ be a holomorphic self-map of a finitely connected, non simply connected domain.

Then C_ϕ is never hypercyclic.

Finally, for the infinitely connected case we have:

Theorem (contd...). Let ϕ be a holomorphic self-map of an infinitely connected domain.

Then C_ϕ is hypercyclic if and only if ϕ is injective and, for every Ω-convex compact subset K of Ω and every $N \in \mathbb{N}$ there is some $n \geq N$ such that $\phi^n(K)$ is Ω-convex and $\phi^n(K) \cap K = \varnothing$.

The sufficiency part relied on Runge's theorem and an extension of the lemma above. The hardest part turned out to be the necessity of the condition.

With this we have a complete characterization of hypercyclic composition operators.

We conclude with an example.

Example. We modify our earlier example.

Let ϕ be the injective holomorphic self-map of $\mathbb{D}$ given by

$$\phi(z) = \frac{z}{4} + \frac{3}{4}.$$

We take out the compact disk

$$K = \{z : |z| \leq \tfrac{1}{2}\}$$

and all its images:

$$\Omega := \mathbb{D} \setminus \bigcup_{n=0}^{\infty} \phi^n(K).$$

Then Ω is a domain of infinite connectivity and

$$\phi : \Omega \to \Omega$$

is injective (note that the pre-image of K lies outside $\mathbb{D}$).

Now, since

$$\phi^n(z) = \frac{1}{4^n} z + 1 - \frac{1}{4^n} \to 1$$

uniformly on compact sets in $\mathbb{D}$, ϕ is run-away, and it obviously satisfies the condition of the theorem.

Therefore, C_ϕ is hypercyclic on Ω.

5. Some highlights of the theory.

One of the pleasant features of the theory of linear dynamics is that it has, by now, led to a considerable number of beautiful and sometimes surprising results that are easy to state but whose proofs are far from trivial.

We want to report on some of these major results.

5.1. Existence of hypercyclic operators.

Rolewicz (1969) had asked if every separable and infinite-dimensional Banach space supports a hypercyclic operator - a very natural question.

If we consider a hypercyclic operator as 'unusual', i.e., 'pathological', then the question was: can a Banach space be so pathological that it admits no pathological operator.

Now, a well-known feature of Banach spaces is that their definition is so broad that it allows for a lot of pathological spaces (e.g., a separable space without a basis).

The solution of Rolewicz' problem came from an unexpected angle.

In 1995, Salas showed that, on the sequence space ℓ^1, *every* perturbation of the identity by a weighted backward shift with non-zero weights is hypercyclic, that is, every operator

$$T : \ell^1 \to \ell^1, \quad T = I + B_w,$$

with (w_k) bounded and $w_k \neq 0$ for all $k \in \mathbb{N}$, is hypercyclic.

The proof is essentially elementary, but certainly non-trivial. Its centre-piece consists in estimating the solutions of a finite system of linear equations.

Ansari (1997) and Bernal (1999) independently realized, that Salas' very special result can be lifted to arbitrary separable, infinite-dimensional Banach spaces.

The crucial point is that, while such a space need not have a basis, it has some kind of weak basis that allows the construction of an operator that is quasi-conjugate to Salas' operator.

Subsequently, Bonet and Peris (1998) realized that some additional considerations are needed to extend the Ansari-Bernal theorem to Fréchet spaces.

Theorem (Ansari, Bernal, Bonet/Peris). Any separable, infinite-dimensional Fréchet space admits a hypercyclic operator.

Based on Salas' fundamental $I + B_w$-theorem, we give here the proof in a special case.

Proof (for a Banach space with a basis).
Let $(X, \|\cdot\|)$ be such a space; that is, there is a sequence (e_n) in X such that every $x \in X$ has a unique representation of the form

$$x = \sum_{k=1}^{\infty} x_k e_k$$

with scalar coefficients $x_k, k \in \mathbb{N}$. W.l.o.g. we can assume that

$$\|e_k\| = 1, \quad k \in \mathbb{N}.$$

The coefficient functionals

$$e_k^* : x \to x_k, \quad k \in \mathbb{N},$$

can be shown to be continuous, of norm $\|e_k^*\|$, say.

We now choose a bounded sequence (w_k) of positive numbers such that

$$C := \sum_{k=1}^{\infty} w_{k+1} \|e_{k+1}^*\| < \infty.$$

Then

$$T : X \to X, \quad \sum_{k=1}^{\infty} x_k e_k \to \sum_{k=1}^{\infty} w_{k+1} x_{k+1} e_k$$

defines a (continuous linear) operator on X because, for any $x \in X$,

$$\sum_{k=1}^{\infty} \|w_{k+1} x_{k+1} e_k\| \le \sum_{k=1}^{\infty} w_{k+1} \|e_{k+1}^*\| \, \|x\| \le C\|x\|.$$

Finally, we define

$$\phi : \ell^1 \to X, \quad (a_n) \to \sum_{k=1}^{\infty} a_k e_k.$$

Since $\|e_k\| = 1$ for all $k \in \mathbb{N}$, this is a well-defined operator; and it has dense range because (e_k) is a basis.

We now claim that the diagram

$$\begin{array}{ccc} \ell^1 & \xrightarrow{I+B_w} & \ell^1 \\ \phi \Big\downarrow & & \Big\downarrow \phi \\ X & \xrightarrow{I+T} & X \end{array}$$

commutes.

In fact, for $(a_k) \in \ell^1$ we have

$$\phi \circ (I + B_w)(a_k) = \phi\big((a_k + w_{k+1} a_{k+1})\big) = \sum_{k=1}^{\infty} a_k e_k + \sum_{k=1}^{\infty} w_{k+1} a_{k+1} e_k =$$

$$= (I + T) \sum_{k=1}^{\infty} a_k e_k = (I + T) \circ \phi\,(a_k).$$

Thus we can deduce the hypercyclicity of $I+T$ from Salas' result that $I+B_w$ is hypercyclic. $\Box$

It is natural to ask if every separable infinite-dimensional Banach space also supports a **chaotic** operator?

First, Martínez and Peris noted that one cannot mimic the above proof to obtain a chaotic operator. Since

$$\|e_k^*\| = \sup_{x\neq 0} \frac{|x_k|}{\|x\|} \geq \frac{1}{\|e_k\|} = 1,$$

we always have that $w_k \to 0$.

But Martínez and Peris have shown that in this case Salas' operator $I + B_w$ is not chaotic.

Can one, nonetheless, obtain chaotic operators on arbitrary Banach spaces?

The answer is: NO!

Theorem (Bonet, Martínez, Peris, 2001). There is a separable, infinite-dimensional complex Banach space that admits no chaotic operator.

The proof is based on a celebrated construction of certain pathological Banach spaces by Gowers and Maurey.

We want to report on two more recent results in the direction of existence of hypercyclic operators.

We have seen that, if there is one hypercyclic vector there are automatically many.

Is something similar true for hypercyclic operators? Given that, as we now know, every (separable, infinite-dimensional) Banach space supports a hypercyclic operator, must there then be automatically many?

But what does 'many' mean?

If we interpret is as 'dense' in the usual operator topology then we are quickly disappointed.

Clearly, no operator T of norm $\|T\| \leq 1$ is hypercyclic, for

$$\|T^n x\| \leq \|T\|^n \|x\| \leq \|x\|,$$

so that all orbits under T are bounded.

Thus, the space $L(X)$ of all (continuous linear) operators on X, when endowed with the operator norm, has no hypercyclic operator in its closed unit ball.

But there is another well-known, and weaker, topology on $L(X)$, the strong operator topology (SOT).

The SOT is defined in such a way that a net (T_α) of operators converges to some operator T iff we have pointwise convergence at each $x \in X$:

$$T_\alpha x \to Tx.$$

Bès and Chan (2003) have shown that the set of hypercyclic operators is indeed SOT-dense.

Theorem (Bès, Chan). Let X be any separable, infinite-dimensional Fréchet space. Then the set of hypercyclic operators is dense in the strong operator topology.

While their first proofs were quite involved, they subsequently realized that their result can also be derived from a theorem of Hadwin, Nordgren, Radjavi and Rosenthal (1979).

Let us start from a hypercyclic operator T on X.

Then we already know that there must be many more!

For, whenever $J : X \to X$ is an ismorphism then the operator

$$T_J = JTJ^{-1}$$

is hypercyclic because the diagram

$$\begin{array}{ccc} X & \xrightarrow{T} & X \\ {\scriptstyle J}\big\downarrow & & \big\downarrow{\scriptstyle J} \\ X & \xrightarrow{T_J} & X \end{array}$$

commutes.

Now the mentioned result of Hadwin, Nordgren, Radjavi and Rosenthal tells us that the set

$$\{T_J : J : X \to X \text{ an ismorphism}\} \subset L(X)$$

is SOT-dense in $L(X)$ if, for any $n \in \mathbb{N}$, there are $x_1, \ldots, x_n \in X$ such that

$$x_1, \ldots, x_n, \quad Tx_1, \ldots, Tx_n$$

is linearly independent.

But these x_k are easy to come by:

Let x be a hypercyclic vector for T. Then

$$x, T^2x, T^4x, \ldots, T^{2n-2}x, \quad Tx, T^3x, T^5x, \ldots, T^{2n-1}x$$

is linearly independent as part of a larger linearly independent set, the orbit of x under T (see Section 2.3).

This finishes the proof of the Bès-Chan result.

So, in the SOT-sense, there are always many hypercyclic operators.

For hypercyclic vectors, we have noted yet another notion of bigness: Every vector is the sum of two hypercyclic vectors.

Is the corresponding statement true for operators?

The affirmative answer, in the case of Hilbert space, was obtained by to Grivaux (2003). And one may even use chaotic operators!

Theorem (Grivaux). Let H be a separable infinite-dimensional complex Hilbert space. Then every operator on H is the sum of two chaotic operators.

First, Grivaux gave a 'simple' proof that every operator is the sum of **six** hypercyclic operators. Reducing the six to two was the major problem...

In addition she notes that, again, a Gowers-Maurey space provides a counterexample to show that her result, even for hypercyclic operators, does not extend to all Banach spaces.

5.2. Powers of hypercyclic operators.

Let T by a hypercyclic operator.

Here is a very simple question: Is then also $T^N, N \geq 2,$ hypercyclic?

In case one is inclined to say: 'Yes, why not!', let us issue two warnings:

(1) The statement is trivially false for non-linear maps: Just consider

$$f : \{-1, 1\} \to \{-1, 1\}, \quad x \to -x.$$

Then f has a dense orbit (in fact, both its orbits are dense), but f^2 does not.

(2) Let us consider the question on the vector level. If x is hypercyclic then

$$x, Tx, T^2x, T^3x, T^4x, \dots$$

is dense. If x was also hypercyclic for T^2 then we would have that also

$$x, T^2x, T^4x, T^6x, T^8x, \dots$$

is dense. Is that to be expected? Or, if not, how then would one construct a new hypercyclic vector for y?

Ansari (1995) found that, indeed, x is also hypercyclic for all $N \geq 2$!

Theorem (Ansari). Let T be a hypercyclic operator on a Fréchet space. Then, for any $N \geq 2$, also T^N is hypercyclic.
In fact, T and T^N have the same set of hypercyclic vectors.

Since every hypercyclic vector for T^N is also hypercyclic for T, the theorem amounts to saying:

$$x \quad \text{hypercyclic for } T \quad \Longrightarrow \quad x \quad \text{hypercyclic for } T^N.$$

Remarks. Since every periodic point for T is also periodic for T^N, the theorem extends trivially to chaotic operators.

Ansari's proof is non-trivial and contains, at a crucial point, an argument of connectedness. We shall return to this later.

Also Ansari's theorem had an afterlife.

Let us reconsider the orbit of a T-hypercyclic vector x:

$$x, Tx, T^2x, T^3x, T^4x, \ldots$$

In view of our aim to find hypercyclic vectors for $T^N, N \geq 2,$, let us split this orbit into N orbits under T^N:

$$\begin{array}{l} x, T^N x, T^{2N} x, T^{3N} x, T^{4N} x, \ldots, \\ Tx, T^{N+1} x, T^{2N+1} x, T^{3N+1} x, T^{4N+1} x, \ldots, \\ T^2 x, T^{N+2} x, T^{2N+2} x, T^{3N+2} x, T^{4N+2} x, \ldots, \\ \vdots \\ T^{N-1} x, T^{2N-1} x, T^{3N-1} x, T^{4N-1} x, T^{5N-1} x, \ldots. \end{array}$$

The **union** of these N orbits under T^N is dense. Must then, already, one of these orbits be dense? It would then follow that x itself is hypercyclic for T^N and thus Ansari's theorem.

But Herrero (1992) had earlier posed a more general question:

> Suppose that the union of the orbits of N vectors $x_1, x_2, \ldots, x_N$ under an operator T is dense. Does it follow that the orbit of one of these x_j is dense?

This question was answered affirmatively and independently by Costakis and Peris (2000/01).

Theorem (Costakis, Peris). Let T be a hypercyclic operator on a Fréchet space X, and let $x_1, x_2, \ldots, x_N \in X$. If

$$\bigcup_{j=1}^{N} \text{orb}(x_j, T) \quad \text{is dense in } X$$

then one of the x_j is hypercyclic for T.

Let us start the proof of this result. By hypothesis we have

$$X = \overline{\bigcup_{j=1}^{N} \mathrm{orb}(x_j, T)} = \bigcup_{j=1}^{N} \overline{\mathrm{orb}(x_j, T)}.$$

We may assume that N is chosen minimal.

If $N = 1$ we are done. Otherwise we have that

$$X \setminus \bigcup_{j=1}^{N-1} \overline{\mathrm{orb}(x_j, T)}$$

is non-empty. On the other hand, this set is open and, by necessity, contained in

$$\overline{\mathrm{orb}(x_N, T)}.$$

Hence, the closure of $\mathrm{orb}(x_N, T)$ has an interior point, that is, $\mathrm{orb}(x_N, T)$ is **somewhere dense**.

Now, Peris wondered if this already makes x_N hypercyclic!?

His question was answered affirmatively by Bourdon and Feldman (2003).

Theorem (Bourdon, Feldman). Let T be an operator on a Fréchet space. If the orbit of $x \in X$ under T is somewhere dense then it is dense (and T is hypercyclic).

Remark. Wengenroth (2003) has shown that the theorems of Ansari, Costakis-Peris and Bourdon-Feldman hold under no hypotheses whatsoever: they are unrestrictedly true for any operator on any topological vector space.

In particular, Bourdon-Feldman says: if any orbit under any operator is somewhere dense it must be dense.

5.3. Multiples of hypercyclic operators.

What can one do to a hypercyclic operator so that it remains hypercyclic?

- taking powers, T^N – yes (Ansari)
- taking direct sums, $T \oplus T$ – not necessarily (de la Rosa, Read)
- taking similar operators, JTJ^{-1} – yes (commutative diagram)
- taking inverses (if existent), T^{-1} – yes (note that T^{-1} is transitive if T is.)

How about multiples?

There are hypercyclic operators T so that all their multiples

$$\lambda T, \quad \lambda \neq 0$$

are hypercyclic. Take, for example, the differentiation operator

$$D : H(\mathbb{C}) \to H(\mathbb{C}), \quad f \to f'.$$

Reason: λD commutes with D and hence is hypercyclic (Godefroy-Shapiro).

But such a scenario can only happen in a non-Banach space setting.

For, if $T : X \to X$ is hypercyclic on a Banach space X and

$$|\lambda| \leq \|T\|^{-1}$$

then $\|\lambda T\| \leq 1$, which prevents λT from being hypercyclic.

In fact, on can show that there are hypercyclic operators such that

$$\lambda T \quad \text{is never hypercyclic for } |\lambda| < 1 \text{ or } |\lambda| > 1.$$

Thus one is reduced to the question:

$$T \text{ hypercyclic} \quad \Longrightarrow \quad \lambda T \text{ hypercyclic for } |\lambda| = 1?$$

In fact, the answer is simple - and YES - for **real** spaces.

In that case we only have the question:

$$T \text{ hypercyclic} \quad \Longrightarrow \quad -T \text{ hypercyclic?}$$

Here is the argument:

$$\begin{aligned} T \text{ hypercyclic} \quad &\Longrightarrow \quad T^2 = (-T)^2 \text{ hypercyclic} \quad \text{(Ansari)} \\ &\Longrightarrow \quad -T \text{ hypercyclic.} \end{aligned}$$

In the **complex** case, the same argument shows that

$$T \text{ hypercyclic} \quad \Longrightarrow \quad \lambda T \text{ hypercyclic if } \lambda^n = 1 \text{ for some } n \in \mathbb{N}.$$

Moreover, a simple Baire category argument shows that this implication holds, in fact, for a dense G_δ-set of numbers $\lambda \in \mathbb{T}$.

But it took a completely new, highly non-trivial approach to obtain the full answer, due to León-Saavedra and Müller (2004). Their argument is based, at a crucial point, on a homotopy argument.

Theorem (León-Saavedra, Müller). Let T be a hypercyclic operator on a complex Banach space. Then λT is hypercyclic whenever $|\lambda| = 1$.

5.4. Frequently hypercyclic operators

Another exciting recent development was the introduction of probabilistic methods, more specifically, the methods of ergodic theory, into linear dynamics by Bayart and Grivaux (2004–...)

These investigations led, among other things, to a natural new concept, that of a *frequently hypercyclic operator*.

It is easy to motivate:

If x is a hypercyclic vector for T then

$$\forall\, U \neq \varnothing \ \text{ open } \quad \text{we have that} \quad \{n \in \mathbb{N} : T^n x \in U\} \neq \varnothing,$$

and therefore also

$$\forall\, U \neq \varnothing \ \text{ open}, \qquad \{n \in \mathbb{N} : T^n x \in U\} \quad \text{is infinite.}$$

One may wonder how 'big' the latter set of positive integers can be.

We shall measure it in terms of *lower density*.

Recall that, for $A \subset \mathbb{N}$,

$$\underline{\text{dens}}\,(A) = \liminf_{N\to\infty} \frac{\#\{n \in A : n \leq N\}}{N}.$$

This leads to:

Definition (Bayart, Grivaux). An operator $T : X \to X$ is called *frequently hypercyclic* if there is some $x \in X$ such that, for all $U \neq \varnothing$ open in X,

$$\underline{\mathrm{dens}}\{n \in \mathbb{N} : T^n x \in U\} > 0.$$

Then x is called a *frequently hypercyclic vector*.

This definition came out of an application of the Birkhoff ergodic theorem to linear operators, after having defined an ergodic measure for this operator.

Remark. Here is an equivalent definition of frequent hypercyclicity. We have that

$$x \text{ is hypercyclic} \iff$$
$$\forall U \neq \varnothing \text{ open} \quad \exists (n_k) : T^{n_k} x \in U, \quad k \geq 1.$$

In this spirit one can show that:

$$x \text{ is frequently hypercyclic} \iff$$
$$\forall U \neq \varnothing \text{ open} \quad \exists (n_k), n_k = O(k) : T^{n_k} x \in U, \quad k \geq 1.$$

Exercise. Do that!

First questions immediately arise:
Do frequently hypercyclic operators exist?
How to recognize if an operator is frequently hypercyclic?

In fact, one can prove a sufficient condition for frequent hypercyclicity that is similar to the Kitai-Gethner-Shapiro criterion. It was obtained by Bayart, Grivaux (2006); in the present form it is due to Bonilla, GE (2007).

Theorem (Frequent Hypercyclicity Criterion).
Let X be a separable complete metric vector space.
Suppose there is a dense subset Y_1 of X and a mapping $S : Y_1 \to Y_1$ such that

(i) for all $x \in Y_1$, $\sum_{n=1}^{\infty} T^n x$ converges unconditionally,

(ii) for all $x \in Y_1$, $\sum_{n=1}^{\infty} S^n x$ converges unconditionally,

(iii) for all $x \in Y_1$, $TSx = x$.

Then T is frequently hypercyclic.

Here, a series
$$\sum_{n=1}^{\infty} x_n$$
is said to *converge unconditionally* in X if, for every $\varepsilon > 0$ there is some $N \in \mathbb{N}$ such that
$$F \subset \mathbb{N} \text{ finite, } F \cap \{1, \dots, N\} = \varnothing \text{ then } \quad d\Big(\sum_{n \in F} x_n, 0\Big) < \varepsilon.$$

The proof of the Frequent Hypercyclicity Criterion is much like the 'constructive' proof of the criterion of Kitai-Gethner-Shapiro.

However, what makes the proof much more difficult is that the approximating (n_k) have to come 'quickly', that is, with $n_k = O(k)$.

The way the proof works is that one must and can fix suitable sequences (n_k) beforehand and then construct a frequently hypercyclic vector accordingly.

In order to define these n_k one uses the following:

Lemma. There are pairwise disjoint sets $A(l,\nu),\ l,\nu\in\mathbb{N},$ of positive lower density such that

$$n-m\geq \nu+\mu, \quad \text{if} \quad n\in A(l,\nu), m\in A(k,\mu).$$

Examples.

(a) The multiple of the backward shift operator

$$T=\mu B, \quad (x_1,x_2,x_3,\ldots)\to \mu(x_2,x_3,x_4,\ldots), \quad |\mu|>1,$$

is frequently hypercyclic on $\ell^p, 1\leq p<\infty$, and on c_0.

In fact, for Y_1 one takes, as usual, the set of finite sequences, and $S=\frac{1}{\mu}F$, where F is the forward shift.

Then, for any unit sequence $x=e_N, N\geq 1,$

$$\sum_{n=1}^{\infty} T^n x$$

converges unconditionally as a finite series, and

$$\sum_{n=1}^{\infty} S^n x=\sum_{n=1}^{\infty}\frac{1}{\mu^n}e_{N+n}=\mu^N\sum_{n=N+1}^{\infty}\frac{1}{\mu^n}e_n$$

converges unconditionally (even absolutely) in ℓ^p and c_0.

(b) The differentiation operator

$$D : H(\mathbb{C}) \to H(\mathbb{C}), \quad f \to f'$$

is frequently hypercyclic.

In fact, for Y_1 one takes, as usual, the set of polynomials, and

$$Sf(z) == \int_0^z f(w)dw.$$

Then, for all monomials $x = z^N, N \geq 0,$

$$\sum_{n=1}^{\infty} D^n x$$

converges unconditionally as a finite series, and

$$\sum_{n=1}^{\infty} S^n x = \sum_{n=1}^{\infty} N! \frac{z^{N+n}}{(N+n)!} = N! \sum_{n=N+1}^{\infty} \frac{1}{n!} z^n$$

converges unconditionally in $H(\mathbb{C})$.

(c) The translation operator

$$T : H(\mathbb{C}) \to H(\mathbb{C}), \quad T(z) = f(z+1)$$

is frequently hypercyclic.

This can also be shown using the Frequent Hypercyclicity Criterion, but an easier proof uses the above lemma directly and the Runge approximation theorem.

Thus, the three classical hypercyclic operators are even frequently hypercyclic.

Of course, not every hypercyclic operator is frequently hypercyclic.

Proposition. The (unweighted) backward shift B is hypercyclic, but not frequently hypercyclic on the weighted ℓ^1-space

$$X = \Big\{(x_n) : \|x\| = \sum_{n=1}^{\infty} \frac{|x_n|}{n} < \infty\Big\}.$$

Proof. Since

$$\|e_n\| = \frac{1}{n} \to 0,$$

B is hypercyclic by the characterization in Section 4.1.

Now, if B had a frequently hypercyclic vector $x \in X$, then there would exist an increasing sequence (n_k) of positive integers with $n_k = O(k)$ such that

$$\|B^{n_k-1}x - e_1\| \leq \frac{1}{2},$$

hence

$$|x_{n_k} - 1| \leq \frac{1}{2},$$

thus

$$|x_{n_k}| \geq \frac{1}{2}.$$

But then

$$\|x\| \geq \sum_{k=1}^{\infty} \frac{|x_{n_k}|}{n_k} \geq \frac{1}{2} \sum_{k=1}^{\infty} \frac{1}{n_k} = \infty,$$

because $n_k = O(k)$. □

Remark. Every operator satisfying the Frequent Hypercyclicity Criterion is even chaotic.
In fact, for $x \in Y_1$ and $N \in \mathbb{N}$ set

$$y_N = \sum_{k=1}^{\infty} S^{kN}x + x + \sum_{k=1}^{\infty} T^{kN}x.$$

Then

$$T^N y_N = y_N, \quad y_N - x \to 0 \quad \text{as } N \to \infty;$$

thus, periodic points are dense.

Remark. There is a frequently hypercyclic operator (on c_0) that is not chaotic (Bayart, Grivaux 2007).
In particular, not every frequently hypercyclic operator satisfies the Frequent Hypercyclicity Criterion.

Problem. Is every chaotic operator frequently hypercyclic?

This is only one of many interesting open problems concerning frequently hypercyclic operators.

5.5. Ansari's theorem revisited.

In this final section we shall return to Ansari's theorem, and we shall return to where we started from, the consideration of **non-linear maps**.

The results presented here come from recent joint work with A. Piqueras and F. León-Saavedra.

Ansari says that powers of hypercyclic operators are hypercyclic.

But the simple example

$$f : \{-1, 1\} \to \{-1, 1\}, \quad x \to -x$$

shows that this is not so for non-linear maps.

So the question arises: what do linear mappings have that non-linear ones don't?

Idea: we shall try to get as close as possible to Ansari's result for general non-linear maps. We shall then see which property of a non-linear map is needed so that the full Ansari result holds.

This idea was first put forward in a paper by Bourdon (1996).

Let us describe his ideas.

Throughout this section, let

$$f : X \to X$$

be an arbitrary continuous map on a Hausdorff topological space X without isolated points.

We shall write

$$\mathcal{D} = \{x \in X : \mathsf{orb}(x, f) \quad \text{is dense in } X\}.$$

Now suppose that x has a dense orbit under f but not under f^2.

Then Bourdon showed that the set $\mathcal{D}$ splits into two sets $\mathcal{D}_0, \mathcal{D}_1$ with respect to which the dynamics of f is easy to describe.

This result allowed him to obtain a new proof of Ansari's theorem in the case $N = 2$.

Bourdon ended his paper by posing the **problem** of finding a similar separation when $N > 2$.

According to Bourdon, subsets $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{k-1}$ of $\mathcal{D}$ will be called a *separation* of $\mathcal{D}$ if they form a partition of $\mathcal{D}$ into non-empty relatively open (or, equivalently, relatively closed) sets.

Here is the solution of his problem:

Theorem (Separation Theorem).
Let x have dense orbit under f but not under $f^N, N > 1$.
Then there is a divisor $k > 1$ of N and a separation $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{k-1}$, of $\mathcal{D}$ such that

(i) $f(\mathcal{D}_0) \subset \mathcal{D}_1, f(\mathcal{D}_1) \subset \mathcal{D}_2, \ldots, f(\mathcal{D}_{k-2}) \subset \mathcal{D}_{k-1}, f(\mathcal{D}_{k-1}) \subset \mathcal{D}_0$;

(ii) for $0 \leq j < k$,

- (a) $f^j(x) \in \mathcal{D}_j$,
- (b) the orbit of $f^j(x)$ under $f^N : \mathcal{D}_j \to \mathcal{D}_j$ is dense in $\mathcal{D}_j$.

For $N = 2$, the result is due to Bourdon. Note that in this case, $k = 2$. Only if k is prime do we have a unique value for k (namely $k = N$). This explains perhaps why the extension of Bourdon's result to general k was not so obvious.

The following gives a sufficient condition, for **general non-linear maps**, for x to have dense orbit under f^N when it has dense orbit under f.

Corollary.
Let x have dense orbit under f.
If x and $f(x)$ belong to the same connected component of $\mathcal{D}$ then x has dense orbit under f^N, for every $N > 1$.

Proof. Suppose that, for some $N > 1$, x did not have dense orbit under f^N.

By the Separation Theorem, we would then obtain a separation of $\mathcal{D}$ into $k > 1$ sets $\mathcal{D}_0, \mathcal{D}_1, \ldots, \mathcal{D}_{k-1}$, where x belongs to $\mathcal{D}_0$ and $f(x)$ belongs to $\mathcal{D}_1$.

Since both these sets are open and closed in $\mathcal{D}$, x and $f(x)$ would belong to different connected components of $\mathcal{D}$, a contradiction. $\square$

This then allows us to deduce Ansari's theorem for linear maps:

Corollary (Ansari's theorem). If T is a hypercyclic operator defined on a Hausdorff topological vector space X and $x \in X$ is hypercyclic for T then it is also hypercyclic for any $T^N, N > 1$.

Proof. By the preceding corollary we only need to show that x and Tx are in the same connected component of $\mathcal{D} = HC(T)$.

In fact, the two vectors can be joined by a straight line of hypercyclic vectors!

To see this, let

$$y = tx + (1-t)Tx, \quad 0 \le t \le 1.$$

We claim that y is hypercyclic.

But we have

$$y \in \text{span orb}(x, T),$$

and we had seen in the proof of Bourdon's theorem (see Section 2.4) that every non-zero vector from span orb(x, T) is hypercyclic (we have given the proof only for some spaces, but it is true in all topological vector spaces).

And $y \neq 0$ because x and Tx are linearly independent.$\square$

We shall now sketch the proof of the Separation Theorem.

The central idea is taken from the work of León-Saavedra and Müller on multiples of hypercyclic operators (see Section 5.3).

In fact, we shall link the dynamics of f to properties of certain sets on the complex unit circle.

We fix $N > 1$, and we set

$$\alpha = e^{\frac{2\pi i}{N}},$$

an Nth root of unity.

For arbitrary $u, v \in X$ let us consider the following subset of the unit circle:

$$\begin{aligned}\Gamma(u,v) &= \{\alpha^j : v \in \overline{\{f^{pN+j}(u) : p \geq 0\}}, j \geq 0\} \\ &= \{\alpha^j : v \in \overline{\text{orb}(f^j(u), f^N)}, j \geq 0\}.\end{aligned}$$

We then have the following two lemmas:

Lemma 1. If u has dense orbit under f then $\Gamma(u,v) \neq \varnothing$, for all $v \in X$.

Lemma 2. For all $u, v, w \in X$, $\Gamma(u,v)\Gamma(v,w) \subset \Gamma(u,w)$.

Let now x be a point with dense orbit under f but not under f^N.

It follows from Lemmas 1 and 2 that $\Gamma(x,x)$ is a subgroup of $\{1, \alpha, \ldots, \alpha^{N-1}\}$.

Thus there is some divisor k of N such that

$$\Gamma(x,x) = \{1, \alpha^k, \alpha^{2k}, \ldots, \alpha^{(\nu-1)k}\}$$

where $\nu k = N$, $\alpha^{\nu k} = 1$.

One can then prove the following.

Lemma 3. Let x and k be as stated. Then, for every $y \in X$ with dense orbit there is $0 \leq j < k$ such that

$$\Gamma(x,y) = \{\alpha^j, \alpha^{j+k}, \ldots, \alpha^{j+(\nu-1)k}\} = \alpha^j \Gamma(x,x).$$

We then define the sets $\mathcal{D}_j$ by

$$\mathcal{D}_j = \{y \in \mathcal{D} : \alpha^j \in \Gamma(x,y)\}, \qquad 0 \le j < k.$$

By Lemma 3, the subsets $\mathcal{D}_j$ of $\mathcal{D}$ are pairwise disjoint and their union is $\mathcal{D}$, so that they form a partition of $\mathcal{D}$.

It remains to show that each set $\mathcal{D}_j$ is closed, and hence also open, in $\mathcal{D}$ and that the properties (i) and (ii) of the Separation Theorem are satisfied.

Finally, $k = 1$ is not possible because then x would have dense orbit under f^N (by (ii)(b)).

Exercise. Finish the proof as indicated above.

A simple (if discrete) example shows that the divisors k of N in the Separation Theorem appear by necessity and cannot be replaced, in general, by N.

Example. Let $N > 1$ and $k > 1$ a divisor of N.

We consider the discrete space

$$X = \{0, 1, \ldots, k-1\}$$

and the continuous map

$$f : n \to n + 1 \ (\text{mod } k).$$

Let $x = 0$, which has dense orbit under f.

Then the separation theorem holds with

$$\mathcal{D}_j = \{j\}, \quad 0 \le j < k,$$

and no other number k works here.

One unpleasant feature of the Separation Theorem is that it splits the set $\mathcal{D}$ of points with dense orbit in several parts. But this set is rarely known so that its splitting might not be so interesting.

One would rather want a partition of the whole space X into parts with respect to which the dynamics of f becomes easy.

We give here a corollary of the Separation Theorem in that direction. For $N = 2$ it is a theorem of Bourdon.

We make the same assumptions on f as before.

Theorem (Decomposition Theorem). Let x have dense orbit under f but not under $f^N, N > 1$.
Then there is a divisor $k > 1$ of N and open subsets $S_0, S_1, \ldots, S_{k-1}$ of X with the following properties:

(i) the sets $S_j, 0 \leq j < k$, are pairwise disjoint and $S := S_0 \cup S_1 \cup \ldots \cup S_{k-1}$ is dense in X;

(ii) $f(S_0) \subset S_1, f(S_1) \subset S_2, \ldots, f(S_{k-2}) \subset S_{k-1}$ and $f(S_{k-1}) \subset S_0 \cup (X \setminus S)$;

(iii) $X \setminus S$ is invariant under f;

(iv) for $0 \leq j < k$, the orbit of $f^j(x)$ under f^N is contained and dense in S_j.

We do not give the proof here.

Let us only say that the sets $S_0, \dots, S_{k-1}$ can be defined from the sets $\mathcal{D}_0, \dots, \mathcal{D}_{k-1}$ of the Separation Theorem in the following way:

$$\begin{aligned} S_{k-1} &= X \setminus \big(\overline{\mathcal{D}_0} \cup \overline{\mathcal{D}_1} \cup \ldots \cup \overline{\mathcal{D}_{k-2}}\big), \\ S_{k-2} &= f^{-1}(S_{k-1}), \\ S_{k-3} &= f^{-1}(S_{k-2}) = f^{-2}(S_{k-1}), \\ &\vdots \\ S_0 &= f^{-1}(S_1) = f^{-k+1}(S_{k-1}). \end{aligned}$$

The following example, taken from Bourdon, illustrates the Decomposition Theorem.

Example. Let C_0 and C_1 be the touching circles

$$C_0 = \mathbb{T}, \quad \text{and} \quad C_1 = 2\mathbb{T} - 1,$$

with $X = C_0 \cup C_1$. Let J be the map $J : C_0 \to C_1, z \to 2z - 1$.

Then the dynamical system

$$f : X \to X$$

is defined by

$$f(z) = \begin{cases} L(z^2), & \text{if } z \in C_0, \\ (L^{-1}z)^2, & \text{if } z \in C_1. \end{cases}$$

Then f has a dense orbit, while f^2 does not (it leaves C_0 invariant).

In fact, f is even chaotic (why?).

If $x \in C_0$ has dense orbit then we obtain as decomposition

$$\begin{aligned} S_0 &= C_0 \setminus \{-1, 1\}, \\ S_1 &= C_1 \setminus \{1\}, \\ X \setminus S &= \{-1, 1\}. \end{aligned}$$